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Non-deterministic Multi-valued Logics and their Applications

Thesis submitted
for the degree of Doctor of Philosophy

by

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This work was carried out under the supervision of
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Submitted to the Senate of Tel-Aviv University
February 2009

Acknowledgements

First and foremost, I would like to thank my advisor, Prof. Arnon Avron for so many things that there is no space on this page to mention even a small part of them. I thank him for introducing me to the beauty of Logic. For never ceasing to amaze me by his profound knowledge, sharp insights and innovative ideas, always finding original solutions to challenging problems, or just giving me hints where to look for them. For teaching me to look deeper into a problem, to ask the right questions, and to carefully formulate the answers. For respecting my opinions and always giving me the opportunity to express my thoughts. For his patient guidance and his constant encouragement. For all those things, and many many more - I will be forever grateful.

I am also thankful to Dr. Ofer Arieli, a joint work with whom has led to some of the results in this thesis. Working with him has opened new horizons for me and has been a most enjoyable and rewarding experience.

I would also like to express my gratitude to Dr. Agata Ciabattoni, a collaboration with whom has led to some further results in this thesis. Her sharp ideas and precise way of thinking have made a major contribution to our joint research. I also thank her for her constant support and care, which kept me focused on my goal and helped me tremendously through some hard times.

I am also thankful to Prof. Beata Konikowska, who has been a real inspiration to me in many ways. Her valuable feedback and suggestions during our discussions are highly appreciated.

I am grateful to Prof. Nachum Dershowitz, Prof. Alexander Rabinovich, Prof. Boaz Trakhtenbrot and Prof. Yoram Hirshfeld for many fruitful discussions and interesting seminars.

Last, but not least, a special thank you to my dear parents, Alexander and Rita, my brother Arkady and my beloved husband Alik for believing in me, and being there for me every step of the way.

Abstract

Non-deterministic multi-valued matrices (Nmatrices) are a new, fruitful and quickly expanding field of research first introduced a few years ago by Avron and Lev. Since then it has been rapidly developing towards a foundational logical theory and has found numerous applications in different research areas, such as reasoning under uncertainty, automated reasoning, and proof theory. The novelty of Nmatrices is in extending the usual algebraic deterministic semantics of logical systems by importing the idea of non-deterministic computations from Computer Science, and allowing the truth-value of a formula to be chosen non-deterministically out of a given set of options. Nmatrices have proved to be a powerful tool, the use of which preserves all the advantages of ordinary many-valued matrices, but is applicable to a much wider range of logics. Indeed, there are many useful (propositional) non-classical logics, which have no finite many-valued characteristic matrices, but *do* have finite Nmatrices, and thus are decidable. Another important advantage of the framework of Nmatrices is its *modularity*. Each syntactic rule in a proof system corresponds to a certain semantic condition, leading to a refinement of some basic Nmatrix. In many cases the semantics of a complex system can be obtained by straightforwardly combining the semantic effects of each of the added rules. As a result, the semantic effect of a syntactic rule can be analyzed separately. This is impossible in standard multi-valued matrices, where the semantics of a system can only be presented as a whole. Nmatrices have also found important applications in proof theory. In particular, for a natural class of Gentzen-type propositional systems called *canonical systems*, there is a strong connection between the existence of a characteristic 2-valued Nmatrix for a given proof system and the ability to eliminate cuts in it.

So far most of the work on the framework of Nmatrices has been done on the purely propositional level. However, no semantic framework can be considered really useful unless it can be naturally extended to the first-order level and beyond. Accordingly, the main goal of this thesis is to extend the framework of Nmatrices to languages with quantifiers and to explore its applications in different areas. We consider several generalizations of first-order quantifiers: unary, multi-ary and (n, k)-ary ones. We provide interpretations of such quantifiers in the framework of Nmatrices, and resolve various problems related to the principles of α-equivalence, identity and void quantification, using special congruence relations between formulas. Some important properties of the extended framework, such as analycity, are investigated. Then we turn to several appli-

cations of the framework of Nmatrices extended with quantifiers. As one application, we provide non-deterministic semantics for a large family of first-order paraconsistent logics. As another application, we generalize the theory of canonical systems to languages with quantifiers and show that the correspondence between cut-elimination and the existence for a system of a corresponding characteristic 2-valued Nmatrix obtains for such systems.

In addition, we also extend in this thesis previous results on Nmatrices on the propositional level. This includes studying two important syntactic properties of canonical propositional systems: invertibility of rules and axiom expansion, and showing that there is a close connection between these properties and the existence for a system of a finite *deterministic* matrix. Moreover, the theory of canonical systems is extended to *signed calculi*, of which Gentzen-type canonical systems are specific instances. Finally, some steps are made in investigating the usefulness of Nmatrices in distance-based approaches to reasoning under uncertainty.

Contents

Chapter 1

Introduction

1.1 The Concept of Non-deterministic Matrices

The principle of truth-functionality (or compositionality) is a basic principle in many-valued logic in general, and in classical logic in particular. According to this principle, the truth-value of a complex formula is uniquely determined by the truth-values of its sub-formulas. However, real-world information is inescapably incomplete, uncertain, vague, imprecise or inconsistent, and these phenomena are in an obvious conflict with the principle of truth-functionality. One possible solution to this problem is to relax this principle by borrowing from automata and computability theory the idea of non-deterministic computations, and apply it in evaluations of truth-values of formulas. This approach has been implicitly used in [44] for handling inconsistent data. However, this was done in an ad-hoc way. A general framework of *non-deterministic matrices* (Nmatrices) based on this approach was introduced in [28, 29]. Nmatrices are a natural generalization of ordinary multi-valued matrices, in which the truth-value of a complex formula can be chosen *non-deterministically* out of some non-empty set of options.

Below we present a number of natural motivations for introducing non-determinism into the truth-tables of logical connectives. They give rise to two different ways in which non-determinism can be incorporated: the *dynamic* and the *static*[1]. In both the value $v(\diamond(\psi_1, \ldots, \psi_n))$ assigned to the formula $\diamond(\psi_1, ..., \psi_n)$ is selected from a set $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ (where $\widetilde{\diamond}$ is the interpretation of $\diamond$). In the dynamic approach this selection is made separately and independently for each tuple $\langle \psi_1, \ldots, \psi_n \rangle$. Thus the choice of one of the possible values is made at the lowest possible (local) level of computation, or on-line, and $v(\psi_1), \ldots, v(\psi_n)$ do not uniquely determine $v(\diamond(\psi_1, \ldots, \psi_n))$. In contrast,

[1]The dynamic approach was introduced together with the concept of Nmatrices. The static approach was later introduced in [27]

in the static semantics this choice is made globally, system-wide, and the interpretation of $\diamond$ is a function, which is selected before any computation begins. This function is a "determinisation" of the non-deterministic interpretation $\tilde{\diamond}$, to be applied in computing the value of any formula under the given valuation. This limits non-determinism, but still leaves the freedom of choosing the above function among all those that are compatible with the non-deterministic interpretation $\tilde{\diamond}$ of $\diamond$.

Let us start by presenting some of the most natural motivations for introducing Nmatrices.

Syntactic "underspecification":
Consider the standard Gentzen-type system for propositional classical logic (see e.g. [125]). Its introduction rules for $\neg$ and $\vee$ are usually formulated as follows:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad\qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}\ (\Rightarrow \neg)$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}\ (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}\ (\Rightarrow \vee)$$

The corresponding semantics is given by the following classical truth-tables:

	$\neg$
t	**f**
f	**t**

		$\vee$
t	**t**	**t**
t	**f**	**t**
f	**f**	**t**
f	**f**	**f**

Note that each syntactic rule dictates some semantic condition on the connective it introduces: $(\neg \Rightarrow)$ corresponds to the condition $\tilde{\neg}(t) = f$, while $(\Rightarrow \neg)$ corresponds to the condition $\tilde{\neg}(f) = t$, thus completely determining the truth-table for negation. Similarly, $(\vee \Rightarrow)$ dictates the last line of the truth-table for $\vee$, i.e $\tilde{\vee}(f, f) = f$, while $(\Rightarrow \vee)$ dictates the other three lines. Now suppose we want to reject the *law of excluded middle* (LEM), in the spirit of intuitionistic logic. This can most simply be done by discarding the rule $(\Rightarrow \neg)$, which corresponds to LEM, while keeping the rest of the rules unchanged. What is the semantics of the resulting system? Intuitively, by discarding $(\Rightarrow \neg)$, we lose the information concerning the second line of the truth-table for $\neg$. Accordingly, we are left with a problem of *underspecification*. This can be modelled using Nmatrices in a very natural way: in case of underspecification, all possible truth-values are allowed.

The corresponding semantics in the case we consider would be as follows (we use sets of possible truth-values instead of truth-values):

	$\neg$
t	{**f**}
f	{**t**,**f**}

		$\vee$
t	**t**	{**t**}
t	**f**	{**t**}
f	**f**	{**t**}
f	**f**	{**f**}

Linguistic ambiguity:

In many natural languages the meaning of the words "either ... or" is ambiguous. Thus the Oxford English Dictionary explains the meaning of this phrase as follows:

> The primary function of either, etc., is to emphasize the indifference of the two (or more) things or courses, ..., but a secondary function is to emphasize the mutual exclusiveness (i.e. either of the two, but not both).

Following this kind of common-sense intuition about "or", it follows that in many natural languages the word "or" has both an "inclusive" and an "exclusive" sense. For instance, when some mathematician promises: *"I shall either attack problem A or attack problem B"*, then in many cases he might at the end solve the two problems, but there are certainly situations in which what he means is "but do not expect me to attack them both". In the first case the meaning of "or" is inclusive, while in the latter case it is exclusive. Now in many cases one is uncertain whether the meaning of a speaker's "or" is inclusive or exclusive. However, even in cases like this one would still like to be able to make some certain inferences from what has been said. This situation can be captured by dynamic semantics based on the following non-deterministic truth-table for $\vee$:

		$\vee$
t	**t**	{**t**, **f**}
t	**f**	{**t**}
f	**t**	{**t**}
f	**f**	{**f**}

Note that the static semantics is less appropriate here, since the meaning of a speaker's "or" is not predetermined, and he might use both meanings of "or" in two different sentences within the same discourse.

Inherent non-deterministic behavior of circuits:

Nmatrices can be applied to model non-deterministic behavior of various elements of

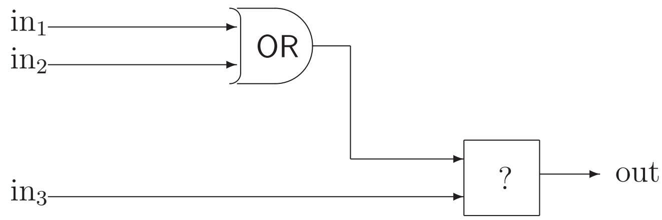

Figure 1.1: The circuit C

electrical circuits. An ideal logic gate performing operations on boolean variables is an abstraction of a physical gate operating with a continuous range of electrical quantity. This electrical quantity is turned into a discrete variable by associating a whole range of electrical voltages with the logical values 1 and 0 (see [114] for further details). There are a number of reasons, due to which the measured behavior of a circuit may deviate from the expected behavior. One reason can be the variations in the manufacturing process: the dimension and device parameters may vary, affecting the electrical behavior of the circuit. The presence of disturbing noise sources, temperature and other conditions are another source of deviations in the circuit response. The exact mathematical form of the relation between input and output in a given logical gate is not always known, and so it can be approximated by a non-deterministic truth-table. For instance, suppose that the circuit C given in Figure 4.2 consists of a standard OR gate and a faulty AND gate, which responds correctly if the inputs are similar, and unpredictably otherwise. The behavior of the gate can be described by the following truth-table, equipped with the dynamic semantics:

		AND
t	**t**	$\{\mathbf{t}\}$
t	**f**	$\{\mathbf{f}, \mathbf{t}\}$
f	**t**	$\{\mathbf{f}, \mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

Computation with unknown functions:
Let us return to Figure 4.2, and suppose that this time it represents a circuit about which only some partial information is known. Namely, it is known that the gate labelled with "?" is either an XOR gate or an OR gate, but it is not known which one. Thus the function describing the second gate is deterministic, but unknown to us. This situation can be represented by using the non-deterministic truth-table for $\vee$ given in the "linguistic ambiguity" example, equipped with the static semantics.

Verification with unknown evaluation models:

There are two well-known three-valued logics for describing different types of computational models. The first, which captures parallel evaluation, was described in the context of computational mathematics by Kleene ([95]); the second, programming oriented method, in which evaluation proceeds sequentially, was proposed by McCarthy ([106]). Below are the corresponding truth-tables for $\vee$:

(Kleene)

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**t**
t	**t**	**t**	**t**

(McCarthy)

$\widetilde{\vee}$	**f**	**e**	**t**
f	**f**	**e**	**t**
e	**e**	**e**	**e**
t	**t**	**t**	**t**

Now suppose we are sending an expression $\psi \vee \varphi$ for evaluation to some distant computer, for which it is not known whether it performs parallel or sequential computations. Hence we know that $\psi \vee \varphi$ will be evaluated using a deterministic function $\tilde{\vee}$, defined by either Kleene's or McCarthy's truth-table for $\vee$, but we have no information which of the two. Again this can be captured by using a static interpretation of the following "truth-table":

$\widetilde{\vee}$	**f**	**e**	**t**
f	$\{\mathbf{f}\}$	$\{\mathbf{e}\}$	$\{\mathbf{t}\}$
e	$\{\mathbf{e}\}$	$\{\mathbf{e}\}$	$\{\mathbf{e},\mathbf{t}\}$
t	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

According to this static interpretation, the function $f_\vee : \{\mathbf{t},\mathbf{f},\mathbf{e}\}^2 \to \{\mathbf{t},\mathbf{f},\mathbf{e}\}$ used by the computer satisfies either $f_\vee(\mathbf{t},\mathbf{e}) = \mathbf{t}$ (in case the computation is parallel) or $f_\vee(\mathbf{t},\mathbf{e}) = \mathbf{e}$ (in case it is sequential). However, it is not known which of these two conditions is satisfied.

Incompleteness and inconsistency:

This example is taken from [24, 25]. Suppose we have a framework for information collecting and processing, which consists of a set S of information sources and a processor P. The sources provide information about formulas over $\{\neg, \vee\}$, and we assume that for each such formula ψ a source $s \in S$ can say that ψ is true (i.e., assigned the truth-value 1), ψ is false (i.e., assigned the truth-value 0), or that it has no knowledge about ψ. In turn, the processor collects information from the sources, combines it according to some strategy and defines the resulting combined valuation of formulas. Thus for every formula ψ the processor can encounter one of the four possible situations: (a) it has information that ψ is true, but no information that ψ is false, (b) it has information that ψ is false, but no information that ψ is true, (c) it has both information that ψ is true and information that it is false, and (d) it has no information on ψ at all. In view of this, it was suggested by Belnap in [47] (following works and ideas of Dunn, e.g. [71]) to account for incomplete and contradictory information by using the following four logical truth values:

$$\mathbf{t} = \{1\}, \mathbf{f} = \{0\}, \top = \{0,1\}, \bot = \emptyset$$

Here 1 and 0 represent "true" and "false" respectively, and so $\top$ represents inconsistent information, while $\bot$ represents absence of information.

