

Color-Critical Graphs Have Logarithmic Circumference

Asaf Shapira *

Robin Thomas †

Abstract

A graph G is k -critical if every proper subgraph of G is $(k - 1)$ -colorable, but the graph G itself is not. We prove that every k -critical graph on n vertices has a cycle of length at least $\log n / (100 \log k)$, improving a bound of Alon, Krivelevich and Seymour from 2000. Examples of Gallai from 1963 show that the bound cannot be improved to exceed $2(k - 1) \log n / \log(k - 2)$. We thus settle the problem of bounding the minimal circumference of k -critical graphs, raised by Dirac in 1952 and Kelly and Kelly in 1954.

1 Introduction

All graphs in this paper are finite and simple; that is, they have no loops or multiple edges. Paths and cycles have no “repeated” vertices. A graph G is k -critical, where $k \geq 1$ is an integer, if every proper subgraph of G is $(k - 1)$ -colorable, but the graph G itself is not. There is an easy description of k -critical graphs for $k \leq 3$, but for $k \geq 4$ their structure appears complicated and no meaningful characterization is known.

The study of k -critical graphs was introduced in the 1940s by Dirac as part of his PhD Thesis. Since then k -critical graphs have been studied extensively, as documented for instance in [8, Chapter 5]. In this paper we study the circumference of k -critical graphs, where the circumference of a graph G is the length of the longest cycle in G . The only 3-critical graphs are odd cycles, but for $k \geq 4$ the circumference problem is more complicated. For integers $k \geq 4$ and $n > k + 1$ let $L_k(n)$ denote the largest integer l such that every k -critical graph on n vertices has circumference at least l . Elementary constructions show that the function $L_k(n)$ is well-defined for all integers $k \geq 4$ and $n > k + 1$. The study of the function $L_k(n)$ originated in the work of Dirac [5] and Kelly and Kelly [9].

As every k -critical graph has minimum degree at least $k - 1$, we have $L_k(n) \geq k$. Dirac [5] showed that $L_k(n) \geq 2k - 2$ for all $n \geq 2k - 2$ and conjectured that k -critical graphs should contain much

*School of Mathematics and School of Computer Science, Georgia Institute of Technology, Atlanta, GA 30332. Supported in part by NSF Grant DMS-0901355.

†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160. Supported in part by NSF Grant number DMS-0739366. 21 August 2009.

longer cycles. Specifically, he conjectured that for every fixed k we have $\lim_{n \rightarrow \infty} L_k(n) = \infty$ and that actually $L_k(n) \geq c\sqrt{n}$.

The first non-trivial bounds on $L_k(n)$ were obtained in 1954 by Kelly and Kelly [9] who showed that

$$\lim_{n \rightarrow \infty} L_k(n) = \infty, \quad (1)$$

thus confirming the first conjecture of Dirac mentioned above. According to [1] Kelly and Kelly [9] actually proved that

$$L_k(n) \geq \sqrt{\frac{\log \log n}{\log \log \log n}} \quad (2)$$

for every fixed $k \geq 4$ and all sufficiently large n . They also showed that

$$\liminf_{n \rightarrow \infty} L_4(n) / \log^2 n \leq 3 / \log^2(27/4), \quad (3)$$

thus disproving Dirac's second conjecture for $k = 4$. Dirac [6] later extended the upper bound of [9] to all $k \geq 4$, and Read [12] later improved the upper bound by showing that

$$\liminf_{n \rightarrow \infty} L_k(n) / (\log n \cdot \log^{(2)} n \cdot \log^{(3)} n \cdots \log^{(k-4)} n \cdot (\log^{(k-3)} n)^2) \leq (2 / \log 4)^{k-2}, \quad (4)$$

where $\log^{(i)}(x)$ is the i -times iterated logarithm function. The best known upper bound on $L_k(n)$ was obtained in 1963 by Gallai [7], who improved and significantly simplified the previous constructions by showing that for every integer $k \geq 4$ there are infinitely many integers n such that

$$L_k(n) \leq \frac{2(k-1)}{\log(k-2)} \log n. \quad (5)$$

We present Gallai's examples in Section 6, and we also point out that the same graphs establish the related fact that for every integer $k \geq 4$ there are infinitely many integers n such that

$$\text{there exists a } k\text{-critical graph on } n \text{ vertices with no path of length exceeding } \frac{4(k-1)}{\log(k-2)} \log n. \quad (6)$$

As for lower bounds on $L_k(n)$, the first improvement of the result of Kelly and Kelly [9] came after almost 50 years, when Alon, Krivelevich and Seymour [1] obtained the following (exponential) improvement of (2) for all integers $k \geq 4$ and all integers $n \geq k + 2$:

$$L_k(n) \geq 2 \sqrt{\frac{\log(n-1)}{\log(k-2)}}. \quad (7)$$

The proof of (7) in [1] is based on a result (implicit in [1]) which says that every k -critical graph on n vertices has a path of length at least $\log n / (\log(k-2))$. For completeness we state and prove it as Lemma 2.2 below. This is asymptotically best possible by (6).

The main result of this paper is the following improvement of the theorem of [1].

Theorem 1.1 *For every integer $k \geq 4$ and every integer $n \geq k + 2$ we have*

$$L_k(n) \geq \frac{\log n}{100 \log k} .$$

The following corollary solves, for every fixed $k \geq 4$, the problem of determining the order of magnitude of $L_k(n)$. The problem originated in the work of Dirac [5] and Kelly and Kelly [9], and is also stated in [8, Problem 5.11]. The lower bound follows immediately from Theorem 1.1; the upper bound follows by a minor modification of Gallai's proof of (5) and is presented as Theorem 6.1.

Corollary 1.2 *For every integer $k \geq 4$ and every integer $n \geq k + 2$*

$$\frac{\log n}{100 \log k} \leq L_k(n) \leq \frac{2(k-1)}{\log(k-2)} \log n + 2k .$$

The corollary raises the obvious question whether there exist a function f and absolute constants c_1, c_2 such that $c_1 f(k) \log n \leq L_k(n) \leq c_2 f(k) \log n$. This remains an interesting open problem. The related (and perhaps easier) question, where we ask for the length of the longest path, is also open. Currently, the best known bounds for the latter problem are given by (6) and Lemma 2.2.

There is a related problem, formulated by Nešetřil and Rödl at the International Colloquium on Finite and Infinite Sets in Keszthely, Hungary in 1973; see [10] for a detailed history of the problem. Nešetřil and Rödl asked whether it is true that for every two integers $k, n \geq 4$ there exists an integer N such that every k -critical graph on at least N vertices has a $(k-1)$ -critical subgraph on at least n vertices. For $k = 4$ the answer is yes, for the following reason. By (1) a large enough k -critical graph G has a long cycle C . Since G is not bipartite, it has an odd cycle, say C' . The graph G is 2-connected by Lemma 3.2(i) below. Now an elementary argument using just the 2-connectivity of G shows that G has an odd cycle of length at least $|V(C)|/2$. (The details may be found in [1].) This argument and Theorem 1.1 imply the following corollary.

Corollary 1.3 *Let $k, n \geq 4$ be integers. Then every k -critical graph on n vertices has a 3-critical subgraph on at least $\log n / (200 \log k)$ vertices.*

This is an improvement over the bound $\sqrt{\log n / \log(k-1)}$ of Alon, Krivelevich and Seymour [1]. The problem of Nešetřil and Rödl is open for all $k \geq 5$.

The rest of the paper is organized as follows. In Section 2 we prove a lemma that is implicit in [1] and deduce (7) from it, and give an overview of the proof of Theorem 1.1. In Section 3 we prove some basic results concerning k -critical graphs which will be used in the proof of Theorem 1.1. In Section 4 we prove a variation of a theorem of Bondy and Locke [2], stated as Theorem 2.4 below. The proof of Theorem 1.1 appears in Section 5. In Section 6 we present Gallai's construction that leads to the upper bound (5) and statement (6), and point out how to deduce the upper bound in Corollary 1.2 from it.

