MATH 7016 Combinatorics (Spring ’09)

Instructor: Asaf Shapira

Home Assignment 3

Due date: 03/5/09

Please submit organized and well written solutions!

Problem 1. Let E be a homogenous linear equation $\sum_{i=1}^{k} a_i x_i = 0$ and denote by $R_E(n)$ the size of the largest subset of $[n]$ containing so solution to E with all x_i being distinct.

- Use the removal lemma for directed graphs to show that if the coefficients of E satisfy $\sum_{i=1}^{k} a_i = 0$ then $R_E(n) = o(n)$.
- Show that if the coefficients of E satisfy $\sum_{i=1}^{k} a_i \neq 0$ then $R_E(n) = \Omega(n)$.

Problem 2. Let E be a linear equation of the form $\sum_{i=1}^{k} a_i x_i = (\sum_{i=1}^{k} a_i) x_{k+1}$, where the a_is are positive integers. A solution of such an equation is trivial if $x_1 = x_2 = \ldots = x_{k+1}$, otherwise it is non-trivial. Prove that there is a constant C (that depends on $a_1, \ldots, a_k$) such that for every large enough n, there exists a subset $X \subseteq [n]$ of size $n/C^{\sqrt{\log n}}$ that contains no non-trivial solution of E.

Problem 3. For integers $g, k \geq 3$ let $n = n(g,k)$ be the smallest integer for which there exists a graph on n vertices with girth at least g and chromatic number at least k. Show that $n(g,k) = k^{\Theta(g)}$, that is, that there are absolute constant c, C such that $k^{cg} \leq n(g,k) \leq k^{Cg}$.

Problem 4. A *kernel* in a directed graph is an independent set of vertices D such that for every vertex $v \notin D$ there is at least one edge pointing from v to one of the vertices of D.

- Show that every strongly connected directed graph without odd directed cycles is bipartite.
- Use the previous item to show that every strongly connected directed graph without odd directed cycles has a kernel.
- Use the previous item to show that every directed graph without odd directed cycles has a kernel (this is *Richardson’s Theorem*).