The above scenario has many ramifications, corresponding to various assumptions regarding the kind of information provided by the sources and the strategy used by the processor to combine it. We assume that the processor respects at least the deterministic consequences (in both ways) of each of the classical truth tables. This assumption means that the values assigned by the processor to complex formulas and those it assigns to their immediate subformulas are interrelated according to the following principles derived from the classical truth-tables of $\neg$ and $\vee$:

1. The processor ascribes 1 to $\neg\varphi$ iff it ascribes 0 to φ.

2. The processor ascribes 0 to $\neg\varphi$ iff it ascribes 1 to φ.

3. If the processor ascribes 1 to either φ or ψ, then it ascribes 1 to $\varphi \vee \psi$.

4. The processor ascribes 0 to $\varphi \vee \psi$ iff it ascribes 0 to both φ and ψ.

Here the statement "the processor ascribes 0 to ψ" means that 0 is included in the subset of $\{0,1\}$ which is assigned by the processor to ψ (recall that the truth-values used by the processor correspond to subsets of $\{0,1\}$). It is crucial to note that the converse of (3) does *not* hold, since some source might inform the processor that $\varphi \vee \psi$ is true, without providing information about the truth/falsehood of either φ or ψ. Under the above assumptions, there can be a number of possible scenarios concerning the type of formulas evaluated by the sources. The case when the sources provide information only about atomic formulas has been considered in [47]. This case is deterministic, and leads to the famous Dunn-Belnap four-valued logic. Now consider the case when the sources provide information about arbitrary formulas (also complex ones), but not necessarily all of them. In this case the assumptions above are reflected in the following *non-deterministic* truth-tables:

$\widetilde{\vee}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\{\mathbf{f}, \top\}$	$\{\mathbf{t}, \bot\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\bot$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}, \bot\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
$\top$	$\{\top\}$	$\{\mathbf{t}\}$	$\{\top\}$	$\{\mathbf{t}\}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

	$\widetilde{\neg}$
$\mathbf{f}$	$\{\mathbf{f}\}$
$\bot$	$\{\bot\}$
$\top$	$\{\top\}$
$\mathbf{t}$	$\{\mathbf{f}\}$

Note that the table for negation reflects the principles 1 and 2, while the table for disjunction reflects the principles 3 and 4. To see this, let us examine one of the most peculiar cases: the entry $\mathbf{f}\widetilde{\vee}\mathbf{f} = \{\mathbf{f}, \top\}$. Suppose that ψ and φ are both assigned the truth-value $\mathbf{f} = \{0\}$. Then by principle 4 above, the truth-value of $\psi \vee \varphi$ (which is a subset of $\{0, 1\}$) must include 0. If in addition one of the sources assigned 1 to $\psi \vee \varphi$, then the processor ascribes 1 to $\psi \vee \varphi$ too, and so the truth-value it assigned to $\psi \vee \varphi$ is in this case $\top$. Otherwise it is $\mathbf{f}$. This justifies the two options in the truth-table. The rest of the entries can be explained in a similar way.

These are just some of the motivations for introducing the framework of Nmatrices. Nmatrices have proved to be a powerful tool, the use of which preserves all the advantages of ordinary many-valued matrices (such as decidability and compactness), but is applicable to a much wider range of logics. Indeed, there are many useful non-classical logics, which have no finite many-valued characteristic matrices, but *do* have finite Nmatrices, and thus are decidable. Another very important advantage of the framework of Nmatrices is its *modularity*. Each syntactic rule in a proof system corresponds to a certain semantic condition, leading to a refinement of some basic Nmatrix. Thus the semantics of a complex system is obtained by straightforwardly combining the semantic effects of each of the added rules. As a result, frequently the semantic effect of a syntactic rule taken separately can be analyzed. This is impossible in standard multi-valued matrices, where the semantics of a system can only be presented as a whole. Nmatrices have been used in [19, 20, 21, 17] to provide simple and modular semantics for thousands of non-classical logics, in particular for paraconsistent logics of the Brazilian school of da Costa (see Section 2.3 for details). Nmatrices have also been shown to have far-reaching applications in the proof theory of the important class of canonical Gentzen-type systems ([28, 29]).

So far most of the work on the framework of Nmatrices has been done on the purely propositional level. However, no semantic framework can be considered really useful unless it can be naturally extended to the first-order level and beyond. Accordingly, the main goal of this thesis is to extend the framework of Nmatrices to languages with quantifiers and to explore its applications in different areas. A number of extensions and applications on the propositional level are also presented in the thesis, as described below.

1.2 Thesis Outline

The structure of this thesis is as follows. Chapter 2 is devoted to introducing the framework of Nmatrices and presenting some of the previous work done on the propositional level. After providing some preliminaries in Section 2.1, we review the basic definitions of the framework of Nmatrices in section 2.2. The modularity of the framework of Nmatrices is demonstrated in Section 2.3, taking as an example a large family of paraconsistent logics.

Chapter 3 describes our main results in the theory of propositional canonical systems. We characterize two important syntactic properties in canonical Gentzen-type systems, namely invertibility of logical rules and axiom expansion. Then we extend the theory of Gentzen-type canonical systems to signed calculi. We provide modular semantics for every canonical signed calculus satisfying a simple and constructive condition. Different notions of cut-elimination in signed canonical calculi are investigated and a strong connection is shown between the existence of a characteristic Nmatrix for such calculi, and the ability to eliminate cuts in them.

In Chapter 4 we investigate an application of Nmatrices for distance-based reasoning. Combining the framework of Nmatrices with distance-based considerations leads to a framework for non-monotonic reasoning with inconsistent information. We study the basic properties of the obtained entailment relations and apply the framework on some examples based on logical circuits.

In Chapter 5 we focus on quantification in Nmatrices and consider three types of generalized quantifiers: unary, multi-ary and (n, k)-ary quantifiers. This chapter includes a general discussion on what such quantifiers mean in the context of Nmatrices and how they should be interpreted. Some problems with incorporating non-determinism into the interpretation of quantifiers, which were not evident on the propositional level, are described and solved.

In Chapter 6 we apply the extended framework of Nmatrices developed in the previous chapter to provide modular semantics for a large family of first-order paraconsistent logics (LFIs).

In Chapter 7 we return to the theory of canonical calculi and further generalize it to the levels of multi-ary and (n, k)-ary quantifiers. We show that the correspondence between the ability to eliminate cuts in a given canonical calculus and its corresponding characteristic 2Nmatrix can be reestablished also on the level of quantifiers by considering a stronger version of cut-elimination.

Finally, in Chapter 8 we conclude with a discussion of some directions for further research.

Chapter 2

Propositional Non-deterministic Matrices

In this chapter we describe the framework of propositional Nmatrices and briefly summarize some of the most important results from [29, 18, 19, 20].

2.1 Preliminaries

In what follows, $\mathcal{L}$ is a propositional language and $Frm_{\mathcal{L}}$ is its set of wffs. The metavariables ψ, φ range over $\mathcal{L}$-formulas, and Γ, Δ over sets of $\mathcal{L}$-formulas. For an $\mathcal{L}$-formula ψ, we denote by $\mathsf{Atoms}(\psi)$ the set of atomic formulas in ψ. We denote by $SF(\Gamma)$ the set of all subformulas of Γ.

2.1.1 Logics, Consequence Relations and Abstract Rules

Definition 2.1.1. 1. A *Scott consequence relation* (*scr* for short) for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following three conditions:

strong reflexivity:	if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.
monotonicity:	if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.
transitivity (*cut)*:	if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

2. A Tarskian consequence relation (tcr) $\vdash^1$ for a language $\mathcal{L}$ is a binary relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, that satisfies the following conditions:

strong reflexivity:	if $\psi \in \Gamma$ then $\Gamma \vdash^1 \psi$.
monotonicity:	if $\Gamma \vdash^1 \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash^1 \psi$.

transitivity (*cut*): if $\Gamma \vdash^1 \psi$ and $\Gamma', \psi \vdash^1 \varphi$ then $\Gamma, \Gamma' \vdash^1 \varphi$.

3. A tcr $\vdash$ for $\mathcal{L}$ is *structural* if for every uniform $\mathcal{L}$-substitution σ and every Γ and ψ, if $\Gamma \vdash \psi$ then $\sigma(\Gamma) \vdash \sigma(\psi)$. $\vdash$ is *finitary* if whenever $\Gamma \vdash \psi$, there exists some finite $\Gamma' \subseteq \Gamma$, such that $\Gamma' \vdash \psi$. $\vdash$ is *consistent* (or *non-trivial*) if whenever $p \neq q$, $p \not\vdash q$ for every two atoms p and q. ψ s.t. $\Gamma \not\vdash \psi$. Similar properties can be defined for an scr.

4. *A Tarskian propositional logic* (*propositional logic*) is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural and consistent[1] tcr (scr) for $\mathcal{L}$. The logic $\langle \mathcal{L}, \vdash \rangle$ is finitary if $\vdash$ is finitary.

For the rest of this section, we focus on scrs. However, the properties below can be formulated in the context of tcrs as well.

There are several ways of defining consequence relations for a language $\mathcal{L}$. The two most common ones are the proof-theoretical and the model-theoretical approaches. In the former, the definition of a consequence relation is based on some notion of a *proof* in some formal calculus.

Example 2.1.2. $LK^+_\neg$ denotes the positive classical logic taken over $\{\wedge, \vee, \supset, \neg\}$. $\mathbf{G}[LK^+_\neg]$, the standard Gentzen-type (canonical) for $LK^+_\neg$, is given in Figure 2.1.

The second approach to defining consequence relations is based on a notion of a *semantics* for $\mathcal{L}$. The general notion of an abstract semantics is rather opaque. One usually starts by defining a notion of a *valuation* as a certain type of partial functions from $Frm_\mathcal{L}$ to some set. Then ones defines what it means for a valuation to *satisfy* a formula (or to be a *model* of a formula). A semantics is then some set S of valuations, and the consequence relation induced by S is defined as follows: $\Gamma \vdash_\mathsf{S} \Delta$ if every *total* valuation in S which satisfies all the formulas in Γ, satisfies some formula in Δ as well (note that this always defines an scr). We say that a semantics S is *analytic*[2] if every partial valuation in S, whose domain is closed under subformulas, can be extended to a full (i.e. total) valuation in S. This implies that the exact identity of the language $\mathcal{L}$ is not important, since analycity allows us to focus on some subset of its connectives. (See Remark 2.1.14 below for another important consequence of analycity.) We shall shortly

[1]Note that usually consistency is not required of a propositional logic, but it is convenient not to take into account trivial logics.

[2]The term 'effective' was used in [20, 34, 31] instead of 'analytic'.

Axioms:

$$A \Rightarrow A$$

Structural Rules:

Cut, Weakening

Logical Rules:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}\ (\supset\Rightarrow) \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}\ (\Rightarrow\supset)$$

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}\ (\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \wedge \varphi, \Delta}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \psi, \varphi, \Delta}{\Gamma \Rightarrow \psi \vee \varphi, \Delta}\ (\Rightarrow \vee)$$

Figure 2.1: The Gentzen-type system $\mathbf{G}[LK^+_\neg]$

see that both ordinary many-valued semantics and non-deterministic semantics based on propositional Nmatrices are always analytic. However this is not necessarily the case in general. [3]

Definition 2.1.3. 1. A *pure (abstract) rule* in a propositional language $\mathcal{L}$ is any ordered pair $\langle \Gamma, \Delta \rangle$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ (We shall usually denote such a rule by $\Gamma \Rightarrow \Delta$ rather than by $\langle \Gamma, \Delta \rangle$).

2. Let $\mathbf{L} = \langle \mathcal{L}, \vdash_1 \rangle$ be a propositional logic, and let S be a set of rules in a propositional language $\mathcal{L}'$. The *extension* $\mathbf{L}[S]$ of $\langle \mathcal{L}, \vdash_1 \rangle$ by S is the logic $\langle \mathcal{L}^*, \vdash^* \rangle$, where $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}'$, and $\vdash^*$ is the least *structural* scr $\vdash$ such that $\Gamma \vdash \Delta$ whenever $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$.[4]

Remark 2.1.4. It is easy to see that $\vdash^*$ is the closure under cuts and weakenings of the set of all pairs $\langle \sigma(\Gamma), \sigma(\Delta) \rangle$, where σ is a uniform substitution in $\mathcal{L}^*$, and either $\Gamma \vdash_1 \Delta$

[3]For instance, in the bivaluations semantics and the possible translations semantics described in [55, 59, 62] no general theorem securing analycity is available. Hence analycity should be proved from scratch for every useful instance of these types of semantics.

[4]Obviously, the extension of $\langle \mathcal{L}, \vdash_1 \rangle$ by S is well-defined (i.e. a logic) only if $\vdash^*$ is consistent. In all the cases we consider below this will easily be guaranteed by the semantics we provide (and so we shall not even mention it).

or $\langle\Gamma,\Delta\rangle \in S$. This in turn implies that an extension of a finitary logic by a set of pure rules is again finitary.

Convention 2.1.5. To emphasize the fact that the presence of a rule in a system means the presence of all its instances, we shall usually describe a rule using the metavariables φ,ψ,θ rather than the atomic formulas $p_1, p_2, \ldots$. Thus although formally $(\supset\Rightarrow)$ is the rule $p_1, p_1 \supset p_2 \Rightarrow p_2$, we shall write it as $\varphi, \varphi\supset\psi \Rightarrow \psi$.