2 Proof Overview

If T is a tree and $x, y \in V(T)$, then there is a unique path in T with ends x and y , and we will denote it by xTy . Let G be a graph, let T be a spanning tree of G , and let $v \in V(T)$. The tree T is called a *depth-first search (DFS) spanning tree rooted at v* if for every edge $xy \in E(G)$ either $x \in V(vTy)$ or $y \in V(vTx)$. It is easy to see that for every connected graph G and every vertex $v \in V(G)$ there is a DFS spanning tree in G rooted at v . For $X = \emptyset$ the following lemma is implicit in [1]. The straightforward generalization will be needed later in the paper.

Lemma 2.1 *Let G be a k -critical graph on n vertices, let $X \subseteq V(G)$ have size s , and let T be a DFS spanning tree of $G \setminus X$ rooted at a vertex t_0 . Then for every integer $j \geq 1$ the number of vertices of T at distance exactly j from t_0 is at most $(k-1)^s$ if $j = 1$ and $(k-2)^{j-2}(k-1)^s$ otherwise.*

Proof. Let G, X, T, t_0 be as stated, let $X = \{x_1, x_2, \dots, x_s\}$, and let $t \in V(T)$ be a vertex at distance $j \geq 1$ from t_0 in T . We wish to define a sequence $Q(t)$. Let $t_0, t_1, \dots, t_{j-1}, t_j = t$ be the vertices of the path tTt_0 , listed in order. Since G is k -critical, the graph $G \setminus tt_{j-1}$ obtained from G by deleting the edge tt_{j-1} is $(k-1)$ -colorable, and since t_0 is adjacent to t_1 it has a $(k-1)$ -coloring ϕ such that $\phi(t_0) = 1$ and $\phi(t_1) = 2$. We define $Q(t) := (\phi(t_2), \phi(t_3), \dots, \phi(t_{j-1}), \phi(x_1), \phi(x_2), \dots, \phi(x_s))$. Since for $i = 1, 2, \dots, j$ the vertex t_i is adjacent to t_{i-1} , there are at most $(k-2)^{j-2}(k-1)^s$ sequences that arise this way (at most $(k-1)^s$ if $j = 1$). It follows that if there are more than $(k-2)^{j-2}(k-1)^s$ vertices at distance j from t_0 in T (or more than $(k-1)^s$ if $j = 1$), then there are two vertices t, t' at distance j from t_0 such that $Q(t) = Q(t')$. Let $t'_0 = t_0, t'_1, \dots, t'_{j-1}, t'_j = t'$ be the vertices of the path t_0Tt' , let p be the largest integer such that $t_p = t'_p$, and let the set Z consist of the vertex t_{p+1} and all its descendants in the rooted tree (T, t_0) . Let ϕ be a coloring as above, and let ϕ' be the analogous coloring of $G \setminus t't'_{j-1}$. The fact that $Q(t) = Q(t')$ implies that

$$\phi(u) = \phi'(u) \text{ for every } u \in X \cup V(t_0Tt_p). \quad (8)$$

We now define a coloring ψ by $\psi(u) = \phi(u)$ for every $u \in V(G) - Z$ and $\psi(u) = \phi'(u)$ for every $u \in Z$. Since T is a DFS spanning tree of $G \setminus X$, it follows that every edge of G with one end in Z and the other end in $V(G) - Z$ has the other end in $X \cup V(t_0Tt_p)$. It follows from (8) that ψ is a valid $(k-1)$ -coloring of G , contrary to the k -criticality of G . ■

The above lemma is the main tool in the proof of (7). Alon, Krivelevich and Seymour [1] use it to deduce that every k -critical graph has a long path, as follows.

Lemma 2.2 *For every integer $k \geq 4$ every k -critical graph on n vertices has a path of length at least $\log n / \log(k-2)$.*

Proof. Let $k \geq 4$ be an integer, let G be a k -critical graph on n vertices, and let $v \in V(G)$ be some vertex in G . Since G is connected, it has a DFS spanning tree T rooted at v . Let h be the length of a longest path in T with one end v . By Lemma 2.1 applied with $X = \emptyset$ we deduce that

$$n \leq 1 + 1 + 1 + (k - 2) + (k - 2)^2 + \cdots + (k - 2)^{h-2} \leq \sum_{j=0}^{h-1} (k - 2)^j \leq (k - 2)^h,$$

because $h \geq 2$ and $k \geq 4$. It follows that $h \geq \log n / \log(k - 2)$, as desired. \blacksquare

The next lemma is due to Dirac and Voss. A proof may be found in [1, 11].

Lemma 2.3 *If a 2-connected graph has a path of length l , then it has a cycle of length at least $2\sqrt{l}$.*

Since every k -critical graph is 2-connected by Lemma 3.2 below, the lower bound (7) of Alon, Krivelevich and Seymour follows immediately from Lemmas 2.2 and 2.3. However, the bound in Lemma 2.3 is tight and the bound in Lemma 2.2 is asymptotically tight by (6), and so an improvement of (7) requires a different strategy. For 3-connected graphs the bound can be dramatically improved, as shown by Bondy and Locke [2]:

Theorem 2.4 *If a 3-connected graph has a path of length l then it has a cycle of length at least $2l/5$.*

So combining Lemma 2.2 and Theorem 2.4 we get that every 3-connected k -critical graph has a cycle of length at least $2 \log n / (5 \log k)$. Unfortunately, not all k -critical graphs are 3-connected, but those that are not can be constructed from two smaller k -critical graphs. This is a result of Dirac [4] and is described in Lemma 3.2; a proof may also be found in [11, Problem 9.22]. Thus one might hope that we could use this result of Dirac and apply induction. That is indeed our strategy, but it turns out that it is not enough (at least in our proof) to just decompose the graph once and apply induction; instead, we need to break the graph repeatedly by (non-crossing) cutsets of size two and use the resulting tree-structure. That brings us to the notion of tree-decomposition, which formalizes this break up process.

Definition 2.5 A *tree decomposition* of a graph G is a pair (T, \mathcal{W}) where T is a tree and $\mathcal{W} = (W_t : t \in V(T))$ is a collection of subsets of $V(G)$ such that

- $\bigcup_{t \in V(T)} W_t = V(G)$ and every edge of G has both ends in some W_t , and
- If $t, t', t'' \in V(T)$ and t' belongs to the unique path in T connecting t and t'' , then $W_t \cap W_{t''} \subseteq W_{t'}$.

For $t \in V(T)$ we define the *torso of (G, T, \mathcal{W}) at t* to be the graph with vertex-set W_t in which $u, v \in W_t$ are adjacent if either they are adjacent in G or $u, v \in W_{t'}$ for some neighbor t' of t in T . We say that the tree-decomposition (T, \mathcal{W}) is *standard* if $|W_t \cap W_{t'}| = 2$ for every edge $tt' \in E(T)$

and each torso of (G, T, \mathcal{W}) is 3-connected or a cycle. (A graph G is t -connected if it has at least $t + 1$ vertices and $G \setminus X$ is connected for every set $X \subseteq V(G)$ of size at most $t - 1$).

We will see in Lemma 3.1 that every k -critical graph has a standard tree-decomposition. The torsos are not necessarily critical, but they are very close, so that is not really an issue, and so for the purpose of this outline we can pretend that every torso satisfies the conclusion of Lemma 2.2. By Theorem 2.4 we deduce that each torso has a sufficiently long cycle, but we need more. We actually need a “linkage”, a set of two disjoint paths with prescribed ends so that we can combine these linkages in individual torsos to produce a cycle in the original graph. We deduce the existence of such a linkage of desired length from Theorem 2.4 in Lemma 4.1, but only under the assumption that the two sets of prescribed ends are disjoint from each other; otherwise Lemma 4.1 is false. Thus when the two sets of prescribed ends are not disjoint we need a different method. In that case we are really looking for one path rather than a linkage, and we use Lemma 2.1 to find it.

3 Some Preliminary Lemmas

In this section we prove some preliminary lemmas and introduce several concepts that will be used later on in the proof of Theorem 1.1 in Section 5. Our first lemma is well-known and appears in [3, Exercise 12.20].

