CS 6550 Algorithms (Fall '10)

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Home Assignment 2

Due date: 10/12/10

Please submit organized and well written solutions!

Problem 1. Give a *deterministic* 7/8-approximation algorithm for MAX-3-CNF.

Problem 2. Let S be a sample space of n random variables which are uniformly distributed over $\mathbb{F}_4$ and 4-wise independent. Show how to use S in order to construct a sample space of size $|S|$ consisting of $2n$ random variables which are uniformly distributed over $\mathbb{F}_2$ and 4-wise independent.

Hint: Recall the representation of elements of $\mathbb{F}_4$ as elements of $(\mathbb{F}_2)^2$.

Problem 3. Let M be an $r \times n$ matrix over $\mathbb{F}_2$. Let $I(M)$ denote the largest integer I such that every set of I columns of M are linearly independent over $\mathbb{F}_2$. In the following items all computations are over $\mathbb{F}_2$.

- Suppose we pick x from $(\mathbb{F}_2)^r$ uniformly at random and define a collection of random variables $\{X_1, \ldots, X_n\}$, by setting X_i to be the i^{th} entry of xM.

  Show that $\{X_1, \ldots, X_n\}$ are k-wise independent unbiased 0/1 bits if and only if $I(M) \geq k$.

- Let $C(M) = \{x \in (\mathbb{F}_2)^n : Mx = 0\}$ and $\mathbf{wt}(x)$ denote the number of non-zero entries of x.

  Show that $I(M) + 1 = \min_{0 \neq x \in C(M)} \mathbf{wt}(x)$.

- Show that $I(M) + 1 = \min_{x,y \in C(M)} d_H(x, y)$, where $d_H(x, y)$ denotes the Hamming distance between x and y.

Problem 4. We have shown that if $X_1, \ldots, X_n$ are ± 1 random-variables which are 4-wise independent, then $\mathbb{P}[|\sum_{i=1}^{n} X_i| \geq c\sqrt{n}] \geq c$ for some absolute $c > 0$. We will show here that one cannot replace the 4-wise independence condition with 3-wise independence.

- Let M' be the $k \times 2^k$ matrix whose columns are all possible 0/1 vectors. Let M be the $(k+1) \times 2^k$ matrix obtained by adding to M a row whose entries are all 1s. Suppose we randomly pick $x \in (\mathbb{F}_2)^{k+1}$ and set the random variables $X_1, \ldots, X_{2^k}$ to be the entries of xM. Show that $X_1, \ldots, X_{2^k}$ are 3-wise independent.

  Hint: Use the first item from the previous question.

- Let $Y_i = 1 - 2X_i$. Show that $Y_1, \ldots, Y_{2^k}$ are 3-wise independent ± 1 random variables and yet $\mathbb{P}[|\sum_{i}^{2^k} Y_i| > 0] = 2^{-k}$.