Remark 2.1.6. Suppose that the formula θ occurs in a pure rule of a logic $\mathcal{L}$, and we decide to select θ as the "principal formula" of that rule. Assume e.g. that the rule is of the form $\varphi_1,\ldots,\varphi_n \Rightarrow \psi_1,\ldots,\psi_k,\theta$ (the consideration in the other case is similar). Suppose further that $\Gamma_i \vdash \Delta_i, \varphi_i$ for $i = 1,\ldots,n$ and $\psi_j, \Gamma_j \vdash \Delta_j$ for $j = 1,\ldots,k$. Then $\Gamma_1,\ldots,\Gamma_n \vdash \Delta_1,\ldots,\Delta_k,\theta$ (by n+k cuts). It follows that $\mathcal{L}$ is closed in this case under the Gentzen-type rule:

$$\frac{\Gamma_i \Rightarrow \Delta_i, \varphi_i \;\; (i = 1,\ldots,n) \qquad \psi_j, \Gamma_j \Rightarrow \Delta_j \;\; (j = 1,\ldots,k)}{\Gamma_1,\ldots,\Gamma_n \Rightarrow \Delta_1,\ldots,\Delta_k,\theta}$$

Conversely, if $\mathcal{L}$ is closed under this Gentzen-type rule then by applying it to the reflexivity axioms $\varphi_i \vdash \varphi_i$ $(i = 1,\ldots,n)$ and $\psi_j \vdash \psi_j$ $(j = 1,\ldots,k)$ we get $\varphi_1,\ldots,\varphi_n \vdash \psi_1,\ldots,\psi_k,\theta$. It follows that every pure rule in the sense of Definition 2.1.3 is equivalent to some *multiplicative* (in the terminology of [80])Girard, J. Y. or *pure* (in the terminology of [15]) Gentzen-type rule. Moreover: it is easy to see that most standard rules used in Gentzen-type systems are equivalent to finite sets of pure rules in the sense of Definition 2.1.3. For example: the usual $(\supset\Rightarrow)$ rule of classical logic is equivalent by what we have just shown to the pure rule $\varphi, \varphi\supset\psi \Rightarrow \psi$. The classical $(\Rightarrow\supset)$, in turn, can be split into the following two rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi\supset\psi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\supset\psi}$$

Hence $(\Rightarrow\supset)$ is equivalent to the set $\{\psi \Rightarrow \varphi\supset\psi, \quad \Rightarrow \varphi, \varphi\supset\psi\}$. [5]

2.1.2 Many-valued Matrices

The most standard general method for defining propositional logics is by using many-valued (deterministic) matrices ([116, 51, 105, 82, 88, 126]):

Definition 2.1.7. 1. A matrix for $\mathcal{L}$ is a tuple $\mathcal{P} = \langle\mathcal{V},\mathcal{D},\mathcal{O}\rangle$, where:

[5] Recall that formally we should have written here $\{p_2 \Rightarrow p_1 \supset p_2, \quad \Rightarrow p_1, p_1 \supset p_2\}$.

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.
- For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$.

We say that $\mathcal{P}$ is (in)finite if so is $\mathcal{V}$.

2. A partial valuation in $\mathcal{P}$ is a function v to $\mathcal{V}$ from some subset W of $Frm_{\mathcal{L}}$ which is closed under subformulas, such that for each n-ary connective $\diamond$ of $\mathcal{L}$, the following holds for all $\psi_1, \ldots, \psi_n \in W$:

$$v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$$

A partial valuation in $\mathcal{P}$ is a (full) *valuation* if its domain is $Frm_{\mathcal{L}}$. A partial valuation v in $\mathcal{P}$ satisfies a formula ψ ($v \models \psi$) if $v(\psi) \in \mathcal{D}$.

3. Let $\mathcal{P}$ be a matrix. We say that $\Gamma \vdash_{\mathcal{P}} \Delta$ if whenever a valuation in $\mathcal{P}$ satisfies all the formulas of Γ, it satisfies also at least one of the formulas of Δ. We say that $\Gamma \vdash^1_{\mathcal{P}} \psi$ if $\Gamma \vdash_{\mathcal{P}} \{\psi\}$. For a family of matrices F, we say that $\Gamma \vdash_F \Delta$ if $\Gamma \vdash_{\mathcal{P}} \Delta$ for every $\mathcal{P}$ in F. We say that $\Gamma \vdash^1_F \psi$ if $\Gamma \vdash_F \{\psi\}$.

4. A logic $\mathbf{L}$ is sound for a matrix $\mathcal{P}$ if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{P}}$. $\mathbf{L}$ is complete for a matrix $\mathcal{P}$ if $\vdash_{\mathcal{P}} \subseteq \vdash_{\mathbf{L}}$. $\mathcal{P}$ is a characteristic matrix for a logic $\mathbf{L}$ if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{P}}$. F is a characteristic set of matrices for $\mathbf{L}$ if $\vdash_{\mathbf{L}} = \vdash_F$.

The following is a well-known fact:

Theorem 2.1.8. *For every matrix $\mathcal{P}$ for $\mathcal{L}$, $\vdash_{\mathcal{P}}$ and $\vdash^1_{\mathcal{P}}$ are propositional logics.*

In the converse direction, matrices-based semantics can be used for a representation of any logic:

Theorem 2.1.9. *([129, 130]) Every logic is induced by some set of matrices.*

Theorem 2.1.10. *([102, 129, 130, 126]) Every Tarskian logic $\langle \mathcal{L}, \vdash \rangle$ has a (single) characteristic matrix iff it satisfies the following condition of uniformity: if $T, S \vdash \varphi$ and S is the union of $\vdash$-consistent sets that have no atomic formulas in common with one another or with T or φ, then $T \vdash \varphi$ (a set S of formulas is $\vdash$-consistent if there exists a formula ψ such that $S \nvdash \psi$).*

Remark 2.1.11. Although every Tarskian uniform structural logic has a characteristic matrix, it is often the case that this matrix is infinite, and is hard to find and use. We will shortly see that finite characteristic Nmatrices exist for many logics which have only infinite characteristic matrices (see Theorem 2.2.12).

Theorem 2.1.12. (Compactness) ([121]) *If $\mathcal{P}$ is a finite matrix then $\vdash_{\mathcal{P}}$ and $\vdash^1_{\mathcal{P}}$ are finitary.*

The next important result is again very easy to prove:

Proposition 2.1.13. (Analycity) *Any partial valuation in a matrix $\mathcal{P}$ for $\mathcal{L}$, which is defined on a set of $\mathcal{L}$-formulas closed under subformulas, can be extended to a full valuation in $\mathcal{P}$.*

Remark 2.1.14. At this point the importance of analycity should again be stressed. Because of this property $\vdash_{\mathcal{S}}$ is decidable whenever $\mathcal{S}$ is a finite matrix. Moreover, analycity guarantees semi-decidability of non-theoremhood even if a matrix $\mathcal{P}$ is infinite, provided that $\mathcal{P}$ is effective (i.e, the set of truth-values is countable, the interpretation functions of the connectives are computable, and the set of designated truth-values is decidable). Note that this implies decidability in case $\vdash_{\mathcal{S}}$ also has a corresponding sound and complete proof system.

2.2 Introducing Propositional Nmatrices

Definition 2.2.1. A *non-deterministic matrix (Nmatrix)* for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.
- For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$.

A *2Nmatrix* is any Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$.

Definition 2.2.2. Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix for $\mathcal{L}$.

1. A partial *dynamic valuation* in $\mathcal{M}$ (or an $\mathcal{M}$-legal partial dynamic valuation) is a function v to $\mathcal{V}$ from some subset of $Frm_{\mathcal{L}}$, which is closed under subformulas, such that for each n-ary connective $\diamond$ of $\mathcal{L}$, the following holds for all $\psi_1, \ldots, \psi_n \in Frm_{\mathcal{L}}$:

   $$\text{(SLC)} \qquad v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

   A partial valuation in $\mathcal{M}$ is called a *valuation* if its domain is $Frm_{\mathcal{L}}$.

2. A (partial) *static valuation* in $\mathcal{M}$ (or an $\mathcal{M}$-legal (partial) static valuation) is a (partial) dynamic valuation which satisfies also the following compositionality (or functionality) principle: for each $\diamond$ of $\mathcal{L}$ and for every $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n \in Frm_{\mathcal{L}}$,

$$\text{(CMP) } v(\diamond(\psi_1, \ldots, \psi_n)) = v(\diamond(\varphi_1, \ldots, \varphi_n)) \text{ if } v(\psi_i) = v(\varphi_i) \ (i = 1 \ldots n)$$

Remark 2.2.3. Ordinary (deterministic) matrices correspond to the case when each $\widetilde{\diamond}$ is a function taking singleton values only (then it can be treated as a function $\widetilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$). In this case there is no difference between static and dynamic valuations, and we have full determinism.

Remark 2.2.4. Like in usual multi-valued semantics, the principle here is that each formula has a definite logical value. This is why we exclude $\emptyset$ from being a value of $\widetilde{\diamond}$. However, the absence of any logical value for a formula can still be simulated in our formalism by introducing a special logical value $\perp$ representing exactly this case (this is a well-known procedure in the framework of partial logics ([50])).

To understand the difference between ordinary matrices and Nmatrices, recall that in the deterministic case (see Defn. 2.1.7), the truth-value assigned by a valuation v to a complex formula is defined as follows: $v(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$. Thus the truth-value assigned to $\diamond(\psi_1, ..., \psi_n)$ is uniquely determined by the truth-values of its subformulas: $v(\psi_1), ..., v(\psi_n)$. This, however, is not the case in dynamic valuations in Nmatrices: in general the truth-values assigned to $\psi_1, ..., \psi_n$ do not uniquely determine the truth-value assigned to $\diamond(\psi_1, ..., \psi_n)$ because v makes a non-deterministic choice out of the set of options $\tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$. Therefore the non-deterministic semantics is non-truth-functional, as opposed to the deterministic one.

Definition 2.2.5.

1. A (partial) valuation v in $\mathcal{M}$ *satisfies* a formula ψ ($v \models \psi$) if ($v(\psi)$ is defined and) $v(\psi) \in \mathcal{D}$. It *satisfies* a set of formulas Γ ($v \models \Gamma$) if it satisfies every formula in Γ.

2. We say that ψ is dynamically (statically) valid in $\mathcal{M}$, in symbols $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$), if $v \models \psi$ for each dynamic (static) valuation v in $\mathcal{M}$.

3. A logic $\mathbf{L}$ is *dynamically (statically) weakly sound for an Nmatrix* $\mathcal{M}$ if $\vdash_{\mathbf{L}} \psi$ implies $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$). A logic $\mathbf{L}$ is *dynamically (statically) weakly complete for* $\mathcal{M}$ if $\models^d_{\mathcal{M}} \psi$ ($\models^s_{\mathcal{M}} \psi$) implies $\vdash_{\mathbf{L}} \psi$. $\mathcal{M}$ is a *dynamically (statically) weakly characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is dynamically (statically) both weakly sound and weakly complete for $\mathcal{M}$.

4. $\vdash^d_{\mathcal{M}}$ ($\vdash^s_{\mathcal{M}}$), the *dynamic (static) consequence relation induced by* $\mathcal{M}$, is defined as follows: $\Gamma \vdash^d_{\mathcal{M}} \Delta$ ($\Gamma \vdash^s_{\mathcal{M}} \Delta$), if every dynamic (static) model v in $\mathcal{M}$ of Γ satisfies some $\psi \in \Delta$.

5. A logic $\mathbf{L} = \langle \vdash_{\mathbf{L}}, \mathcal{L} \rangle$ is dynamically (statically) sound for an Nmatrix $\mathcal{M}$ for $\mathcal{L}$ if $\vdash_{\mathbf{L}} \subseteq \vdash^d_{\mathcal{M}}$ ($\vdash_{\mathbf{L}} \subseteq \vdash^s_{\mathcal{M}}$). $\mathbf{L}$ is dynamically (statically) complete for $\mathcal{M}$ if $\vdash^d_{\mathcal{M}} \subseteq \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} \subseteq \vdash_{\mathbf{L}}$). $\mathcal{M}$ is dynamically (statically) characteristic for $\mathbf{L}$ if $\vdash^d_{\mathcal{M}} = \vdash_{\mathbf{L}}$ ($\vdash^s_{\mathcal{M}} = \vdash_{\mathbf{L}}$).

The notion of a consequence relation can be generalized to the context of sequents (i.e., expressions of the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are finite sets of formulas). Given a Gentzen-type calculus G, denote by $\Theta \vdash_G \Omega$ when a sequent Ω has a proof from a set of sequents Θ in G.

Definition 2.2.6. 1. A valuation v *satisfies* a sequent $\Gamma \Rightarrow \Delta$ if whenever $v \models \Gamma$, there is some $\psi \in \Delta$, such that $v \models \psi$. v satisfies a set of sequents Θ if it satisfies every sequent in Θ.

2. Let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. For a set of sequents Θ and a sequent Ω, $\Theta \vdash^d_{\mathcal{M}} \Omega$ ($\Theta \vdash^s_{\mathcal{M}} \Omega$) if every dynamic (static) model v in $\mathcal{M}$ of Θ satisfies Ω.

3. An Nmatrix $\mathcal{M}$ is *dynamically (statically) strongly characteristic* for a calculus G if for every set of sequents Θ and every sequent Ω: $\Theta \vdash^d_{\mathcal{M}} \Omega$ ($\Theta \vdash^s_{\mathcal{M}} \Omega$) iff $\Theta \vdash_G \Omega$.

Note that if $\mathcal{M}$ is dynamically (statically) strongly characteristic for G, then $\mathcal{M}$ is dynamically (statically) sound and complete for the logic $\vdash_G$ (i.e., $\mathcal{M}$ is dynamically (statically) characteristic for $\vdash_G$ according to Definition 2.2.5).

Notation 2.2.7. We shall denote $\mathcal{F} = \mathcal{V} \setminus \mathcal{D}$, and shall usually identify singletons of truth-values with the truth-values themselves.

Example 2.2.8. Assume that $\mathcal{L}$ has binary connectives $\vee$, $\wedge$, and $\supset$ interpreted classically, and a unary connective $\neg$, for which the law of contradiction obtains, but not necessarily the law of excluded middle. This leads to the Nmatrix $\mathcal{M}^2 = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ for $\mathcal{L}$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}, \mathcal{D} = \{\mathbf{t}\}$, and $\mathcal{O}$ is given by:

		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$
t	**t**	**t**	**t**	**t**
t	**f**	**t**	**f**	**f**
f	**t**	**t**	**f**	**t**
f	**f**	**f**	**f**	**t**

	$\tilde{\neg}$
t	**f**
f	$\{\mathbf{t}, \mathbf{f}\}$

Note that classical negation can be defined in $\mathcal{M}^2$ by: $\sim\psi = \psi \supset \neg\psi$ (this is a semantic counterpart of the observation made in [49]).

Example 2.2.9. Consider the following two 3-valued Nmatrices $\mathcal{M}^3_L, \mathcal{M}^3_S$. In both we have $\mathcal{V} = \{\mathbf{f}, \top, \mathbf{t}\}, \mathcal{D} = \{\top, \mathbf{t}\}$. Also the interpretations of disjunction, conjunction and implication are the same in both of them, and correspond to those in positive classical logic:

$$a\widetilde{\vee}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a\widetilde{\wedge}b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \end{cases}$$

$$a\widetilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

However, negation is interpreted differently: more liberally in $\mathcal{M}^3_L$, and more strictly in $\mathcal{M}^3_S$:

$\mathcal{M}^3_L$:

	$\widetilde{\neg}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\mathcal{V}$
$\mathbf{f}$	$\mathbf{t}$

$\mathcal{M}^3_S$:

	$\widetilde{\neg}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\mathcal{D}$
$\mathbf{f}$	$\mathbf{t}$

It is shown in [27] that the dynamic semantics for $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ induce the same logic (i.e., consequence relation). (However, the developed proof mechanisms give a deeper insight into the matter: the sets of *3-sequents* which are derivable in the respective proof systems developed there for $\mathcal{M}^3_L$ and $\mathcal{M}^3_S$ do differ from each other.)