Lemma 3.1 *Every 2-connected graph has a standard tree-decomposition.*

Let G be a graph, and let x, y be distinct vertices of G . If x, y are not adjacent, then we define $G + xy$ to be the graph obtained from G by adding an edge joining x and y , and if x, y are adjacent, then we define $G + xy$ to be G . We define G/xy to be the graph obtained from G by deleting the edge xy (if it exists), identifying the vertices x and y and deleting all resulting parallel edges. Actually, in all applications of the operation G/xy in this paper the vertices x and y will not be adjacent, and they will have no common neighbors, so the clause about deleting parallel edges will not be needed. Statement (iv) of the following lemma is due to Dirac [4], and a proof may also be found in [11, Problem 9.22].

Lemma 3.2 *Let $k \geq 4$ be an integer, let G be a k -critical graph, and let $u, v \in V(G)$ be such that $G \setminus \{u, v\}$, the graph obtained from G by deleting the vertices u and v , is disconnected. Then*

- (i) $u \neq v$, and hence G is 2-connected,
- (ii) u is not adjacent to v ,
- (iii) $G \setminus \{u, v\}$ has exactly two components, and
- (iv) there are unique proper induced subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u, v\}$, the graphs $G_1 \setminus \{u, v\}$ and $G_2 \setminus \{u, v\}$ are the two components of $G \setminus \{u, v\}$, u, v have no common neighbor in G_2 , and $G_1 + uv$ and G_2/uv are k -critical.

Proof. Since $G \setminus \{u, v\}$ is disconnected, there is an integer $s \geq 2$ such that the graph G can be expressed as $G = G_1 \cup G_2 \cup \dots \cup G_s$, where the graphs G_i are pairwise edge-disjoint, each has at least one edge, and $V(G_i \cap G_j) = \{u, v\}$ for distinct integers $i, j \in \{1, 2, \dots, s\}$. Since G is k -critical, each G_i is $(k-1)$ -colorable. If $u = v$, then each G_i has a $(k-1)$ -coloring that gives the vertex u color 1. Those colorings can be combined to produce a $(k-1)$ -coloring of G , contrary to the k -criticality of G . Thus $u \neq v$, and statement (i) follows.

We now prove (ii) and (iii) simultaneously. If one of them does not hold, then we may assume that $s = 3$. (If (ii) does not hold, then G_3 may be chosen to consist of u, v , and the edge joining them.) Let us say that G_i is of type one if some $(k-1)$ -coloring of G_i gives u and v the same color, and let us say it is of type two if some $(k-1)$ -coloring of G_i gives u and v different colors. Thus each G_i is of type one or type two. We claim that no G_i is of both types. For suppose for a contradiction that G_3 is of both types. But G_1 and G_2 are of the same type, because $G_1 \cup G_2$ is $(k-1)$ -colorable by the k -criticality of G , and hence it follows that $G_1 \cup G_2 \cup G_3 = G$ is $(k-1)$ -colorable, a contradiction. Thus no G_i is of both types. We may assume that G_1 and G_2 are of the same type and that G_3 is of different type. But then $G_1 \cup G_3$ is not $(k-1)$ -colorable, contrary to the k -criticality of G . This proves (ii) and (iii). In particular, $s = 2$.

We now prove (iv). We may assume from the symmetry that G_1 is of type one and G_2 is of type two. Since G is not k -colorable, it follows that G_1 is not of type two and G_1 is not of type one. It follows that neither $G_1 + uv$ nor G_2/uv is $(k-1)$ -colorable. It remains to show that u, v have no common neighbor in G_2 and that every proper subgraph of $G_1 + uv$ and G_2/uv is $(k-1)$ -colorable, because the remaining properties are clear or follow from (iii). Suppose for a contradiction that $w \in V(G_2) - \{u, v\}$ is a neighbor of both u and v in G . Then the graph $G \setminus uw$ has a $(k-1)$ -coloring ϕ by the k -criticality of G . Since G_1 is not of type two we deduce that $\phi(u) = \phi(v)$. But $\phi(v) \neq \phi(w)$, because v is adjacent to w in $G \setminus uw$. Thus ϕ is $(k-1)$ -coloring of G , a contradiction. This proves that u, v have no common neighbor in G_2 . Let e be an edge of $G_1 + uv$. If $e \neq uv$, then let ψ be a $(k-1)$ -coloring of $G \setminus e$. We have $\psi(u) \neq \psi(v)$, because G_2 is not of type one, and hence ψ is a $(k-1)$ -coloring of $(G_1 + uv) \setminus e$. For $e = uv$ we note that G_1 is $(k-1)$ -colorable by the k -criticality of G . Finally, let f be an edge of G_2/uv , and let λ be a $(k-1)$ -coloring of $G \setminus f$. Since G_1 is not of type two, $\lambda(u) = \lambda(v)$, and hence λ can be converted to a $(k-1)$ -coloring of $(G_2/uv) \setminus f$, as desired. ■

The first three statements of Lemma 3.2 have the following consequence.

Lemma 3.3 *Let $k \geq 4$ be an integer, let G be a k -critical graph, and let (T, \mathcal{W}) be a standard tree-decomposition of G .*

- (i) *If t_0, t_1 are adjacent in T , then the two vertices in the set $W_{t_0} \cap W_{t_1}$ are not adjacent, and*
- (ii) *if t_1, t_2 are distinct neighbors of t_0 in T , then $W_{t_0} \cap W_{t_1} \neq W_{t_0} \cap W_{t_2}$.*

Proof. We prove only (ii), leaving (i) to the reader. Suppose for a contradiction that $W_{t_0} \cap W_{t_1} = W_{t_0} \cap W_{t_2}$, and let X denote this 2-element set. Let $i \in \{0, 1, 2\}$. Since (T, \mathcal{W}) is standard, W_{t_i} has at least three elements, and hence there exists a vertex $v_i \in W_{t_i} - X$. Then v_0, v_1, v_2 belong to three different components of $G \setminus X$, contrary to Lemma 3.2(iii). ■

Part (iv) of Lemma 3.2 leads to the following construction, which will modify each torso of a tree-decomposition of a k -critical graph and turn it into a k -critical graph. Let G be a k -critical graph and let (T, \mathcal{W}) be a standard tree-decomposition of G such that each W_t has at least three elements. Let $t \in V(T)$, and let $u, v \in W_t$ be distinct. We say that the pair uv is a virtual edge of W_t if $W_t \cap W_{t'} = \{u, v\}$ for some neighbor t' of t in T . Thus Lemma 3.3 asserts that the virtual edges of each W_t are pairwise distinct, and that they are not edges of G (but they are edges of the torso at t , by definition of torso). We now classify virtual edges of W_t into additive and contractive, as follows. Let uv be a virtual edge of W_t , and let t' be the neighbor of t in T such that $W_t \cap W_{t'} = \{u, v\}$. Since $G \setminus \{u, v\}$ is disconnected, there exist graphs G_1, G_2 as in Lemma 3.2(iv). Then W_t is a subset of exactly one of $V(G_1), V(G_2)$; if $W_t \subseteq V(G_1)$, then we say that the virtual edge uv is *additive*; otherwise we say that it is *contractive*. We now define a graph N_t as the graph obtained from $G[W_t]$ by adding the edge uv for every additive virtual edge uv of W_t , and identifying the vertices u and v for every contractive virtual edge uv of W_t . In other words, N_t can be regarded as being obtained from the torso of (G, T, \mathcal{W}) at t by contracting all contractive virtual edges of W_t . We call N_t the *nucleus* of (G, T, \mathcal{W}) at t . The next lemma shows that the nucleus is well-defined in the sense that the vertex identifications used during the construction do not produce loops or parallel edges.

