Topics in Extremal Combinatorics (0366.4996)- **Fall** ′21

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Home Assignment 2

Due date: 7/12/21

Please submit organized and well written solutions!

Problem 1. Let G be an n-vertex graph of average degree t. Suppose we construct an independent set by repeatedly (i) removing a vertex v of minimum degree (ii) adding v to the independent set (iii) removing v's neighbors. Show that we are guaranteed to get an independent set of size at least $n/(1+t)$.

Problem 2. Let T be a k-partite graph where each of the classes $V_1, \ldots, V_k$ is of size t^{k-1}.

- Suppose none of the bipartite graphs (V_1, V_i) $(i > 1)$ contains an empty $t \times t$ bipartite graph. Conclude that some vertex $v \in V_1$ is adjacent to at least t^{k-2} vertices in each of the sets $V_2, \ldots, V_k$.
- Conclude that if T has no copy of K_k then it contains an empty $t \times t$ bipartite graph.
- Conclude that for any H on k vertices, if T has no induced copy of H with one vertex in each class V_i, then it contains either an empty $t \times t$ bipartite graph or a complete one, and hence that every n-vertex induced H-free graph contains either an empty or complete bipartite graph of size at least $(n/k)^{\frac{1}{k-1}}$.

Problem 3. Show that if H is a cograph then every n-vertex graph without an induced copy of H has a homogenous set of size n^{ϵ} for some $\epsilon = \epsilon(H) > 0$.

Problem 4. Show that for every graph H there is a graph G so that every 2-coloring of G contains an induced monochromatic copy of H. By this we mean that there is a set S of $|V(H)|$ vertices so that if we keep only the edges inside S of one of the colors, we will get an induced copy of H. Note that this is weaker than what is usually referred to as the Induced Ramsey Theorem.
Hint: Use the Erdős-Hajnal Theorem together with Erdős's lower bound for Ramsey's theorem.