Example 2.2.10. After considering 2-valued Nmatrices and 3-valued Nmatrices, our last example is the 4-valued Nmatrix $\mathcal{M}^B_4 = \langle \mathcal{V}_4, \mathcal{D}_4, \mathcal{O}_4 \rangle$ for $\mathcal{L} = \{\wedge, \vee, \supset, \neg\}$ defined as follows. For a pair $p = \langle x, y \rangle$ of elements, let $P_1(p) = x$ and $P_2(p) = y$.

- $\mathcal{V}_4 = \{t, \top, \bot, f\}$[6] where:

$$\begin{array}{rcl} t & = & \langle 1, 0 \rangle \\ \top & = & \langle 1, 1 \rangle \\ \bot & = & \langle 0, 0 \rangle \\ f & = & \langle 0, 1 \rangle \end{array}$$

- $\mathcal{D}_4 = \{a \in \mathcal{V}_4 \mid P_1(a) = 1\} = \{t, \top\}$

[6] The intuition behind these four truth-values is like in Dunn-Belnap's logic mentioned in the Introduction.

- Let $\mathcal{V} = \mathcal{V}_4$, $\mathcal{D} = \mathcal{D}_4$, $\mathcal{F} = \mathcal{V}_4 - \mathcal{D}$. The operations in $\mathcal{O}_4$ are:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \quad (\text{i.e. } a \in \{f, \top\}) \\ \mathcal{F} & \text{if } P_2(a) = 0 \quad (\text{i.e. } a \in \{t, \bot\}) \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$\mathcal{M}_4^{B\mathbf{ff}}$ (for $\mathcal{L}_{\mathbf{ff}}$) is obtained from $\mathcal{M}_4^B$ by adding the condition: $\tilde{\mathbf{ff}} \in \mathcal{F}$.

Theorem 2.2.11. *([20]) $\mathcal{M}_4^B$ is a characteristic Nmatrix for $LK_{\neg}^{+}$.*

At this point it is natural to ask whether finite Nmatrices can be used for characterizing logics that cannot be characterized by finite ordinary matrices. The next theorem provides a positive answer to this question:

Theorem 2.2.12. *Let $\mathcal{M}$ be a two-valued Nmatrix which has at least one proper non-deterministic operation. Then there is no finite family of finite ordinary matrices F, such that $\vdash_{\mathcal{M}} = \vdash_F$. If in addition $\mathcal{M}$ includes the classical implication, then there is no finite family of ordinary matrices F, such that $\vdash_{\mathcal{M}} \psi$ iff $\vdash_F \psi$.*

Proof: A straightforward modification of the proof of Theorem 3.4 in [29].

As the next easy theorem shows, things are different in the case of the static semantics:

Theorem 2.2.13. *For every (finite) Nmatrix $\mathcal{M}$, there is a (finite) family of ordinary matrices, such that $\vdash^s_{\mathcal{M}} = \vdash_F$.*

Thus only the expressive power of the dynamic semantics based on Nmatrices is stronger than that of ordinary matrices. For this reason (after providing general proof theory for both kinds of semantics in the next subsection) our main focus will be on this semantics and what it induces. Accordingly, *we shall usually write simply $\vdash_{\mathcal{M}}$ instead of $\vdash^d_{\mathcal{M}}$.*

The following theorem from [29] is a generalization of Theorem 2.1.12 to the case of Nmatrices:

Theorem 2.2.14. (Compactness) $\vdash_{\mathcal{M}}$ *is finitary for any finite Nmatrix* $\mathcal{M}$.

The proof of the next important result is as easy for Nmatrices as it is for ordinary matrices:

Proposition 2.2.15. (Analycity) *Let* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *be an Nmatrix for* $\mathcal{L}$, *and let* v' *be a partial valuation in* $\mathcal{M}$. *Then* v' *can be extended to a (full) valuation in* $\mathcal{M}$.

It is easy to show that like in the case of ordinary matrices (see Remark 2.1.14), Proposition 2.2.15 implies the following Theorem:

Theorem 2.2.16. *Non-theoremhood of a logic which has an effective characteristic Nmatrix* $\mathcal{M}$ *is semi-decidable. If* $\mathcal{M}$ *is finite, or* L *also has a sound and complete formal proof system, then* L *is decidable.*

The following is an easy analogue for Nmatrices of Theorem 2.1.8:

Proposition 2.2.17. *For any Nmatrix* $\mathcal{M}$, $\vdash_{\mathcal{M}}$ *is uniform.*

Finally, we introduce the notion of a *refinement*:

Definition 2.2.18. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$.

1. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$ such that:

   (a) For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$.

   (b) $F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n, y \in \mathcal{V}_1$ such that $y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$.

2. $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.

Theorem 2.2.19. *If* $\mathcal{M}_1$ *is a refinement of* $\mathcal{M}_2$ *then* $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Remark 2.2.20. An important case in which $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is when $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\widetilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{V}_1^n$. It is easy to see that the identity function on $\mathcal{V}_1$ is in this case a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. A refinement of this sort will be called *simple*.

2.3 Application: Nmatrices for Logics of Formal Inconsistency

The concept of paraconsistency was introduced more than half a century ago, when several philosophers questioned the validity of classical logic with regard to its *ex contradictione quodlibet* (ECQ) principle. According to this counterintuitive principle, any proposition can be inferred from any inconsistent set of assumptions. Now the philosophical objections to this principle have recently been reinforced by practical considerations concerning information systems. Classical logic simply fails to capture the fact that information systems which contain some inconsistent pieces of information may produce useful answers to queries. The obvious conclusion from this state of affairs is that a more appropriate logic is needed for such systems.

A paraconsistent logic is a logic which allows non-trivial inconsistent theories. One of the oldest and best known approaches to the problem of designing useful paraconsistent logics is da Costa's approach. This approach is based on two main ideas. The first is to limit the applicability of the classical (and intuitionistic) rule $\neg\varphi, \varphi \vdash \psi$ to the case where φ is "consistent". The second is to express this assumption of consistency of φ within the language. The easiest way to implement these ideas is to include in the language a special connective $\circ$, with the intended meaning of $\circ\varphi$ being "φ is consistent". Then one can explicitly add the assumption of the consistency of φ to the problematic (from a paraconsistent point of view) rule, getting the rule called (**b**) below. Other rules concerning $\neg$ and $\circ$ can then be added, leading to a large family of logics known as "Logics of Formal Inconsistency" (LFIs - see [70, 59, 62]). Although the syntactic formulations of the LFIs are relatively simple, already on the propositional level the problem of finding useful semantic interpretations for them is rather complicated. Thus the vast majority of the propositional LFIs *cannot* be characterized by means of finite multi-valued matrices. What is more, for almost all of them no useful infinite characteristic matrix is known either. Therefore other types of semantics, like bivaluations semantics and possible translations semantics, have been proposed for them ([59, 62]). However, it is not clear how to extend these types of semantics to the first-order level.

In this chapter we briefly summarize the results from [20, 19, 18] and demonstrate how the framework of Nmatrices can be used to provide modular semantics for many propositional LFIs. In Chapter 6 we show that the framework can be naturally extended to the first-order level, preserving the property of modularity.

We shall focus on the main rules involving the consistency operator that have been studied in the literature on LFIs. These rules are listed in Figure 2.2 (in which $\diamond \in \{\wedge, \vee, \supset\}$).

Name of rule	Abstract form	Hilbert-style axiom
$(\mathbf{n})$	$\vdash \varphi, \neg\varphi$	$\varphi \vee \neg\varphi$
$(\mathbf{b})$	$\circ\varphi, \neg\varphi, \varphi \vdash$	$(\circ\varphi \wedge \neg\varphi \wedge \varphi) \supset \psi$
$(\mathbf{c})$	$\neg\neg\varphi \vdash \varphi$	$\neg\neg\varphi \supset \varphi$
$(\mathbf{e})$	$\varphi \vdash \neg\neg\varphi$	$\varphi \supset \neg\neg\varphi$
$(\mathbf{w})$	$\vdash \circ(\neg\varphi)$	$\circ(\neg\varphi)$
$(\mathbf{k1})$	$\vdash \circ\varphi, \varphi$	$\circ\varphi \vee \varphi$
$(\mathbf{k2})$	$\vdash \circ\varphi, \neg\varphi$	$\circ\varphi \vee \neg\varphi$
$(\mathbf{i1})$	$\neg\circ\varphi \vdash \varphi$	$\neg\circ\varphi \supset \varphi$
$(\mathbf{i2})$	$\neg\circ\varphi \vdash \neg\varphi$	$\neg\circ\varphi \supset \neg\varphi$
$(\mathbf{a}_\neg)$	$\circ\varphi \vdash \circ\neg\varphi$	$\circ\varphi \supset \circ\neg\varphi$
$(\mathbf{a}_\diamond)$	$\circ\varphi, \circ\psi \vdash \circ(\varphi \diamond \psi)$	$\circ\varphi \supset (\circ\psi \supset \circ(\varphi \diamond \psi))$
$(\mathbf{o}^1_\diamond)$	$\circ\varphi \vdash \circ(\varphi \diamond \psi)$	$\circ\varphi \supset \circ(\varphi \diamond \psi)$
$(\mathbf{o}^2_\diamond)$	$\circ\psi \vdash \circ(\varphi \diamond \psi)$	$\circ\psi \supset \circ(\varphi \diamond \psi)$
$(\mathbf{v}_\diamond)$	$\vdash \circ(\varphi \diamond \psi)$	$\circ(\varphi \diamond \psi)$
$(\mathbf{l})$	$\neg(\varphi \wedge \neg\varphi) \vdash \circ\varphi$	$\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

Figure 2.2: Schemata involving $\circ$

Throughout this section, we fix the language $\mathcal{L}_C = \{\neg, \circ, \supset, \wedge, \vee\}$.

Our starting point will be the basic logic $LK^+_\neg$ from Example 2.1.2.

Definition 2.3.1. 1. Let $LFIR$ be the set of all the rules from Figure 2.2 *except the last one* $(\mathbf{l})$. We denote by $HLFIR$ the set of Hilbert-style axioms corresponding to the rules in $LFIR$. We shall write $(\mathbf{i})$ instead of the combination of $(\mathbf{i1})$ and $(\mathbf{i2})$, $(\mathbf{a})$ instead of $\{(\mathbf{a}_\diamond) \mid \diamond \in \{\wedge, \vee, \supset\}\}$ and similarly for $(\mathbf{o})$.

2. For $S \subseteq LFIR$ let $LK^+_\neg[S]$ be the extension of $LK^+_\neg$ by S.

The basic idea in providing semantics for $LK^+_\neg[S]$ (where $S \subseteq LFIR$) is to let the value assigned to a sentence φ provide information not only about the truth/falsity of φ and $\neg\varphi$ (like in Dunn-Belnap logic), but also about the truth/falsity of $\circ\varphi$. This leads to the use of elements from $\{0,1\}^3$ as our truth-values, where the intended intuitive meaning of $v(\varphi) = \langle x, y, z\rangle$ is now:

- $x = 1$ iff φ is "true" (i.e. $v(\varphi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).

- $z = 1$ iff $\circ\varphi$ is "true" (i.e. $v(\circ\varphi) \in \mathcal{D}$).

This interpretation of the truth-values dictates the following conditions in the context of Nmatrices (where $P_i(\langle x_1, x_2, x_3\rangle) = x_i$):

$$\text{(NEG)} \qquad \tilde{\neg} a \subseteq \{y \mid P_1(y) = P_2(a)\}$$

$$\text{(CON)} \qquad \tilde{\circ} a \subseteq \{y \mid P_1(y) = P_3(a)\}$$

We start our semantic investigation of $LFIR$ with the weakest Nmatrix which satisfies both (NEG) and (CON). Then we show that every logic which is defined by some subset of $LFIR$ is characterized by some (easily computable) simple refinement of that Nmatrix.

Definition 2.3.2. The Nmatrix $\mathcal{M}_8^B = \langle \mathcal{V}_8, \mathcal{D}_8, \mathcal{O}_8\rangle$ is defined as follows:

- $\mathcal{V}_8 = \{0, 1\}^3$
- $\mathcal{D}_8 = \{a \in \mathcal{V}_8 \mid P_1(a) = 1\}$
- Let $\mathcal{V} = \mathcal{V}_8$, $\mathcal{D} = \mathcal{D}_8$, $\mathcal{F} = \mathcal{V}_8 - \mathcal{D}$. The operations in $\mathcal{O}_8$ are:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \\ \mathcal{F} & \text{if } P_2(a) = 0 \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } P_3(a) = 1 \\ \mathcal{F} & \text{if } P_3(a) = 0 \end{cases}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Definition 2.3.3. 1. The general refining conditions induced by the conditions in $LFIR$ are:

C(**n**): If $P_1(a) = 0$ then $P_2(a) = 1$

C(**b**): If $P_1(a) = 1$ and $P_2(a) = 1$ then $P_3(a) = 0$

C(**c**): If $P_1(a) = 0$ then $\tilde{\neg} a \subseteq \{x \mid P_2(x) = 0\}$

C(**e**): If $P_1(a) = 1$ then $\tilde{\neg} a \subseteq \{x \mid P_2(x) = 1\}$

C(**w**): $\tilde{\neg} a \subseteq \{x \mid P_3(x) = 1\}$

C(**k1**): If $P_1(a) = 0$ then $P_3(a) = 1$

C(**k2**): If $P_2(a) = 0$ then $P_3(a) = 1$

C(**i1**): If $P_1(a) = 0$ then $\tilde{\circ} a \subseteq \{x \mid P_2(x) = 0\}$

C(**i2**): If $P_2(a) = 0$ then $\tilde{\circ} a \subseteq \{x \mid P_2(x) = 0\}$

C($\mathbf{a}_\neg$): If $P_3(a) = 1$ then $\tilde{\neg} a \subseteq \{x \mid P_3(x) = 1\}$

C($\mathbf{a}_\diamond$): If $P_3(a) = 1$ and $P_3(b) = 1$ then $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$

C($\mathbf{o}^1_\diamond$): If $P_3(a) = 1$ then $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$

C($\mathbf{o}^2_\diamond$): If $P_3(b) = 1$ then $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$

C($\mathbf{v}_\diamond$): $a \tilde{\diamond} b \subseteq \{x \mid P_3(x) = 1\}$

2. For $S \subseteq LFIR$, let $C(S) = \{\mathrm{C}r \mid r \in S\}$, and let $\mathcal{M}_S$ be the weakest simple refinement of $\mathcal{M}^B_8$ in which the conditions in $C(S)$ are all satisfied (again it is not difficult to check that this is well-defined for every $S \subseteq LFIR$).