Lemma 3.4 *Let $k \geq 4$ be an integer, let G be a k -critical graph, let (T, \mathcal{W}) be a standard tree-decomposition of G , let $t \in V(T)$ and let H denote the torso of (G, T, \mathcal{W}) at t . Then*

- (i) *the subgraph of H induced by contractive virtual edges of W_t is a forest, and for every component R of this forest and every $v \in V(H) - V(R)$, at most one vertex of R is adjacent to v in H , and*
- (ii) *the nucleus N of (G, T, \mathcal{W}) at t is k -critical.*

Proof. We proceed by induction on the number of vertices of T . If T has only one vertex, then there are no virtual edges and $N_t = G$, and hence both statements of the lemma hold. We may therefore assume that T has more than one vertex, and that the lemma holds for all k -critical graphs that have a standard tree-decomposition using a tree with strictly fewer than $|V(T)|$ vertices. Let t' be a neighbor of t in T , and let $W_t \cap W_{t'} = \{u, v\}$, so that uv is a virtual edge of W_t . Let T' be the component of $T \setminus tt'$ containing t , let $\mathcal{W}' = (W_r : r \in V(T'))$, and let G' be the subgraph of G induced by the union of all W_r over all $r \in V(T')$.

Assume first that uv is an additive virtual edge. Then $G' + uv$ is k -critical by Lemma 3.2(iv) and (T', \mathcal{W}') is a standard tree-decomposition of $G' + uv$, where T' has strictly fewer vertices than

T . Furthermore, H is equal to the torso of $(G' + uv, T', \mathcal{W}')$ at t , and N is equal to the nucleus of $(G' + uv, T', \mathcal{W}')$ at t . Thus both conclusions follow by induction applied to $G' + uv$ and the tree-decomposition (T', \mathcal{W}') . This completes the case when uv is an additive virtual edge.

We may therefore assume that uv is a contractive virtual edge. In this case we proceed analogously, applying induction to the graph G'/uv and the tree-decomposition obtained from (T', \mathcal{W}') by replacing each occurrence of u or v by the new vertex of G'/uv that resulted from the identification of u and v . In the proof of (i) we take advantage of the provision in Lemma 3.2(iv) that guarantees that u, v have no common neighbor in G' . ■

Lemma 3.5 *If a graph N is obtained from a graph H by repeatedly contracting edges, each time contracting an edge that belongs to no triangle, and H has minimum degree at least three, then $|E(H)| \leq 3|E(N)|$.*

Proof. Let d_1, d_2, \dots, d_n be the degree sequence of N , and let us consider the reverse process that produces H starting from N . Then the i^{th} vertex of N gives rise to at most $d_i - 3$ new edges of H . Thus

$$|E(H)| \leq |E(N)| + (d_1 - 3) + (d_2 - 3) + \dots + (d_n - 3) \leq 3|E(N)|.$$

as desired. ■

Lemma 3.6 *Let $k \geq 4$ be an integer, let G be a k -critical graph, let (T, \mathcal{W}) be a standard tree-decomposition of G , and let $t \in V(T)$. Then the torso of (G, T, \mathcal{W}) at t is 3-connected.*

Proof. Let H denote the torso of (G, T, \mathcal{W}) at t . If H is not 3-connected, then it is a cycle by the definition of standard tree-decomposition. But then the nucleus of (G, T, \mathcal{W}) at t is a cycle, because it is obtained from H by contracting edges, contrary to Lemma 3.4(ii). ■

Lemma 3.7 *Let $k \geq 4$ be an integer, let G be a k -critical graph, let (T, \mathcal{W}) be a standard tree-decomposition of G , let $t \in V(T)$, and let N be the nucleus of (G, T, \mathcal{W}) at t . Then $\deg_T(t) \leq 3|E(N)|$.*

Proof. Let H be the torso of (G, T, \mathcal{W}) at t . We first notice that $\deg_T(t) \leq |E(H)|$, because each neighbor of t in T gives rise to a unique virtual edge of G at t by Lemma 3.3(ii), and each virtual edge belongs to H . The graph H is 3-connected by Lemma 3.6. By Lemma 3.4(i) the graph N is obtained from H as in Lemma 3.5, and hence $|E(H)| \leq 3|E(N)|$ by that lemma, as desired. ■

Lemma 3.8 *Let $k \geq 4$ be an integer, let G be a k -critical graph, let (T, \mathcal{W}) be a standard tree-decomposition of G , and let $t \in V(T)$. Then the nucleus of (G, T, \mathcal{W}) at t has at least as many edges as $G[W_t]$.*

Proof. This follows from the fact that no edges of $G[W_t]$ are lost during the construction of the nucleus. ■

Lemma 3.9 *Let $k \geq 4$ be an integer, let G be a k -critical graph, let (T, \mathcal{W}) be a standard tree-decomposition of G , let $t \in V(T)$, and let N be the nucleus of (G, T, \mathcal{W}) at t . Then the torso of (G, T, \mathcal{W}) at t has a path of length at least $\frac{1}{2} \log |E(N)| / \log k$.*

Proof: Let H denote the torso of (G, T, \mathcal{W}) at t . By Lemma 3.4 the graph N is k -critical, and so by Lemma 2.2 it has a path of length at least $\log |V(N)| / \log k \geq \frac{1}{2} \log |E(N)| / \log k$. Since N is obtained from H by contracting edges, H has a path at least as long. ■

Finally we need an easy lemma about trees. If G is a graph, $\phi : V(G) \mapsto \{0, 1, 2, 3, \dots\}$ is a mapping, and H is a subgraph of G , then we define $\phi(H) := \sum_{v \in V(H)} \phi(v)$.

Lemma 3.10 *Let $k \geq 2$ be an integer, let T be a tree, let $r \in V(T)$, and assume that for every integer $l \geq 1$ there are at most k^l vertices at distance exactly l from r in T . Let $\phi : V(T) \mapsto \{0, 1, \dots\}$ be a weight function with $\phi(r) = 0$ and $\phi(t) \neq 0$ for at least one vertex in $t \in V(T)$. Then there exists a vertex $t \in V(T)$ at distance exactly l from r in T such that $\phi(t) > 0$ and*

$$2l \log k + \log \phi(t) \geq \log \phi(T) .$$

Proof. For every integer $l \geq 0$ let D_l be the set of vertices of T at distance exactly l from r . Since $\phi(r) = 0$ we have that for some $l \geq 1$

$$\sum_{t \in D_l} \phi(t) \geq \phi(T) / 2^l ,$$

since if this is not the case, then

$$\phi(T) = \sum_{l \geq 1} \sum_{t \in D_l} \phi(t) < \phi(T) \sum_{l \geq 1} 2^{-l} \leq \phi(T) ,$$

a contradiction. Since $|D_l| \leq k^l$, we deduce that there is a vertex $t \in D_l$ satisfying

$$\phi(t) \geq \phi(T) / 2^l k^l \geq \phi(T) / k^{2l} , \tag{9}$$

because $k \geq 2$. It follows that

$$2l \log k + \log \phi(t) \geq 2l \log k + \log \phi(T) - 2l \log k = \log \phi(T) ,$$

as desired. ■

4 An Application of the Theorem of Bondy and Locke

Let G be a graph, and let $X, Y \subseteq V(G)$ be disjoint sets of size two. A *linkage* in G from X to Y is a set $\{P_1, P_2\}$ of two disjoint paths, each with one end in X and the other end in Y . The *length* of the linkage is defined to be $|E(P_1)| + |E(P_2)|$. The following is the main result of this section.

Lemma 4.1 *Let G be a 3-connected graph, let $X, Y \subseteq V(G)$ be disjoint sets of size two, and suppose that G has a path of length at least l . Then G has a linkage from X to Y of length at least $l/25$.*

The assumption that the sets X, Y be disjoint is necessary, as the following example shows. Let $t \geq 2$ be an integer, and let H be the graph obtained from a path with vertex-set $\{v_0, v_1, \dots, v_{t^2}\}$ in order by adding an edge joining v_{it} and $v_{(i+2)t}$ for every $i = 0, 1, \dots, t-2$. Finally, let G be obtained from H by adding a vertex u joined to all vertices of H . One can easily verify that G is 3-connected, has a path of length t^2 , and yet every linkage from $\{u, v_0\}$ to $\{u, v_2\}$ has length at most linear in t .

In the proof of Lemma 4.1 we will make use of the following lemma, which follows from the standard ‘‘augmenting path’’ proof of Menger’s theorem or the Max-Flow Min-Cut Theorem; see, for instance [3, Section 3.3].