Theorem 2.3.4. *$\mathcal{M}_S$ ($S \subseteq LFIR$) is a characteristic Nmatrix for $LK^+_\neg[S]$.*

Corollary 2.3.5. *$LK^+_\neg[S]$ is decidable for every $S \subseteq LFIR$.*

Example 2.3.6. Let $\mathbf{B} = LK^+_\neg[\{(\mathbf{n}), (\mathbf{b})\}]$. This logic is the basic logic of formal inconsistency from [59, 62] (where it is called *mbC*). By Theorem 2.3.4, the following Nmatrix $\mathcal{M}^B_5 = \langle \mathcal{V}_5, \mathcal{D}_5, \mathcal{O}_5 \rangle$ is characteristic for it:

- $\mathcal{V}_5$ is the set $\{t, t_I, I, f_I, f\}$ where:

$$\begin{array}{rcl} t & = & \langle 1,0,1\rangle \\ t_I & = & \langle 1,0,0\rangle \\ I & = & \langle 1,1,0\rangle \\ f & = & \langle 0,1,1\rangle \\ f_I & = & \langle 0,1,0\rangle \end{array}$$

  Note that the axiom $(\mathbf{n})$ leads to the deletion from $\mathcal{V}_8$ of the truth-values $\langle 0,0,1\rangle$ and $\langle 0,0,0\rangle$, while the axiom $(\mathbf{b})$ leads to the deletion of $\langle 1,1,1\rangle$.

- $\mathcal{D}_5 = \{t, I, t_I\}$ $(= \{\langle x,y,z\rangle \in \mathcal{V}_5 \mid x = 1\})$.

- Let $\mathcal{D} = \mathcal{D}_5$, $\mathcal{F} = \mathcal{V}_5 - \mathcal{D}$. The operations in $\mathcal{O}_5$ are defined by:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } a \in \{I, f, f_I\} \\ \mathcal{F} & \text{if } a \in \{t, t_I\} \end{cases}$$

$$\widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{I, t_I, f_I\} \end{cases}$$

The rest of the operations are defined like in Definition 2.3.2.

Example 2.3.7. Let **Cia** $= \{(\mathbf{n}), (\mathbf{b}), (\mathbf{c}), (\mathbf{i}), (\mathbf{a})\}$. $\mathcal{M}_{Cia} = \langle \mathcal{V}_{Cia}, \mathcal{D}_{Cia}, \mathcal{O}_{Cia} \rangle$, where:

- $\mathcal{V}_{Cia} = \{t, I, f\}$
- $\mathcal{D}_{Cia} = \{t, I\}$
- $a \widetilde{\supset} b = \begin{cases} \{f\} & \text{if } a \in \{t, I\} \text{ and } b = f \\ \{t\} & \text{if either } a = f,\ b \in \{f, t\} \text{ or } a = t,\ b = t \\ \{t, I\} & \text{otherwise} \end{cases}$
- $a \widetilde{\vee} b = \begin{cases} \{f\} & \text{if } a = f \text{ and } b = f \\ \{t\} & \text{if either } a = t,\ b \in \{f, t\} \text{ or } b = t,\ a \in \{f, t\} \\ \{t, I\} & \text{otherwise} \end{cases}$
- $a \widetilde{\wedge} b = \begin{cases} \{f\} & \text{if } a = f \text{ or } b = f \\ \{t\} & \text{if } a = t \text{ and } b = t \\ \{t, I\} & \text{otherwise} \end{cases}$
- $\widetilde{\neg} t = \{f\} \quad \widetilde{\neg} I = \{I\} \quad \widetilde{\neg} f = \{t\}$
- $\widetilde{\circ} t = \widetilde{\circ} f = \{t\} \quad \widetilde{\circ} I = \{f\}$

The family of LFIs for which we provided semantics in the previous subsection does not include the well-known da Costa's original logic C_1 from ([70]). Now C_1 is just the $\circ$-free fragment of **Cila**, the logic which is obtained by adding the rule **(l)** from Figure 2.2 to the system **Cia** from Example 2.3.7. This rule is problematic, because of the following theorem:

Theorem 2.3.8. *No logic between* **Bl** *and* **Bl**$[(\mathbf{i}), (\mathbf{o})]$ *has a finite characteristic Nmatrix (and so also a finite characteristic ordinary matrix).*

It follows that the method used in the previous subsection cannot work for logics like **Cila**. As a reasonable useful substitute, *infinite* (but still effective) Nmatrices can be used for a family of such systems (which includes **Cila**).

Definition 2.3.9. Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$. The Nmatrix $\mathcal{M}_{\mathbf{Bl}} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined as follows:

1. $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$ and $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$.

2. $\mathcal{O}$ is defined by:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$\widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

Theorem 2.3.10. *$\mathcal{M}_{\mathbf{Bl}}$ is a characteristic Nmatrix for* **Bl**.

As for extending **Bl** with axioms from the set $LFIR$, like in the previous subsection, each of the schemata corresponds to some easily computed semantic condition, this time on simple refinements of the basic Nmatrix $\mathcal{M}_{\mathbf{Bl}}$. These conditions are in fact identical to the conditions that correspond to these axioms in refinements of $\mathcal{M}_{\{(\mathbf{b}),(\mathbf{n}),(\mathbf{k1}),(\mathbf{k2})\}}$, but with t replaced by $\mathcal{T}$, and I replaced by $\mathcal{I}$ (see [20] for further details).

Example 2.3.11. *da Costa's system C_1 is decidable, and it has a characteristic Nmatrix $\mathcal{M}_{C_1}$, in which the sets of truth-values and designated truth-values are like in $\mathcal{M}_{\mathbf{Bl}}$, and the interpretations of the connectives are defined as follows:*

$$a \widetilde{\supset} b = \begin{cases} \mathcal{F} & a \in \mathcal{D},\ b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{F},\ b \notin \mathcal{I} \\ \mathcal{T} & b \in \mathcal{T},\ a \notin \mathcal{I} \\ \mathcal{D} & \textit{otherwise} \end{cases} \qquad a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & a \in \mathcal{F} \textit{ or } b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T},\ b \in \mathcal{T} \\ \mathcal{T} & a = I_i^j,\ b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & a \in \mathcal{T} \\ \mathcal{T} & a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & a = I_i^j \end{cases} \qquad a \widetilde{\vee} b = \begin{cases} \mathcal{F} & a \in \mathcal{F},\ b \in \mathcal{F} \\ \mathcal{T} & a \in \mathcal{T},\ b \notin \mathcal{I} \\ \mathcal{T} & b \in \mathcal{T},\ a \notin \mathcal{I} \\ \mathcal{D} & \textit{otherwise} \end{cases}$$

Chapter 3

Nmatrices for Canonical Calculi

In this chapter we apply the propositional framework of Nmatrices presented in the previous chapter for characterizing a very natural family of *canonical systems* and investigating the phenomena of cut-elimination in such systems. The idea of "canonical" systems implicitly underlies a long tradition in the philosophy of logic, established by G. Gentzen in his classical paper [78]. According to this tradition, the meaning of a connective is determined by the introduction and the elimination rules which are associated with it (see, e.g., [135, 136]). The supporters of this thesis usually have in mind Natural Deduction systems of an ideal type. In this type of "canonical systems" each connective $\diamond$ has its own introduction and elimination rules, in each of which $\diamond$ is mentioned exactly once, and no other connective is involved. The rules should also be pure in the sense of [15]. Unfortunately, already the handling of negation requires rules which are not canonical in this sense. This problem was solved by Gentzen himself by moving to what is now known as (multiple-conclusion) Gentzen-type calculi, which instead of introduction and elimination rules use left and right introduction rules. The intuitive notion of a "canonical rule" can be adapted to such systems in a straightforward way, and it is well-known that the usual classical connectives can indeed be fully characterized in this framework by such rules. Moreover, the cut-elimination theorem obtains in all the usual Gentzen-type calculi for propositional classical logic (or some fragment of it) which employ only rules of this type. These facts were generalized in [28, 29], where the notion of a canonical propositional Gentzen-type system was defined in precise terms. It was shown that semantics for such systems can be provided using two-valued non-deterministic matrices (2Nmatrices). Moreover, there is an exact *triple correspondence* between cut-elimination in such systems, the existence of a characteristic 2Nmatrix for them, and a constructive syntactic property called *coherence*. We briefly summarize these results in Section 3.1.
This chapter has two main goals. First of all, we show that 2Nmatrices play an important role not only in the phenomena of cut-elimination, but also in two other important

properties of sequent calculi: *invertibility of logical rules* and *axiom expansion*. We provide a full characterization of these properties in canonical coherent Gentzen-type calculi and show that for a coherent calculus G in *normal form* (to which every calculus can be transformed), another triple correspondence can be established: (i) the connectives of G admit axiom expansion, iff (ii) the rules of G are invertible, iff (iii) G has a finite deterministic characteristic matrix.

The second goal of this chapter is to extend the theory of canonical systems to a considerably more general class of systems: *signed calculi* (of which Gentzen-type calculi are particular instances). For this we first extend the notion of "canonical systems" to signed calculi. Then, using finite Nmatrices, we provide modular non-deterministic semantics for signed canonical calculi. Finally, we show that the extended criterion of coherence fully characterizes strong analytic cut-elimination in such calculi, while for characterizing strong and standard cut-elimination a stronger criterion of density is required.

The new results of this chapter are mainly based on [26, 134].

3.1 Canonical Gentzen-type Systems

In this section we briefly summarize the main results from previous works on canonical Gentzen-type propositional calculi from [28, 29].

By a *sequent* we shall mean here an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are *finite* sets of $\mathcal{L}$-formulas. A *clause* is a sequent consisting of atomic formulas.

Definition 3.1.1. *A canonical rule of arity n is an expression of the form $[\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C]$, where $m \geq 0$, C is either $\diamond(p_1, ..., p_n) \Rightarrow$ or $\Rightarrow \diamond(p_1, ..., p_n)$ for some n-ary connective $\diamond$, and for all $1 \leq i \leq m$: $\Pi_i, \Sigma_i \subseteq \{p_1, ..., p_n\}$.*

An application of a canonical rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\diamond(p_1, ..., p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, ..., \psi_n) \Rightarrow \Delta}$$

where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i respectively by substituting ψ_j for p_j for all $1 \leq j \leq n$, and Γ, Δ are any sets of formulas.

An application of $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\Rightarrow \diamond(p_1, ..., p_n)$ is defined symetrically.

An application is an identity application *if $\Sigma_i^* = \Sigma_i$ and $\Pi_i^* = \Pi_i$ for all $1 \leq i \leq n$.*

Example 3.1.2. The standard Gentzen-style introduction rules for the classical conjunction are formulated as follows:

$$[\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow] \quad [\{\Rightarrow p_1 \; ; \; \Rightarrow p_2\}/\Rightarrow p_1 \wedge p_2]$$

Their applications have the forms:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

The precise notion of a "canonical calculus" is defined as follows:

Definition 3.1.3. *A Gentzen-type calculus G is* canonical *if in addition to the standard axioms of the form $\psi \Rightarrow \psi$ and the standard structural rules, it has only canonical logical rules.*

Any set of canonical rules (together with logical axioms and structural rules) constitutes a canonical calculus. However, our quest is for calculi which also have a well-defined semantics in terms of 2Nmatrices. For characterizing such calculi the syntactic criterion of *coherence* is introduced:

Definition 3.1.4. *A canonical calculus G is* coherent *if for every pair of rules $[\Theta_1/ \Rightarrow \diamond(p_1, ..., p_n)]$ and $[\Theta_2/\diamond(p_1, ..., p_n) \Rightarrow]$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent (and so the empty set can be derived from it using cuts).*

For instance, the canonical calculus from Example 3.1.2 is coherent, as one can derive the empty sequent from $\{p_1, p_2 \Rightarrow; \Rightarrow p_1; \Rightarrow p_2\}$ using cuts.

Remark 3.1.5. [69] investigates a general class of two-sided (sequent) calculi with generalized quantifiers, which include any set of structural rules (so canonical calculi are a particular instance, which includes all of the standard structural rules). The reductivity condition of [69] can be shown to be equivalent to coherence. Note, however, that unlike coherence, reductivity is not constructive.

The following theorem provides an exact correspondence between the existence of a 2Nmatrix for a canonical calculus, cut-elimination in this calculus, its consistency and coherence:

Theorem 3.1.6. *([29]) Let G be a canonical calculus. The following statements concerning G are equivalent:*

1. *G admits cut-elimination.*
2. *$\vdash_G$ is consistent[1].*

[1] Recall Definition 2.1.1.

3. *G is coherent.*

4. *G has a characteristic 2Nmatrix.*

3.2 Invertibility, Axiom Expansion, Determinism

In this section we investigate invertibility and axiom expansion in canonical calculi and establish a connection between these two important properties and deterministic 2Nmatrices. Invertibility of logical rules (see Definition 3.2.16 below) is a key property of many deduction formalisms, such as analytic tableaux [86, 87, 38] and Rasiowa-Sikorski (R-S) systems [115, 96], also known as dual tableaux. This property induces an algorithm for finding a proof of a complex formula in a deduction system, if such a proof exists. Axiom expansion (see Definition 3.2.26 below) is another important property of sequent calculi, often considered crucial when designing "well-behaved" systems (see e.g. [81]). This property allows for the reduction of logical axioms to the atomic case.

We start by defining the notion of a "normal form" for a canonical calculus. In general, a canonical calculus may have a number of right (and left) introduction rules for the same connective. However, we show that any canonical calculus can be "normalized", i.e. transformed into a calculus with at most one right and one left introduction rule for each connective, which also satisfy the properties described below.

Definition 3.2.1. 1. *A sequent $\Gamma \Rightarrow \Delta$ is* subsumed *by a sequent $\Gamma' \Rightarrow \Delta'$ if $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.*

2. *An* extended axiom *is any sequent of the form $\Gamma \Rightarrow \Delta$, where $\Gamma \cap \Delta \neq \emptyset$. An extended axiom is atomic if $\Gamma \cap \Delta$ contains an atomic formula.*

3. *A canonical calculus G is in* normal form *if (i) G has at most one left and at most one right introduction rule for each connective, (ii) its introduction rules have no extended axioms as their premises, and (iii) its introduction rules have no clauses in their premises which are subsumed by some other clause in their premises.*

Below we show that for every calculus has a calculus in normal form, which is equivalent to it in the following sense:

Definition 3.2.2. *Two sets of canonical rules S_1 and S_2 are* equivalent *if the conclusion of every application of $R \in S_1$ is derivable from its premises using rules from S_2 and weakening, and vice versa. Two canonical calculi G_1 and G_2 are* cut-free equivalent *if their sets of canonical rules are equivalent.*

The following easy proposition follows from the definition of coherence:

Proposition 3.2.3. *If a canonical calculus G is coherent, then so is any canonical calculus G' which is cut-free equivalent to G.*

The proof of the following proposition is an adaptation of proofs from [27] and [41].