Lemma 4.2 *Let $r \geq 1$ be an integer, let G be an r -connected graph, let S and T be two subsets of the vertex-set of G , each of size at least r , and let P_1, P_2, \dots, P_{r-1} be disjoint paths such that for $i = 1, 2, \dots, r-1$, the path P_i has ends $s_i \in S$ and $t_i \in T$. Then there exist disjoint paths Q_1, Q_2, \dots, Q_r in G between S and T in such a way that all but one of the paths Q_i has an end in $\{s_1, s_2, \dots, s_{r-1}\}$, and all but one of the paths Q_i has an end in $\{t_1, t_2, \dots, t_{r-1}\}$.*

The proof of Lemma 4.1 will consist of three steps. In the first step we will obtain either a required linkage, or a similar structure we call *hammock*, which we introduce next. In the second step we show that if a 3-connected graph has a long hammock, then it has either a long linkage, or a long ‘‘non-singular’’ hammock. Finally, we show how to get a required long linkage from the existence of a long non-singular hammock.

A *hammock* in G from X to Y is a quadruple $\eta = (P_1, P_2, R_1, R_2)$, where

- $\{P_1, P_2\}$ is a linkage from X to Y , where P_i has ends $x_i \in X$ and $y_i \in Y$,
- R_i is a path with ends $s_i \in V(P_1)$ and $t_i \in V(P_2)$, and is otherwise disjoint from $P_1 \cup P_2$,
- the paths R_1, R_2 are disjoint, except possibly $s_1 = s_2$,
- the vertices x_1, s_1, s_2, y_1 occur on P_1 in the order listed (but are not necessarily distinct), and
- the vertices x_2, t_1, t_2, y_2 occur on P_2 in the order listed (but are not necessarily distinct).

The *length* of the hammock η is defined to be $|E(R_1)|$. (It may seem more natural to define the length of η to be $|E(R_1)| + |E(R_2)|$. Indeed, by doing so it is possible to improve the constant 25 in Lemma 4.1 to 17.5, but only at the expense of more extensive case analysis. The extra effort did not seem justified.) We say that η is *singular* if $s_1 = s_2$, and *non-singular* otherwise.

Let us recall that if P is a path and $u, v \in V(P)$, then by uPv we denote the unique subpath of P with ends u and v .

Lemma 4.3 *Let G be a 3-connected graph, let $X, Y \subseteq V(G)$ be disjoint sets of size two, and assume that G has a cycle C of length l . Then G has either a linkage from X to Y of length at least $l/5$, or a hammock from X to Y or from Y to X of length at least $2l/5$.*

Proof: Assume first that there exist four disjoint paths P_1, P_2, P_3, P_4 , each with one end in $X \cup Y$ and the other end in $V(C)$. For $i = 1, 2, 3, 4$ let u_i and v_i be the ends of P_i such that $u_i \in V(C)$, $v_1, v_2 \in X$ and $v_3, v_4 \in Y$. If u_1, u_3, u_2, u_4 occur on C in the order listed, then let C_{13} denote the subpath of $C \setminus \{u_2, u_4\}$ with ends u_1 and u_3 , and let C_{23}, C_{24} , and C_{14} be defined analogously. Then either $P_1 \cup P_2 \cup P_3 \cup P_4 \cup C_{13} \cup C_{24}$ or $P_1 \cup P_2 \cup P_3 \cup P_4 \cup C_{23} \cup C_{14}$ is a linkage from X to Y of length at least $l/2$, as desired. Thus we may assume that u_1, u_2, u_3, u_4 occur on C in the order listed. Using analogous notation, if $|E(C_{14})| + |E(C_{23})| \geq l/5$, then $P_1 \cup P_2 \cup P_3 \cup P_4 \cup C_{14} \cup C_{23}$ is a linkage from X to Y in G of length at least $l/5$. Thus we may assume that $|E(C_{12})| + |E(C_{34})| \geq 4l/5$, and so from the symmetry we may assume that $|E(C_{12})| \geq 2l/5$. Then $(P_1 \cup C_{14} \cup P_4, P_2 \cup C_{23} \cup P_3, C_{12}, C_{34})$ is a hammock from X to Y in G of length at least $2l/5$, as desired. This completes the case when G has four disjoint paths from $X \cup Y$ to $V(C)$.

We may therefore assume that those four paths do not exist, and hence by Menger's theorem G can be expressed as $G_1 \cup G_2$, where $|V(G_1) \cap V(G_2)| = 3$, $X \cup Y \subseteq V(G_1)$ and $V(C) \subseteq V(G_2)$. Since G is 3-connected there exist three disjoint paths P_1, P_2, P_3 from $X \cup Y$ to $V(C)$ with no internal vertices in $X \cup Y \cup V(C)$. By symmetry, we may assume that P_i has ends u_i and v_i , where $u_i \in V(C)$, $v_1, v_2 \in X$ and $v_3 \in Y$. Then for $i = 1, 2, 3$ the set $V(G_1) \cap V(G_2) \cap V(P_i)$ includes a unique vertex, say w_i . Thus $V(G_1) \cap V(G_2) = \{w_1, w_2, w_3\}$. Please note that the sets $\{w_1, w_2, w_3\}$ and $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ may intersect.

By Lemma 4.2 applied to the path $v_3P_3w_3$ there exist two disjoint paths Q_1, Q_2 in G from Y to $V(P_1 \cup P_2) \cup \{w_3\}$, with no internal vertices in $Y \cup V(P_1 \cup P_2)$ and such that one of them, say Q_2 , ends in w_3 . From the symmetry we may assume that Q_1 ends in $V(P_1)$. Similarly as before, let C_{12} denote the subpath of $C \setminus u_3$ with ends u_1 and u_2 , and let C_{13} and C_{23} be defined similarly. If C_{23} has at least $l/5$ edges, then the disjoint subgraphs $P_1 \cup Q_1$ and $P_2 \cup Q_2 \cup C_{23} \cup w_3P_3w_3$ include a linkage from X to Y of length at least $l/5$, as desired. Thus we may assume that $|E(C_{23})| < l/5$. If $u_1 \notin V(Q_1)$, then replacing C_{23} by $C_{12} \cup C_{13}$ above results in a linkage from X to Y of length at least $4l/5$. Thus we may assume that $V(Q_1) \cap V(P_1) = \{w_1\}$ and $w_1 = u_1$. But now either $(P_1 \cup Q_1, P_2 \cup C_{23} \cup Q_2, C_{12}, C_{13})$ is a hammock from X to Y of length at least $2l/5$, or $(P_1 \cup Q_1, P_2 \cup C_{23} \cup Q_2, C_{13}, C_{12})$ is a hammock from Y to X of length at least $2l/5$, as desired. ■

Lemma 4.4 *Let G be a 3-connected graph, let $X, Y \subseteq V(G)$ be disjoint sets of size two, and let G have a hammock from X to Y of length l . Then G has either a non-singular hammock from X to Y*

or from Y to X of length at least $l/2$, or a linkage from X to Y of length at least $l/2$.

Proof: Let $\eta = (P_1, P_2, R_1, R_2)$ be a hammock in G of length l , and let $x_1, x_2, y_1, y_2, s_1, s_2, t_1, t_2$ be as in the definition of hammock. We may assume that $s_1 = s_2$, for otherwise η is non-singular, and hence satisfies the conclusion of the lemma. Since $x_1 \neq y_1$, at least one of the sets $A := V(x_1P_1s_1) - \{s_1\}$ and $B := V(y_1P_1s_2) - \{s_2\}$ is not empty.

Assume first that $A \neq \emptyset$. Since G is 3-connected, there is a path Q in $G \setminus \{s_1, t_1\}$ with ends $a \in A$ and $b \in V(P_2 \cup R_1 \cup R_2 \cup s_2P_1y_1)$. If $b \in V(R_2 \cup t_1P_2y_2)$, then $x_1P_1s_1 \cup Q \cup R_2 \cup t_1P_2y_2$ includes a path from x_1 to y_2 that together with $x_2P_2t_1 \cup R_1 \cup s_2P_1y_1$ forms a linkage from X to Y of length at least l . If $b \in V(x_2P_2t_1)$, then (P_1, P_2, R_1, Q) is a non-singular hammock from Y to X of length l . If $b \in V(s_1P_1y_1)$, then the paths $x_1P_1a \cup Q \cup bP_1y_1$ and $x_2P_2t_1 \cup R_1 \cup R_2 \cup t_2P_2y_2$ form a linkage from X to Y of length at least l . Thus we may assume that $b \in V(R_1)$. If the path t_1R_1b has at least $l/2$ edges, then $(P_1, P_2, Q \cup t_1R_1b, R_2)$ is a non-singular hammock from X to Y of length at least $l/2$, and if the path s_1R_1b has at least $l/2$ edges, then the paths P_2 and $x_1P_1a \cup Q \cup bR_1s_1 \cup s_1P_1y_1$ form a linkage from X to Y of length at least $l/2$. This completes the case $A \neq \emptyset$.

Thus we may assume that $B \neq \emptyset$. We take a path in $G \setminus \{s_2, t_2\}$ connecting a vertex in B to a vertex in $V(P_2 \cup R_1 \cup R_2 \cup x_1P_1s_1)$ and proceed similarly as in the previous paragraph. The details are analogous to the case $A \neq \emptyset$ and are left to the reader. \blacksquare

Lemma 4.5 *Let G be a 3-connected graph, let $X, Y \subseteq V(G)$ be disjoint sets of size two, and assume that G has a non-singular hammock from X to Y of length l . Then G has a linkage from X to Y of length at least $l/2$.*

Proof: Let $\eta = (P_1, P_2, R_1, R_2)$ be a non-singular hammock in G , and let $x_1, x_2, y_1, y_2, s_1, s_2, t_1, t_2$ be as in the definition of hammock. Since G is non-singular, (P_2, P_1, R_1, R_2) is also a hammock from X to Y of the same length as η , and hence there is symmetry between P_1 and P_2 . Let $A := x_1P_1s_1 \cup R_1 \cup x_2P_2t_1$ and $B := y_1P_1s_2 \cup R_2 \cup y_2P_2t_2$. Then A and B are paths in G . By Lemma 4.2 applied to the sets $V(A)$ and $V(B)$ and paths $s_1P_1s_2$ and $t_1P_2t_2$ there exist three disjoint paths Q, Q_1, Q_2 from $V(A)$ to $V(B)$ such that two of them have ends in $\{s_1, t_1\}$, and two have ends in $\{s_2, t_2\}$. From the symmetry we may assume that s_1 is an end of Q_1 ; let s'_2 be the other end of Q_1 . Similarly we may assume that t_1 is an end of Q_2 ; let t'_2 be the other end of Q_2 . Now the path $x_1P_1s_1 \cup Q_1 \cup s'_2By_1$ can play the role of P_1 , the path $x_2P_2t_1 \cup Q_2 \cup t'_2By_2$ can play the role of P_2 , and the path $s'_2Bt'_2$ can play the role of R_2 . In other words, we may assume (by changing the hammock η but not changing its length) that there exists a path Q from $V(A)$ to $V(B)$ that is disjoint from the paths P_1 and P_2 . Let a be the end of Q in $V(A)$, and let b be the end of Q in $V(B)$. From the symmetry we may assume that either $a \in V(x_1P_1s_1)$, or $a \in V(R_1)$ and the path aR_1t_1 has at least $l/2$ edges.

Assume first that $a \in V(x_1P_1s_1)$. If $b \in V(s_2P_1y_1)$, then the paths $x_1P_1a \cup Q \cup bP_1y_1$ and $x_2P_2t_1 \cup R_1 \cup s_1P_1s_2 \cup R_2 \cup t_2P_2y_2$ form a linkage from X to Y of length at least l , and if $b \in V(R_2 \cup t_2P_2y_2)$, then the path $x_2P_2t_1 \cup R_1 \cup s_1P_1y_1$ and a subpath of $x_1P_1a \cup Q \cup R_2 \cup t_2P_2y_2$ form a linkage from X to Y of length at least l . This completes the case when $a \in V(x_1P_1s_1)$.

We may therefore assume that $a \in V(R_1)$ and the path aR_1t_1 has at least $l/2$ edges. If $b \in V(s_2P_1y_1)$, then the paths $x_1P_1s_2 \cup R_2 \cup t_2P_2y_2$ and $x_2P_2t_1 \cup t_1R_1a \cup Q \cup bP_1y_1$ form a linkage from X to Y of length at least $l/2$, and if $b \in V(R_2 \cup t_2P_2y_2)$, then the path P_1 and a subpath of $x_2P_2t_1 \cup t_1R_1a \cup Q \cup R_2 \cup t_2P_2y_2$ form a linkage from X to Y of length at least $l/2$. ■

Proof of Lemma 4.1. Let G, X, Y be as stated, and assume that G has a path of length l . Then G has a cycle of length at least $2l/5$ by Theorem 2.4. By Lemma 4.3 we may assume that G has a hammock from X to Y of length at least $4l/25$, for otherwise the theorem holds. Similarly, by Lemma 4.4 we may assume that G has a non-singular hammock from X to Y of length at least $2l/25$. By Lemma 4.5 the graph G has a linkage from X to Y of length at least $l/25$, as desired. ■

5 Proof of Theorem 1.1

Notation. Throughout this section we will assume the following notation. Let $k \geq 4$ be an integer, let G be a k -critical graph on n vertices, and let (T, \mathcal{W}) be a standard tree-decomposition of G . One exists by Lemma 3.1. For $t \in V(T)$ let H_t denote the torso of (G, T, \mathcal{W}) at t , and let N_t denote the nucleus of (G, T, \mathcal{W}) at t . We select a vertex $r \in V(T)$ of degree one that we will regard as the root of T . Thus a *descendant* of a vertex $t \in V(T)$ is any vertex $t' \in V(T) - \{t\}$ such that t belongs to the path from r to t' in T . For $t \in V(T)$ we denote by T_t the subtree of T induced by t and all its descendants. We define a weight function $w : V(T) \rightarrow \{0, 1, \dots\}$ by $w(t) := |E(N_t)|$. Thus $w(t) \geq 6$ for every $t \in V(T)$ by Lemma 3.4(ii). According to the convention introduced prior to lemma 3.10, $w(T_t)$ means $\sum_{v \in V(T_t)} w(v)$. We now define, for every $t \in V(T)$, a set $X_t \subseteq W_t$ of size two. If $t \neq r$, then let t' be the parent of t in the rooted tree (T, r) and we set $X_t = W_t \cap W_{t'}$. If $t = r$ and T has at least two vertices, then let t' be the unique child of r in T and let $X_r \subseteq W_r$ be any set disjoint from $W_t \cap W_{t'}$ that consists of two vertices that are adjacent in G . Such a set exists because H_t is 3-connected by Lemma 3.6 and the elements of $W_t \cap W_{t'}$ form the only edge of H_t that does not belong to G . Finally, if T has only one vertex we choose X_t arbitrarily. For $t \in V(T)$ we denote by G_t the graph induced in G by the set of vertices $\bigcup_{t' \in V(T_t)} W_{t'}$.

In order to be able to apply Lemma 3.10 we prove the following lemma.

Lemma 5.1 *Let $t \in V(T) - \{r\}$, and let $X_t = \{x, x'\}$. Then the graph $G_t \setminus x$ has a spanning tree R such that for every integer $l \geq 0$ there are at most k^l vertices of R at distance exactly l from x' .*

Proof. Let G'_t be the subgraph of G induced by the union of all $W_{t'}$ over all $t' \in V(T) - V(T_t)$. Then $G_t \cap G'_t = X_t = W_t \cap W_{t'}$, where t' is the ancestor of t in the rooted tree (T, r) . By Lemma 3.2(iv) applied to G and the vertices x, x' the graph G_t was obtained from some k -critical graph H by either (i) deleting the edge xx' , or (ii) splitting a vertex of H into the two vertices x, x' .

Assume first that G_t was obtained by deleting the edge xx' . Since H is 2-connected by Lemma 3.2(i), the graph $H \setminus x$ has a DFS spanning tree R rooted at x' . We deduce from Lemma 2.1 applied to H and $X = \{x\}$ that R satisfies the conclusion of the lemma.