Proposition 3.2.4. *Every canonical calculus G has a cut-free equivalent calculus G^n in normal form.*

Proof. Let us describe the transformation of G into a calculus G^n in normal form. Take a pair of rules in G of the forms $R_1 = \{\Sigma_i^1 \Rightarrow \Pi_i^1\}_{1 \leq i \leq m} / \Rightarrow \diamond(p_1 \ldots p_n)$ and $R_2 = \{\Sigma_j^2 \Rightarrow \Pi_j^2\}_{1 \leq j \leq l} / \Rightarrow \diamond(p_1 \ldots p_n)$. Replace R_1 and R_2 in G by $R = \{\Sigma_i^1, \Sigma_j^2 \Rightarrow \Pi_i^1, \Pi_j^2\}_{1 \leq i \leq m, 1 \leq j \leq l} / \Rightarrow \diamond(p_1 \ldots p_n)$. Clearly, any application of R can be simulated by applying R_1 and R_2. Moreover, any application of R_1 and of R_2 can be simulated by weakening and R. Hence, $\{R\}$ and $\{R_1, R_2\}$ are cut-free equivalent. By repeatedly applying this step, we get at most one left and one right introduction rule for each connective. Next, discard the premises which are extended axioms. (Indeed, if R' is a rule obtained from R by discarding an extended axiom, then any application of R' can be simulated by an application of R on the premises with the addition of an axiom). Clearly, G and the resulting calculus G' are cut-free equivalent. Finally, discard any premise $\Gamma \Rightarrow \Delta$ subsumed by any other premise $\Gamma' \Rightarrow \Delta'$ in each rule (since $\Gamma \Rightarrow \Delta$ can be derived from $\Gamma' \Rightarrow \Delta'$ using weakening, the resulting calculus is cut-free equivalent to G'). □

Example 3.2.5. Consider the canonical calculus G_X with four introduction rules for the binary connective X, representing XOR:

$$[\{\Rightarrow p_1 \ ; \ p_2 \Rightarrow\} / \Rightarrow p_1 X p_2] \quad [\{\Rightarrow p_2 \ ; \ p_1 \Rightarrow\} / \Rightarrow p_1 X p_2]$$
$$[\{\Rightarrow p_1 \ ; \ \Rightarrow p_2\} / p_1 X p_2 \Rightarrow] \quad \{[p_1 \Rightarrow \ ; \ p_2 \Rightarrow\} / p_1 X p_2 \Rightarrow]$$

This calculus can be transformed into an equivalent calculus G_X^n in normal form as follows. We start by replacing the first two rules by the following rule:

$$\{[\Rightarrow p_1, p_2 \ ; \ p_1, p_2 \Rightarrow; \ p_1 \Rightarrow p_1 \ ; \ p_2 \Rightarrow p_2\} / \Rightarrow p_1 X p_2]$$

The second pair of rules can be replaced by:

$$[\{p_1 \Rightarrow p_2 \ ; \ p_2 \Rightarrow p_1; \ p_1 \Rightarrow p_1 \ ; \ p_2 \Rightarrow p_2\} / p_1 X p_2 \Rightarrow]$$

Finally the axioms in the premises are discarded and we get the following cut-free equivalent calculus G_X^n in normal form:

$$[\{\Rightarrow p_1, p_2 \ ; \ p_1, p_2 \Rightarrow\} / \Rightarrow p_1 X p_2 \quad \{p_1 \Rightarrow p_2 \ ; \ p_2 \Rightarrow p_1\} / p_1 X p_2 \Rightarrow]$$

The following well-known fact follows from the completeness of propositional resolution:

Proposition 3.2.6. *A set of clauses is satisfiable (by an atomic valuation) iff it is consistent.*

Notation 3.2.7. *For $i \leq 0$, denote the clause $\Rightarrow p_i$ by S_i^t and the clause $p_i \Rightarrow$ by S_i^f. Let $\overline{a} = \langle a_1, ..., a_n \rangle \in \{t, f\}^n$. $\mathsf{C}_{\overline{a}}$ is the set of clauses $\{S_i^{a_i}\}_{1 \leq i \leq n}$.*

Now we construct a strongly characteristic 2Nmatrix for every coherent canonical calculus. For this we shall need the following easy lemma:

Lemma 3.2.8. *Let Θ be a set of clauses over $\{p_1, ..., p_n\}$. Let $\overline{a} = \langle a_1, ..., a_n \rangle \in \{t, f\}^n$ and let v be any atomic valuation, such that $v(p_i) = a_i$ for all $1 \leq i \leq n$. Then $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent iff v is satisfies Θ.*

Definition 3.2.9. *Let G be a coherent canonical calculus. The Nmatrix $\mathcal{M}_G$ is defined as follows for every n-ary connective $\diamond$ and $\overline{a} = \langle a_1, ..., a_n \rangle \in \{t, f\}^n$:*

$$\tilde{\diamond}(a_1, ..., a_n) = \begin{cases} \{t\} & \textit{there is some } [\Theta / \Rightarrow \diamond(p_1, ..., p_n)] \in G, \textit{ where } \Theta \cup \mathsf{C}_{\overline{a}} \textit{ is consistent.} \\ \{f\} & \textit{there is some } [\Theta / \diamond(p_1, ..., p_n) \Rightarrow] \in G, \textit{ where } \Theta \cup \mathsf{C}_{\overline{a}} \textit{ is consistent.} \\ \{t, f\} & \textit{otherwise} \end{cases}$$

Note that the coherence of G guarantees that $\mathcal{M}_G$ is well-defined. Indeed, if G is coherent, then there is no pair of rules $[\Theta_1 / \Rightarrow \diamond(p_1, \ldots, p_n)]$ and $[\Theta_2 / \diamond(p_1, \ldots, p_n) \Rightarrow]$ in G, such that $\Theta_1 \cup \Theta_2$ is consistent.

Theorem 3.2.10. *$\mathcal{M}_G$ is a strongly characteristic Nmatrix for G.*

Proof. The proof is a simplified version of the proof of Theorem 3.3.30 in the sequel. □

Our construction of $\mathcal{M}_G$ is much simpler than the construction carried out in [28]: a canonical calculus G there is first transformed into a cut-free equivalent calculus G', which usually has more rules than G. G' is then used to construct the characteristic Nmatrix. The idea is to transform the calculus so that each rule of G' dictates the interpretation for only one tuple $\langle a_1, \ldots, a_n \rangle$. However, the above definition shows that this transformation is actually not necessary and we can construct $\mathcal{M}_G$ directly from G. Moreover, by the above theorem, Proposition 3.2.11 below and the soundness and completeness theorem of [28], it follows that the 2Nmatrix constructed there for every coherent calculus G is identical to $\mathcal{M}_G$ defined above.

Proposition 3.2.11. *For every two coherent canonical calculi G_1 and G_2 which are cut-free equivalent, $\mathcal{M}_{G_1} = \mathcal{M}_{G_2}$.*

Proof. First we shall need the following technical propositions:

Notation 3.2.12. For a set of formulas Γ, denote by $\mathsf{At}(\Gamma)$ the set of atomic formulas occurring in Γ. For a sequent $\Omega = \Gamma \Rightarrow \Delta$, denote by $\mathsf{At}(\Omega)$ the sequent $\mathsf{At}(\Gamma) \Rightarrow \mathsf{At}(\Delta)$. For a clause Ω (a set of clauses Θ), denote by $mod(\Omega)$ the set of all the atomic valuations which satisfy Ω (Θ).

Lemma 3.2.13. *Let $R = [\Theta/C]$ be a canonical rule, where $\Theta = \{\Sigma_i \Rightarrow \Pi_i\}_{1\leq i\leq m}$. Consider an identity application (Definition 3.1.1) of R with premises $\Omega_1, \dots, \Omega_m$ and conclusion Ω. Then it must hold that $(\bigcap_{1\leq i\leq m} mod(\mathsf{At}(\Omega_i))) \setminus mod(\mathsf{At}(\Omega)) \subseteq mod(\Theta)$.*

Proof. Let Ω be either $\Gamma \Rightarrow \Delta, \diamond(p_1, \dots, p_n)$ or $\diamond(p_1, \dots, p_n), \Gamma \Rightarrow \Delta$. For all $1 \leq i \leq m$, denote the sequent $\Gamma, \Sigma_i \Rightarrow \Pi_i, \Delta$ by Ω_i. Let $v \in (\bigcap_{1\leq i\leq m} mod(\mathsf{At}(\Omega_i))) \setminus mod(\mathsf{At}(\Omega))$. $v \notin mod(\mathsf{At}(\Gamma \Rightarrow \Delta))$ (indeed, otherwise it would be the case that $v \in mod(\mathsf{At}(\Omega))$). Thus v satisfies $\mathsf{At}(\Gamma)$ but does not satisfy any of the formulas in $\mathsf{At}(\Delta)$. Let $1 \leq i \leq m$. If v satisfies Σ_i, then since v satisfies $\mathsf{At}(\Omega_i) = \mathsf{At}(\Gamma), \Sigma_i \Rightarrow \mathsf{At}(\Delta), \Pi_i$, there is some $\psi \in \Pi_i$, of which v is a model. Thus v satisfies $\Sigma_i \Rightarrow \Pi_i$ for all $1 \leq i \leq m$ and so $v \in mod(\Theta)$. □

Corollary 3.2.14. *Let G be a canonical calculus. Suppose that Ω has a derivation in G from extended atomic axioms, which consists only of identity applications of canonical rules. If an atomic valuation v does not satisfy $\mathsf{At}(\Omega)$, then there is some canonical rule $[\Theta/C]$ applied in this derivation, such that $v \in mod(\Theta)$.*

Proof. By induction on the length l of the derivation of Ω. For $l = 1$ the claim trivially holds (v satisfies $\mathsf{At}(\Omega)$). Otherwise, consider the last application in the derivation, which must be an identity application of some canonical rule $[\Theta/C]$, where $\Theta = \{\Sigma_i \Rightarrow \Pi_i\}_{1\leq i\leq m}$. Denote its premises by $\Omega_1, \dots, \Omega_m$ and its conclusion by Ω. Let $v \notin mod(\mathsf{At}(\Omega))$. If v satisfies $\mathsf{At}(\Omega_i)$ for all $1 \leq i \leq m$, then by Lemma 3.2.13, $v \in mod(\Theta)$. Otherwise there is some $1 \leq i \leq m$, such that v does not satisfy $\mathsf{At}(\Omega_i)$. By the induction hypothesis, v satisfies Θ' for some canonical rule $[\Theta'/C']$ applied in the derivation of Ω_i. □

Back to the proof of Proposition 3.2.11, let G_1 and G_2 be two cut-free equivalent coherent calculi. Let $\diamond$ be some n-ary connective and let $\overline{a} = \langle a_1, \dots, a_n\rangle \in \{t, f\}^n$. Suppose that $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{t\}$. Then there is a rule $R = [\Theta/ \Rightarrow \diamond(p_1, \dots, p_n)]$ in G_1, such that $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent. Consider the application of R with the premises Θ and the conclusion $\Rightarrow \diamond(p_1, \dots, p_n)$. Let v be any atomic valuation, such that $v(p_i) = a_i$ for all

$1 \leq i \leq n$. Since $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent, by Lemma 3.2.8, $v \in mod(\Theta)$. Now since G_1 and G_2 are cut-free equivalent, there is a derivation D of $\Rightarrow \diamond(p_1, \dots, p_n)$ from Θ using the rules of G_2 and weakening. Since $\mathsf{At}(\Rightarrow \diamond(p_1, \dots, p_n)) = \emptyset$, $v \notin \mathsf{At}(\Rightarrow \diamond(p_1, \dots, p_n))$, and by Corollary 3.2.14, there is some rule $[\Theta'/S]$ of G_2 applied in D, such that $v \in mod(\Theta')$. Since the derivation of $\Rightarrow \diamond(p_1, \dots, p_n)$ from Θ is cut-free, it must be the case that this application is an identity application and S is the sequent $\Rightarrow (p_1, \dots, p_n)$. By Lemma 3.2.8, $\Theta' \cup \mathsf{C}_{\overline{a}}$ is consistent. Hence, $\tilde{\diamond}_{\mathcal{M}_{G_2}}(\overline{a}) = \{t\}$. The case when $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{f\}$ is handled similarly. If $\tilde{\diamond}_{\mathcal{M}_{G_2}}(\overline{a}) = \{t\}$ or $\tilde{\diamond}_{\mathcal{M}_{G_2}}(\overline{a}) = \{f\}$, the proof that $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{t\}$ or $\tilde{\diamond}_{\mathcal{M}_{G_1}}(\overline{a}) = \{f\}$ respectively is symmetric to the previous cases.

□

Proposition 3.2.15. *Let G be a coherent canonical calculus. $\mathcal{M}_G$ is deterministic iff G has a finite characteristic deterministic matrix.*

Proof. ($\Leftarrow$) Assume that $\mathcal{M}_G$ has at least one non-deterministic operation. Then by Theorem 2.2.12, there is no finite ordinary matrix P, such that $\vdash_P = \vdash_{\mathcal{M}}$. Hence, there is no characteristic finite deterministic matrix for G. The second direction is trivial (recall that we can identify a deterministic matrix with its corresponding deterministic Nmatrix). □

3.2.1 Invertibility and Determinism

Below we provide a full characterization of invertibility in the context of coherent canonical calculi[2].

The usual definition of invertibility of rules (see, e.g. [125]) is the following:

Definition 3.2.16. *A rule R is* invertible *in a calculus G if for every application of R it holds that whenever its conclusion is provable in G, then also each of its premises is provable in G.*

In the context of canonical calculi it is convenient to introduce the following useful notion:

Definition 3.2.17. *Let G be a canonical calculus. A rule R is* canonically invertible *in G if for every $1 \leq i \leq m$: $\Sigma_i \Rightarrow \Pi_i$ has a proof in G from $\Rightarrow \diamond(p_1, ..., p_n)$. Canonical invertibility for left introduction rules is defined similarly.*

[2]Syntactic sufficient conditions for invertibility in sequent calculi with non-standard structural rules were introduced in [63] and [109].

Note that unlike standard invertibility, canonical invertibility is defined for *rules*, and not their instances. Thus canonical invertibility can be checked constructively, as opposed to invertibility (since each rule has infinitely many instances).