We may therefore assume that G_t was obtained from H by splitting a vertex, say z , into the two vertices x, x' . Since G is 2-connected by Lemma 3.2(i), there is an edge $e \in E(G_t)$ joining the vertex x' to a vertex in $V(G_t) - \{x, x'\}$. Then e is also an edge of H . Since H is 2-connected, it has a DFS spanning tree R' rooted at z such that e is the only edge of R' incident with z . The tree R' gives rise to a unique spanning tree R of $G_t \setminus x$ with the same edge-set in the obvious way. It follows from Lemma 2.1 applied to H and $X = \emptyset$ that R satisfies the conclusion of the lemma. ■

To prove Theorem 1.1 we prove, for the sake of induction, the following lemma.

Lemma 5.2 *For every $t \in V(T)$ the graph G_t has a path connecting the vertices of X_t of length at least $\log w(T_t)/(100 \log k)$.*

Let us first derive Theorem 1.1 from Lemma 5.2.

Proof of Theorem 1.1, assuming Lemma 5.2. Apply Lemma 5.2 with $t = r$. Since by definition the vertices of X_r are adjacent, we get a cycle of length at least $\log w(T)/(100 \log k)$. Since distinct nuclei are edge-disjoint by the definition of nucleus, Lemma 3.8 implies $w(T) = \sum_{t \in V(T)} |E(N_t)| \geq |E(G)| \geq n$, and hence G has a cycle of length at least $\log n/(100 \log k)$. ■

The rest of this section is devoted to a proof of Lemma 5.2. We first take care of the following special case.

Lemma 5.3 *Let $t \in V(T)$. The statement of Lemma 5.2 holds for t if $w(t) \geq w(T_t)/5$. In particular, the lemma holds for t if $|V(T_t)| = 1$.*

Proof. The second assertion follows from the first, and so it suffices to prove the first statement. By Lemma 3.9 the torso H_t has a path of length at least $\frac{1}{2} \log w(t)/\log k$ (recall that $w(t)$ is the number of edges in the nucleus N_t). Lemma 3.6 guarantees that H_t is 3-connected, and so by Theorem 2.4 we get that H_t has a cycle C of length at least $\frac{1}{5} \log w(t)/\log k$. Since H_t is 3-connected we get from Menger's Theorem that it contains two disjoint paths connecting X_t to C . Suppose these paths meet C at vertices w, w' . Then one of the two subpaths of C connecting w and w' has length at least $\frac{1}{10} \log w(t)/\log k$. Together with the two paths connecting the vertices of X_t to C we get a path P

in H_t connecting the vertices of X_t of length at least $\frac{1}{10} \log w(t)/\log k \geq \frac{1}{100} \log w(T_t)/\log k$ by the hypothesis of the lemma and the fact that $w(T_t) \geq 6$.

For every edge $e = uv \in E(P) - E(G)$ we do the following. By Lemma 3.3(ii) there is a unique neighbor t' of t in T such that $W_t \cap W_{t'} = \{u, v\}$. If $r \neq t$, then t' is not the parent of t in the rooted tree (T, r) , because $\{u, v\} \neq X_t$ by the choice of P . We claim that there exists a path P_e in $G_{t'}$ with ends u, v . Indeed, since (T, \mathcal{W}) is standard, there exists a vertex $w \in W_{t'} - \{u, v\}$. Since G is 2-connected by Lemma 3.2(i), there exist two paths P_1, P_2 in G with one end w and the other end in $\{u, v\}$, pairwise disjoint, except for w . Then $P_e := P_1 \cup P_2$ is a path in G with ends u, v . It follows that P_e is a path in $G_{t'}$, for otherwise some subpath Q of $P_e \setminus \{u, v\}$ joins the vertex $w \in V(G_{t'}) - \{u, v\}$ to a vertex of $V(G) - V(G_{t'})$. But then Q has an edge with one end in $V(G_{t'}) - \{u, v\}$ and the other end in $V(G) - V(G_{t'})$, contrary to definition of tree-decomposition. This proves our claim that P_e exists. We replace e by the path P_e and repeat the construction for each edge $e \in E(P) - E(G)$. For distinct edges $e, e' \in E(P) - E(G)$ the paths $P_e, P_{e'}$ have no internal vertices in common, because their interiors belong to disjoint subgraphs. We thus arrive at a path in G with ends in X_t of length at least $\log n/(100 \log k)$, as desired. \blacksquare

Proof of Lemma 5.2. We proceed by induction on $|V(T_t)|$. Since Lemma 5.3 establishes the base case $|V(T_t)| = 1$, we can assume henceforth that $|V(T_t)| > 1$ and that the lemma holds for trees of size less than $|V(T_t)|$. Let $X_t = \{x, x'\}$, let N be the children of t in the rooted tree (T, r) and define

$$N_0 = \{t' \in N : W_t \cap W_{t'} \cap X_t = \emptyset\}$$

$$N_1 = \{t' \in N : W_t \cap W_{t'} \cap X_t = \{x\}\}$$

$$N_2 = \{t' \in N : W_t \cap W_{t'} \cap X_t = \{x'\}\}$$

The sets N_0, N_1, N_2 form a partition of N . For $t \neq r$ this follows from Lemma 3.3(ii), and for $t = r$ this follows from the way we picked X_r . Therefore, either

$$\sum_{y \in N_0} w(T_y) \geq \frac{3}{4}(w(T_t) - w(t)), \quad (10)$$

or

$$\sum_{y \in N_1 \cup N_2} w(T_y) \geq \frac{1}{4}(w(T_t) - w(t)). \quad (11)$$

We first deal with the case (10). By Lemma 5.3 we may assume that $w(t) < w(T_t)/5$, and hence

$$\sum_{y \in N_0} w(T_y) \geq \frac{3}{5}w(T_t). \quad (12)$$

By Lemma 3.7 we know that $|N_0| \leq |N| \leq 3w(t)$. Therefore, there is a vertex $t' \in N_0$ for which

$$w(T_{t'}) \geq \frac{w(T_t)}{5w(t)}. \quad (13)$$

By Lemma 3.9 the graph H_t (the torso at t) has a path of length at least $\frac{1}{2} \log w(t) / \log k$. Therefore, by Lemma 3.6, we can apply Lemma 4.1 to the graph H_t and sets X_t and $X_{t'}$ to deduce that H_t has two disjoint paths P_1, P_2 from X_t to $X_{t'}$ satisfying

$$|E(P_1)| + |E(P_2)| \geq \frac{\log w(t)}{50 \log k}. \quad (14)$$

By the induction hypothesis the graph $G_{t'}$ has a path P connecting the pair of vertices of $X_{t'}$ satisfying

$$|E(P)| \geq \frac{\log w(T_{t'})}{100 \log k}. \quad (15)$$

Combining (13), (14) and (15) we get that $P_1 \cup P \cup P_2$ is a path in $G_t \cup H_t$ with ends in X_t of length at least

$$\begin{aligned} |E(P_1)| + |E(P)| + |E(P_2)| &\geq \frac{\log w(t)}{50 \log k} + \frac{\log w(T_{t'})}{100 \log k} \\ &\geq \frac{\log w(t)}{50 \log k} + \frac{\log w(T_t)}{100 \log k} - \frac{\log w(t) + \log 5}{100 \log k} \\ &\geq \frac{\log w(T_t)}{100 \log k}, \end{aligned}$$

because $w(t) \geq 6$. We now convert $P_1 \cup P \cup P_2$ to a path in G_t of length at least $\log w(T_t) / (100 \log k)$ in the same way as in the second paragraph of the proof of Lemma 5.3. This completes the proof when (10) holds.

Thus we may assume (11). From the symmetry between N_1 and N_2 we may assume that

$$\sum_{y \in N_1} w(T_y) \geq (w(T_t) - w(t)) / 8.$$

Again, by Lemma 5.3 we may assume that $w(t) < w(T_t) / 5$, and hence

$$\sum_{y \in N_1} w(T_y) \geq w(T_t) / 10. \quad (16)$$

It follows that $t \neq r$, for otherwise $N_1 = \emptyset$. We need to define a new weight function $\phi : V(G_t) - \{x\} \rightarrow \{0, 1, \dots\}$. Let $v \in V(G_t) - \{x\}$. If $v \in W_t$ and there exists a neighbor t' of t in T_t such that $W_{t'} \cap W_t = \{x, v\}$, then t' is unique by Lemma 3.3(ii), and we define $\phi(v) = w(T_{t'})$. If $v \notin W_t$ or no such t' exists, then we define $\phi(v) = 0$. Thus, in particular, $\phi(x') = 0$ by Lemma 3.3(ii). By Lemma 5.1 the graph $G_t \setminus x$ has a spanning tree R such that for every integer $l \geq 0$ there are at

most k^l vertices of R at distance exactly l from x' . Note that by (16) we have that the total weight of R satisfies

$$\phi(R) = \sum_{y \in N_1} w(T_y) \geq w(T_t)/10. \quad (17)$$

By Lemma 3.10 applied to the tree R and vertex x' there exists a vertex $v \in V(R)$ at distance l from x' in R such that $\phi(v) > 0$ and $2l \log k + \log \phi(v) \geq \log \phi(T)$. It follows that there is a path P in $G_t \setminus x$ from x' to v satisfying

$$2|E(P)| \log k + \log \phi(v) \geq \log \phi(R). \quad (18)$$

Since $\phi(v) > 0$ we deduce that $v \in W_t$ and P has length at least one. Let $t' \in N_1$ be such that $W_{t'} \cap W_t = \{x, v\}$, so that $\phi(v) = w(T_{t'})$. Since P is a path from $x' \in W_t - V(G_{t'})$ to v in $G_t \setminus x$, we deduce that $V(P) \cap V(G_{t'}) = \{v\}$. By the induction hypothesis applied to the graph $G_{t'}$ the graph $G_{t'}$ has a path Q connecting x to v of length at least $\frac{1}{100} \log \phi(v) / \log k$. So $P \cup Q$ is a path in G_t from x to x' of length at least

$$\begin{aligned} |E(P)| + \frac{\log \phi(v)}{100 \log k} &= \frac{1}{50} \left(|E(P)| + \frac{\log \phi(v)}{2 \log k} \right) + \left(1 - \frac{1}{50} \right) |E(P)| \\ &\geq \frac{\log \phi(R)}{100 \log k} + 1 - \frac{1}{50} \\ &\geq \frac{\log(w(T_t)/10)}{100 \log k} + 1 - \frac{1}{50} \\ &= \frac{\log w(T_t)}{100 \log k} - \frac{\log 10}{100 \log k} + 1 - \frac{1}{50} \\ &\geq \frac{\log w(T_t)}{100 \log k}, \end{aligned}$$

where in the first inequality we used (18) and the fact that $|E(P)| \geq 1$, and the second inequality uses (17). This completes the proof of Lemma 5.2. \blacksquare

6 Gallai's Upper Bound

We need to introduce the notion of Hajós sum of two graphs. Let K and L be two graphs with disjoint vertex-sets, and let $k_1 k_2$ and $l_1 l_2$ be edges of K and L , respectively. Let G be the graph obtained from the union of K and L by deleting the edges $k_1 k_2$ and $l_1 l_2$, identifying the vertices k_1 and l_1 , and adding an edge joining k_2 and l_2 . In those circumstances we say that G is a *Hajós sum* of K and L . It is straightforward to check that if K and L are k -critical, then so is G .

We now describe a construction of k -critical graphs with no long path, and hence no long cycle. Let $k \geq 4$ be an integer, let T be a tree of maximum degree at most $k - 1$, and let $(H_t : t \in V(T))$ be a family of k -critical graphs, each containing the same vertex x_0 , and otherwise pairwise disjoint. For every ordered pair t, t' of adjacent vertices of T we select a vertex $v_{tt'} \in V(H_t)$ such that

- $v_{tt'}$ is adjacent to x_0 in H_t , and
- if t' and t'' are distinct neighbors of t in T , then $v_{tt'} \neq v_{tt''}$.

Such a choice is possible, because T has maximum degree at most $k - 1$ and every k -critical graph has minimum degree at least $k - 1$. Let us emphasize that even though tt' and $t't$ denote the same edge of T , the vertices $v_{tt'}$ and $v_{t't}$ are distinct: the first belongs to H_t and the second to $H_{t'}$. We define a graph G to be the graph obtained from $\bigcup_{t \in V(T)} H_t$ by, for every edge $tt' \in E(T)$, deleting the edges $x_0v_{tt'}$ and $x_0v_{t't}$, and adding an edge joining $v_{tt'}$ and $v_{t't}$.

It is easy to see that the graph G can be viewed as being obtained from the graphs H_t by repeatedly taking Hajós sums, and is thus k -critical. Also, it has $1 + \sum_{t \in V(T)} (|V(H_t)| - 1)$ vertices, and for every path P in $G \setminus x_0$ there exists a path R in T such that $|V(P)| \leq \sum_{t \in V(R)} (|V(H_t)| - 1)$. To replicate (a close relative of) Gallai's original construction, let $h \geq 0$ be an integer, and let T be the $(k - 1)$ -branching tree of height h ; that is, a tree T with a vertex $r \in V(T)$ such that every vertex is at distance at most h from r , and each vertex at distance at most $h - 1$ from r has degree exactly $k - 1$. Each of the graphs H_t will be the complete graph on k vertices. Then the graphs resulting from the construction described above with this choice of T and H_t prove the inequality (5) and statement (6), as is easily seen. (In Gallai's original construction the vertex r has degree $k - 2$, but that makes little difference.) However, there exist k -critical graphs on n vertices for every $n \geq k$, except $n = k + 1$. It is easy to deduce the following theorem, by utilizing such k -critical graphs and trees that are not necessarily regular, and the above construction.

Theorem 6.1 *For every integer $k \geq 4$ and every integer $n \geq k + 2$*

$$L_k(n) \leq \frac{2(k-1)}{\log(k-1)} \log n + 2k. \quad (19)$$

Acknowledgment. We are grateful to Michael Krivelevich for answering many questions related to this paper and especially for sharing his English translation of the construction of Gallai from [7].

References

- [1] N. Alon, M. Krivelevich and P. D. Seymour, Long cycles in critical graphs, *J. Graph Theory* 35 (2000), 193-196.
- [2] A. Bondy and S. C. Locke, Relative lengths of paths and cycles in 3-connected graphs, *Discrete Math.* 33 (1981), 111-122.
- [3] R. Diestel, *Graph theory*, third edition, Springer, Berlin, 2005.

- [4] G. A. Dirac, On the structure of 5- and 6-chromatic abstract graphs, *J. Reine Angew. Math.* 214/215 (1964) 43-52.
- [5] G. A. Dirac, Some theorems on abstract graphs, *Proc. of the London Math. Soc.*, 2 (1952), 69-81.
- [6] G. A. Dirac, Circuits in critical graphs, *Monatsh. Math.* 59 (1955) 178-187.
- [7] T. Gallai, Kritische Graphen I, *Publ. Math. Inst. Hungar. Acad. Sci.* 8 (1963), 165-192.
- [8] T. Jensen and B. Toft, **Graph Coloring Problems**, Wiley, New York, 1995.
- [9] J. B. Kelly and L. M. Kelly, Paths and circuits in critical graphs, *Amer. J. Math.* 76 (1954), 786-792.
- [10] J. Bang-Jensen, B. Reed, M. Schacht, R. Šámal, B. Toft and U. Wagner, On six problems posed by Jarik Nešetřil, In: *Topics in Discrete Mathematics. Dedicated to Jarik Nešetřil on the occasion of his 60th birthday.* M. Klazar, J. Kratochvíl, M. Loeb, J. Matoušek, R. Thomas and P. Valtr, editors, *Algorithms and Combinatorics* **26**, Springer Verlag 2006.
- [11] L. Lovász, *Combinatorial Problems and Exercises*, North Holland, Amsterdam, 1979, Problem 10.29.
- [12] R. C. Read, Maximal circuits in critical graphs, *J. London Math. Soc.* 32 (1957), 456-462.

This material is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.