We show below that the two notions defined above are equivalent for canonical calculi:

Proposition 3.2.18. *A canonical rule is invertible in G iff it is canonically invertible in G.*

Proof. ($\Leftarrow$) Assume w.l.o.g. that a rule R is canonically invertible in G. Consider an application of R with the premises $\Gamma, \Sigma_1^* \Rightarrow \Delta, \Pi_1^*;\ \ldots\ ;\ \Gamma, \Sigma_m^* \Rightarrow \Delta, \Pi_m^*$ and the conclusion $\Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$ where for all $1 \leq j \leq m$, Σ_j^*, Π_j^* are obtained from Σ_j, Π_j by replacing each p_k by ψ_k for all $1 \leq k \leq n$. Suppose that $\vdash_G \Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$. We need to show that $\vdash_G \Gamma, \Sigma_j^* \Rightarrow \Delta, \Pi_j^*$ for all $1 \leq j \leq m$. Since R is canonically invertible, there is a proof of $\Sigma_j \Rightarrow \Pi_j$ from $\Rightarrow \diamond(p_1, ..., p_n)$. By replacing in this proof each p_k by ψ_k and adding the contexts Γ and Δ in each step of the derivation, we obtain a proof of $\Gamma, \Sigma_j^* \Rightarrow \Delta, \Pi_j^*$ from $\Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$. Thus if $\Gamma \Rightarrow \Delta, \diamond(\psi_1, ..., \psi_n)$ is provable, so is $\Gamma, \Sigma_j^* \Rightarrow \Delta, \Pi_j^*$. Hence R is invertible.
($\Rightarrow$) Assume that R is invertible in G. Consider the application of R with the conclusion $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$. Since G is canonical, $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ is provable in G. Since R is invertible, each of its premises $\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i$ is provable as well. By applying cut, we have a proof of $\Sigma_i \Rightarrow \Pi_i$ from $\Rightarrow \diamond(p_1, ..., p_n)$ for every $1 \leq i \leq m$ and the claim follows. □

Next we introduce the notion of *expandability* of rules, and show that it is equivalent to invertibility in coherent canonical calculi.

Definition 3.2.19. *A canonical rule $R = [\{\Sigma_i \Rightarrow \Pi_i\}_{1 \leq i \leq m} / \Rightarrow \diamond(p_1, \ldots, p_n)]$ is* expandable *in a canonical calculus G if for every $1 \leq i \leq m$: $\diamond(p_1, ..., p_n), \Sigma_i \Rightarrow \Pi_i$ has a* cut-free *proof in G. The notion of expandability in G for a left introduction rule is defined symmetrically.*

Proposition 3.2.20. *For any canonical calculus G, every expandable rule is invertible. If G is coherent, then every invertible rule is expandable.*

Proof. Let G be any canonical calculus. Assume w.l.o.g. that the rule R is expandable in G. Hence $\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i$ is provable for each $1 \leq i \leq m$. By cut, $\Sigma_i \Rightarrow \Pi_i$ is provable from $\Rightarrow \diamond(p_1, ..., p_n)$. Thus R is canonically invertible, and hence invertible by Proposition 3.2.18. Now assume that G is coherent and R is invertible in G. By

Proposition 3.2.18, R is canonically invertible, and so for all $1 \leq i \leq m$: $\Sigma_i \Rightarrow \Pi_i$ is derivable from $\Rightarrow \diamond(p_1, \ldots, p_n)$. By adding $\diamond(p_1, \ldots, p_n)$ on the left side of all the sequents in the derivation, we obtain a derivation of $\diamond(p_1, \ldots, p_n), \Sigma_i \Rightarrow \Pi_i$ in G. Since G is coherent, by Theorem 3.1.6 it admits cut-elimination, thus we have a cut-free derivation of $\diamond(p_1, \ldots, p_n), \Sigma_i \Rightarrow \Pi_i$ in G, and hence R is expandable. $\square$

Although expandability and invertibility are equivalent for coherent canonical calculi, checking the former is an easier task, as it amounts to checking whether a sequent is cut-free provable.

Not surprisingly, in canonical calculi which are not coherent (and hence do not admit cut-elimination by Theorem 3.1.6), expandability is strictly stronger than invertibility. This is demonstrated by the following example.

Example 3.2.21. Consider the following non-coherent calculus G_B:

$$R_1 = \{p_1 \Rightarrow p_2\}/ \Rightarrow p_1 \star p_2 \qquad R_2 = \{p_1 \Rightarrow p_2\}/p_1 \star p_2 \Rightarrow$$

Neither $p_1 \star p_2, p_1 \Rightarrow p_2$ nor $p_1 \Rightarrow p_2, p_1 \star p_2$ have a cut-free derivation in G_B. Indeed, while trying to find a proof bottom-up, the only rules which could be applied are either introduction rules for $\star$ or structural rules but these do not lead to (extended) axioms. Thus the above rules are not expandable. However, $p_1 \Rightarrow p_2$ has a derivation[3] (using cuts) in G_B:

$$\dfrac{\dfrac{\dfrac{p_1 \Rightarrow p_1}{p_1, p_2 \Rightarrow p_1}\,(w)}{p_1 \Rightarrow p_2 \star p_1}\,(R_1) \qquad \dfrac{\dfrac{p_2 \Rightarrow p_2}{p_2 \Rightarrow p_1, p_2}\,(w)}{p_2 \star p_1 \Rightarrow p_2}\,(R_2)}{p_1 \Rightarrow p_2}\,(cut)$$

Thus the rules are invertible, although not expandable.

Proposition 3.2.22. *Let G be a coherent canonical calculus. If G has an invertible rule for $\diamond$, then $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.*

Proof. Assume w.l.o.g. that $R = [\{\Sigma_i \Rightarrow \Pi_i\}_{1 \leq i \leq m}/ \Rightarrow \diamond(p_1, ..., p_n)]$ is invertible in G. Suppose by contradiction that $\tilde{\diamond}_{\mathcal{M}_G}$ is not deterministic. Then there is some $\overline{a} = \langle a_1, ..., a_n\rangle \in \{t, f\}^n$, such that $\tilde{\diamond}(\overline{a}) = \{t, f\}$. Let v be any $\mathcal{M}_G$-legal valuation, such that $v(p_i) = a_i$ and $v(\diamond(p_1, ..., p_n)) = t$ (clearly, such v exists). By Lemma 3.2.8, $\Theta \cup \mathsf{C}_{\overline{a}}$ is inconsistent (since otherwise by the definition of $\mathcal{M}_G$, it would be the case that $\tilde{\diamond}(\overline{a}) = \{t\}$ due to the rule R). Thus $(*)$ there is some $1 \leq j_v \leq m$, for which v does not satisfy the sequent $\Sigma_{j_v} \Rightarrow \Pi_{j_v}$ (otherwise, since v also satisfies $\mathsf{C}_{\langle a_1, ..., a_n\rangle}$ the set of clauses $\Theta \cup \mathsf{C}_{\overline{a}}$ would be consistent). Since R is invertible, by Proposition 3.2.4 it is also canonically

[3]Note that by Theorem 3.1.6, G_B is trivial as it is not coherent. Hence, for any two atoms p, q: $\vdash_{G_B} p \Rightarrow q$.

invertible. Then for every $1 \leq i \leq m$, $\Sigma_i \Rightarrow \Pi_i$ is provable in G from $\Rightarrow \diamond(p_1, ..., p_n)$. Since $\mathcal{M}_G$ is strongly characteristic for G, $\Rightarrow \diamond(p_1, ..., p_n) \vdash_{\mathcal{M}_G} \Sigma_i \Rightarrow \Pi_i$ for every $1 \leq i \leq m$. Since v satisfies $\Rightarrow \diamond(p_1, ..., p_n)$, it should also satisfy $\Sigma_{j_v} \Rightarrow \Pi_{j_v}$, in contradiction to $(*)$. ☐

The following theorem establishes a correspondence between determinism, invertibility and expandability:

Theorem 3.2.23. *Let G be a coherent canonical calculus in normal form with introduction rules for each connective in $\mathcal{L}$. Then the following statements are equivalent:*

1. *G has an invertible rule for $\diamond$.*
2. *G has an expandable rule for $\diamond$.*
3. *$\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.*

Proof. $1 \Rightarrow 3$ follows by Proposition 3.2.22. $1 \Leftrightarrow 2$ follows by Proposition 3.2.20. It remains to show that $3 \Rightarrow 2$. Suppose that $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic. By the definition of $\mathcal{M}_G$, there must be at least one rule for $\diamond$ (otherwise $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \{t, f\}$ for every $\overline{a} \in \{t, f\}^n$). Let R be any such rule. Suppose for contradiction that R is not expandable in G. Then there is some $1 \leq i \leq m$, such that $\diamond(p_1, ..., p_n), \Sigma_i \Rightarrow \Pi_i$ has no cut-free proof in G. Since G is coherent, by Theorem 3.1.6 it admits cut-elimination, and so $\diamond(p_1, ..., p_n), \Sigma_i \Rightarrow \Pi_i$ is not provable in G. Since $\mathcal{M}_G$ is a characteristic Nmatrix for G, $\Sigma_i, \diamond(p_1, ..., p_n) \not\vdash \Pi_i$. Then there is an $\mathcal{M}_G$-legal valuation, such that $v \models_{\mathcal{M}_G} \{\diamond(p_1, ..., p_n)\} \cup \Sigma_i$ and for every $\psi \in \Pi_i$: $v \not\models \psi$. Let $\overline{a} = \langle v(p_1), ..., v(p_n) \rangle$. By Lemma 3.2.8, $(*)$ $\{\Sigma_i \Rightarrow \Pi_i\}_{1 \leq i \leq m} \cup \mathsf{C}_{\overline{a}}$ is inconsistent. Since $\mathcal{M}_G$ is deterministic, $\tilde{\diamond}(\overline{a}) = \{f\}$. (Indeed, it cannot be the case that $\tilde{\diamond}(\overline{a}) = \{t\}$ by definition of $\mathcal{M}_G$ and the fact that R is the only right introduction rule for $\diamond$). Thus $\tilde{\diamond}(\overline{v}) = \{f\}$, in contradiction to our assumption that $v \models_{\mathcal{M}_G} \diamond(p_1, ..., p_n)$. This means that R is expandable in G. ☐

Corollary 3.2.24. *If a canonical coherent calculus G in normal form has a right (left) invertible rule for $\diamond$, then it also has an invertible left (right) rule for $\diamond$.*

Proof. Let G be a canonical coherent calculus G in normal form with an invertible right rule $[\Theta / \Rightarrow \diamond(p_1, \ldots, p_n)]$ for $\diamond$. By Theorem 3.2.23, $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic. Since Θ cannot be a set of extended axioms (recall that G is in normal form), there is some $v \notin mod(\Theta)$. But since $\tilde{\diamond}(v(p_1), \ldots, v(p_n))$ is deterministic, there must be a rule $[\Theta'/C']$, such that $\Theta \cup \mathsf{C}_{\langle v(p_1), \ldots, v(p_n) \rangle}$ is consistent. Since G is in normal form and $\Theta' \neq \Theta$, this cannot be a right introduction rule for $\diamond$, hence C' is $\diamond(p_1, \ldots, p_n) \Rightarrow$. The proof for the case of a left rule is similar. ☐

The next example demonstrates that the correspondence does not hold for calculi which are not in normal form.

Example 3.2.25. Consider the calculus G_X in Example 3.2.5 and its associated (deterministic) Nmatrix $\mathcal{M}_{G_X}$:

X	t	f
t	$\{f\}$	$\{t\}$
f	$\{t\}$	$\{f\}$

It is easy to see that $\Rightarrow p_1 X p_2 \not\vdash_{\mathcal{M}_{G_X}} \Rightarrow p_1$. Hence $\Rightarrow p_1$ is not derivable in G_X from $\Rightarrow p_1 X p_2$ and so the first rule is not canonically invertible. By Proposition 3.2.4 it is not invertible, and by Proposition 3.2.20, it is also not expandable.

3.2.2 Axiom Expansion and Determinism

Axiom expansion ([68]) can be formalized as follows in the context of canonical calculi:

Definition 3.2.26. *An n-ary connective $\diamond$ admits axiom expansion in a calculus G if whenever the sequent $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ is provable in G, it has a cut-free derivation in G from atomic axioms of the form $\{p_i \Rightarrow p_i\}_{1 \leq i \leq n}$.*

Axiom expansion has been studied in the context of various deduction systems. A semantic characterization (i.e., a necessary and sufficient condition) of axiom expansion in single-conclusioned sequent calculi with arbitrary structural rules was provided in [68] in the framework of phase spaces. In the context of labeled sequent calculi (of which canonical calculi are a particular instance, see Section 3.3), [41] shows that the existence of a finite matrix is a necessary condition for axiom expansion. Below we extend these results (in the context of canonical Gentzen-type calculi) by showing that the existence of a finite matrix for a canonical coherent calculus is also a *sufficient* condition for axiom expansion. Furthermore, we establish an exact correspondence between invertibility and axiom expansion for coherent calculi in normal form.

Proposition 3.2.27. *Let G be a canonical calculus. If G has an expandable rule for $\diamond$, then $\diamond$ admits axiom expansion in G.*

Proof. Suppose w.l.o.g. that G has a rule $R = \{\Sigma_i \Rightarrow \Pi_i\}_{1 \leq i \leq m} / \Rightarrow \diamond(p_1, ..., p_n)$, which is expandable in G. Then $(*)$ $\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i$ has a cut-free derivation in G for every $1 \leq i \leq m$. Note that $\Sigma_i, \Pi_i \subseteq \{p_1, ..., p_n\}$ and hence the sequents denoted by $(*)$ are derivable from atomic axioms $\{p_i \Rightarrow p_i\}_{1 \leq i \leq n}$. By applying R with premises $\{\Sigma_i, \diamond(p_1, ..., p_n) \Rightarrow \Pi_i\}_{1 \leq i \leq m}$, we obtain the required cut-free derivation of $\diamond(p_1, ..., p_n) \Rightarrow \diamond(p_1, ..., p_n)$ in G from atomic axioms. Thus $\diamond$ admits axiom expansion in G. □

Theorem 3.2.28. *Let G be a coherent canonical calculus. $\diamond$ admits axiom expansion in G iff $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.*

Proof. We shall first need the following easy lemma (proved by induction on the length of the proof):

Lemma 3.2.29. *Let G be a canonical calculus. If a sequent Ω has a cut-free proof in G from atomic axioms, then Ω also has a cut-free proof in G from atomic (extended) axioms with no application of weakening.*

($\Rightarrow$) If $\diamond$ admits axiom expansion in G then $\diamond(p_1, \dots, p_n) \Rightarrow \diamond(p_1, \dots, p_n)$ is cut-free derivable from atomic axioms. By Lemma 3.2.29 we can assume that the derivation contains only extended atomic axioms and applications of canonical rules. Since there are no cuts, it is easy to see that the applications of canonical rules in this derivation must be identity applications of introduction rules for $\diamond$. Now since $\mathsf{At}(\diamond(p_1, \dots, p_n) \Rightarrow \diamond(p_1, \dots, p_n))$ is the empty sequent, by Corollary 3.2.14 we have that for every valuation v there is some rule $[\Theta/C]$ (where C is either $\Rightarrow \diamond(p_1, \dots, p_n)$ or $\diamond(p_1, \dots, p_n) \Rightarrow$) used in this derivation, such that $v \in mod(\Theta)$. By Lemma 3.2.8, for every $\overline{a} = \langle a_1, \dots, a_n \rangle \in \{t, f\}^n$ there is some canonical rule $[\Theta/C]$ for $\diamond$ in G, such that $\Theta \cup \mathsf{C}_{\overline{a}}$ is consistent. Thus $\tilde{\diamond}_{\mathcal{M}_G}(a_1, \dots, a_n)$ is a singleton, and so $\tilde{\diamond}_{\mathcal{M}_G}$ is deterministic.
($\Leftarrow$) First transform G into a cut-free equivalent calculus G^n in normal form. By Propositions 3.2.11 and 3.2.3, $\mathcal{M}_{G^n}$ is deterministic and G^n is coherent. By Theorem 3.1.6 and Proposition 3.2.27, $\diamond$ admits axiom expansion in G^n and therefore also in G, since G is cut-free equivalent to G^n (see Definition 3.2.2). $\square$

Remark 3.2.30. An alternative proof of ($\Rightarrow$) can be found in [41] for signed canonical calculi.

Corollary 3.2.31. *Let G be a coherent canonical calculus. G has a finite characteristic deterministic matrix iff every connective of $\mathcal{L}$ admits axiom expansion in G.*

Proof. Follows from the theorem above and Proposition 3.2.15. $\square$

Corollary 3.2.32. *If a coherent canonical calculus G has an invertible rule for $\diamond$, then $\diamond$ admits axiom expansion in G.*

Proof. If G has an invertible rule for $\diamond$, then by Proposition 3.2.20 it is also expandable. By Proposition 3.2.27, $\diamond$ admits axiom expansion in G. $\square$

We finish the paper by summarizing the triple correspondence between determinism, invertibility and axiom expansion.

Corollary 3.2.33. *Let G be a coherent canonical calculus in normal form with introduction rules for each connective in $\mathcal{L}$. The following are equivalent: (1) The rules of G are invertible. (2) G has a characteristic deterministic matrix. (3) Every connective of $\mathcal{L}$ admits axiom expansion in G.*

Proof. By Proposition 3.2.15, the existence of a two-valued characteristic deterministic matrix for G is equivalent to $\mathcal{M}_G$ being deterministic. The rest follows by Theorem 3.2.23, Corollary 3.2.24 and Theorem 3.2.28. □

Remark 3.2.34. Note that the above does not hold for calculi which are not in normal form. For instance, the connective X of Example 3.2.5 admits axiom expansion in the calculus G_X (see Example 3.2.25) although its rules are not invertible.

3.3 Canonical Signed Calculi

Signed calculi ([117, 39, 41]) are deduction systems which manipulate sets of *signed formulas*, where the signs can be thought of as syntactic markers which keep track of the formulas in the course of a derivation.
In what follows $\mathcal{V}$ denotes some finite set of signs. $\mathcal{V}$ is also the set of truth-values of all the Nmatrices used in this section.

Definition 3.3.1. A *signed formula* for $(\mathcal{L}, \mathcal{V})$ is an expression of the form $s : \psi$, where $s \in \mathcal{V}$ and $\psi \in Frm_{\mathcal{L}}$. A signed formula $s : \psi$ is atomic if ψ is an atomic formula. A (signed) *sequent* for $(\mathcal{L}, \mathcal{V})$ is a finite set of signed formulas for $(\mathcal{L}, \mathcal{V})$. A (signed) *clause* is a sequent consisting of atomic signed formulas.

Notation 3.3.2. *Formulas will be denoted by φ, ψ, signed formulas - by $\alpha, \beta, \gamma, \delta$, sets of signed formulas - by Υ, Λ, sequents - by Ω, Σ, Π, sets of sets of signed formulas - by Φ, Ψ and sets of sequents - by Θ, Ξ. We write $s : \Delta$ instead of $\{s : \psi \mid \psi \in \Delta\}$, $S : \psi$ instead of $\{s : \psi \mid s \in S\}$, and $S : \Delta$ instead of $\{s : \psi \mid s \in S, \psi \in \Delta\}$.*

Remark 3.3.3. The usual (two-sided) sequent notation $\Gamma \Rightarrow \Delta$ can be interpreted as $\{f : \Gamma\} \cup \{t : \Delta\}$, i.e. a sequent in the sense of Definition 3.3.1 over the two signs $\{t, f\}$.

Definition 3.3.4. For any function v from the set of formulas of $\mathcal{L}$ to $\mathcal{V}$, v satisfies a signed formula $\gamma = (l : \psi)$, denoted by $v \models (l : \psi)$, if $v(\psi) = l$. v satisfies a set of signed formulas Υ, denoted by $v \models \Upsilon$, if there is some $\gamma \in \Upsilon$, such that $v \models \gamma$.
An *atomic valuation* is a function from atomic formulas of $\mathcal{L}$ to $\mathcal{V}$. Satisfiability of clauses and of sets of clauses by an atomic valuation is defined similarly.

If $\Theta \cup \{\Omega\}$ is a set of clauses, we say that Θ (atomically) follows[4] from Θ, denoted by $\Theta \vdash_a \Omega$, if every atomic valuation which satisfies Θ also satisfies Ω.

Thus sequents are interpreted as a *disjunction* of statements, saying that a particular formula takes a particular truth-value (interpreting sequents in a dual way corresponds to the method of analytic tableaux, see e.g. [38, 87]).

Now we extend the notion of "canonical signed rules and calculi" from Definition 3.1.1 to signed calculi:

Notation 3.3.5. We say that a clause (set of clauses) is *n-canonical* if the only atomic formulas occurring in it are of the form $a : p_i$, where $a \in \mathcal{V}$ and $1 \leq i \leq n$.

Definition 3.3.6. A *signed canonical (propositional) rule of arity n* for $(\mathcal{L}, \mathcal{V})$ is an expression of the form $[\Theta/S : \diamond(p_1, \ldots, p_n)]$, where S is a non-empty subset of $\mathcal{V}$, $\diamond$ is an n-ary connective of $\mathcal{L}$ and $\Theta = \{\Sigma_1, ..., \Sigma_m\}$, where $m \geq 0$ and for every $1 \leq j \leq m$, Σ_j is an n-canonical clause.

An *application* of a rule $[\{\Sigma_1, ..., \Sigma_m\}/S : \diamond(p_1, \ldots, p_n)]$ is any inference of the form:

$$\frac{\Omega \cup \Sigma_1^* \quad \ldots \quad \Omega \cup \Sigma_m^*}{\Omega \cup S : \diamond(\psi_1, ..., \psi_n)}$$

where $\psi_1, ..., \psi_n$ are $\mathcal{L}$-formulas, Ω is a sequent, and for all $1 \leq i \leq m$: Σ_i^* is obtained from Σ_i by replacing p_j by ψ_j for every $1 \leq j \leq n$.

Remark 3.3.7. It is easy to see that the canonical Gentzen-type systems from Definition 3.1.1 are a special case of canonical signed calculi for $\mathcal{V} = \{t, f\}$ (taking $\Gamma \Rightarrow \Delta$ as an abbreviation of the signed set $\{f : \psi \mid \psi \in \Gamma\} \cup \{t : \psi \mid \psi \in \Delta\}$).

Example 3.3.8. 1. Using the notation in Remark 3.3.3, we can write the rules for conjunction from Example 3.1.2 as follows:

$$[\{\{f : p_1, f : p_2\}\}/\{f\} : p_1 \wedge p_2] \qquad [\{\{t : p_1\}, \{t : p_2\}\}/\{t\} : p_1 \wedge p_2]$$

Applications of these rules have the forms:

$$\frac{\Omega \cup \{f : \psi_1, f : \psi_2\}}{\Omega \cup \{f : \psi_1 \wedge \psi_2\}} \qquad \frac{\Omega \cup \{t : \psi_1\} \quad \Omega \cup \{t : \psi_2\}}{\Omega \cup \{t : \psi_1 \wedge \psi_2\}}$$

[4]Note that for any set of clauses Θ and a clause Ω, if Ω atomically follows from Θ for $\mathcal{V}$, then $\Theta \vdash_{\mathcal{M}} \Omega$ in any Nmatrix $\mathcal{M}$.

2. Consider a calculus over $\mathcal{V} = \{a, b, c\}$ with the following rules for a ternary connective $\circ$:

$$[\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\}/\{a, c\} : \circ(p_1, p_2, p_3)]$$

$$[\{\{c : p_2\}, \{a : p_3, b : p_3\}, \{c : p_1\}\}/\{b, c\} : \circ(p_1, p_2, p_3)]$$

Their applications are of the forms:

$$\frac{\Omega \cup \{a : \psi_1, c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_2\}}{\Omega \cup \{a : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

$$\frac{\Omega \cup \{c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_3\} \quad \Omega \cup \{c : \psi_1\}}{\Omega \cup \{b : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}}$$

Definition 3.3.9. Let $\mathcal{V}$ be a finite set of signs.

1. A logical axiom for $\mathcal{V}$ is a sequent of the form: $\{l : \psi \mid l \in \mathcal{V}\}$.

2. The cut and weakening rules for $\mathcal{V}$ are defined as follows:

$$\frac{\Omega \cup \{l : \psi \mid l \in L_1\} \quad \Omega \cup \{l : \psi \mid l \in L_2\}}{\Omega \cup \{l : \psi \mid l \in L_1 \cap L_2\}} \ \mathsf{cut}$$

$$\frac{\Omega}{\Omega, l : \psi} \ \mathsf{weak}$$

where $L_1, L_2 \subseteq \mathcal{V}$ and $l \in \mathcal{V}$.

It is easy to verify the soundness of cut and weakening (in every Nmatrix).

Proposition 3.3.10. *Let Θ be a set of clauses and Ω - a clause. Then $\Theta \vdash_a \Omega$ (see Definition 3.3.4) iff there is some $\Omega' \subseteq \Omega$, such that Ω' is derivable from Θ by cuts*[5].

Proof. For the first direction, assume that there is no $\Omega' \subseteq \Omega$, which is derivable from Θ using cuts. It is a standard matter to show that Ω can be extended to a maximal set Ω^* of atomic formulas, such that for any $\Omega' \subseteq \Omega^*$: Ω' is not derivable from Θ using cuts. Then for every atom p there is some $l \in \mathcal{V}$, such that $l : p \notin \Omega^*$ (otherwise Ω^* would contain a logical axiom). Suppose by contradiction that there is some atom p, such that $l_1 : p \notin \Omega^*$ and $l_2 : p \notin \Omega^*$ for some $l_1, l_2 \in \mathcal{V}$, such that $l_1 \neq l_2$. Then, by the maximality of Ω^*, there is some $\Omega_1 \subseteq \Omega^*$, such that $\Omega_1 \cup \{l_1 : p\}$ is derivable from Θ using cuts. Similarly, there is some $\Omega_2 \subseteq \Omega^*$, such that $\Omega_2 \cup \{l_2 : p\}$ is derivable from Θ using cuts. Then $\Omega_1 \cup \Omega_2 \subseteq \Omega^*$ is derivable using cuts from Θ, in contradiction to our assumption. Thus for every atom p there is exactly one $l_p \in \mathcal{V}$, such that $l_p : p \notin \Omega^*$. Let v be the

[5] This proposition also follows from the completeness of many-valued resolution from [40]. We provide here a different proof.

atomic valuation which satisfies $v(p) = l_p$ for every atom p. Clearly, v does not satisfy Ω^*. Now let $\Sigma \in \Theta$. Then there is some $a : p \in \Sigma$, such that $a : p \notin \Omega^*$ (otherwise $\Sigma \subseteq \Omega^*$ and is derivable from Θ, in contradiction to our assumption). Then $a = l_p = v(p)$, and so v satisfies Σ. We have shown that v satisfies all the clauses in Θ, but does not satisfy Ω. Hence, Ω does not follow from Θ.
The converse direction follows from the soundness of cut.

□

Corollary 3.3.11. *Let Θ be a set of clauses. The empty sequent is derivable from Θ by cuts iff Θ is not satisfiable.*

Now we are ready to define "canonical signed calculi" in precise terms:

Definition 3.3.12. A signed calculus over a language $\mathcal{L}$ and a finite set of signs $\mathcal{V}$ is *canonical* if it consists of:

1. All logical axioms for $\mathcal{V}$.

2. The rules of cut and weakening from Definition 3.3.9.

3. Any number of signed canonical inference rules.

Of course, not all canonical calculi are useful. In fact, our quest is for calculi which "define" the semantic meaning of the logical connectives they introduce. Below we extend the notion of coherence to signed calculi.

Definition 3.3.13. A canonical calculus G is *coherent* if $\Theta_1 \cup ... \cup \Theta_m$ is unsatisfiable whenever $\{[\Theta_1/S_1 : \psi], ..., [\Theta_m/S_m : \psi]\}$ is a set of rules of G such that $S_1 \cap ... \cap S_m = \emptyset$ (here $\psi = \diamond(p_1, \dots, p_n)$ for some n-ary connective $\diamond$ of $\mathcal{L}$).

Note that that it is not sufficient to check only pairs of rules like in the definition for the two-signed case (Definition 3.1.4), as it can be the case that $S_1 \cap S_2 \neq \emptyset$ and $S_2 \cap S_3 \neq \emptyset$, but $S_1 \cap S_2 \cap S_3 = \emptyset$.
Obviously, coherence is a decidable property of canonical calculi. Note that by Corollary 3.3.11, a canonical calculus G is coherent iff whenever $\{[\Theta_1/S_1 : \psi], ..., [\Theta_m/S_m : \psi]\}$ is a set of rules of G, and $S_1 \cap ... \cap S_m = \emptyset$, we have that $\Theta_1 \cup ... \cup \Theta_m$ is inconsistent (i.e. the empty sequent can be derived from it using cuts).

Example 3.3.14. 1. Consider the canonical calculus G_1 over $\mathcal{L} = \{\wedge\}$ and $\mathcal{V} = \{t, f\}$, the canonical rules of which are the two rules for $\wedge$ from Example 3.3.8. We can derive the empty sequent from $\{\{t : p_1\}, \{t : p_2\}, \{f : p_1, f : p_2\}\}$ as follows:

$$\dfrac{\dfrac{\{t : p_1\} \quad \{f : p_1, f : p_2\}}{\{f : p_2\}}\,\mathsf{cut} \qquad \{t : p_2\}}{\emptyset}\,\mathsf{cut}$$

Thus G_1 is coherent.

2. Consider the canonical calculus G_2 over $\mathcal{V} = \{a, b, c\}$ with the following introduction rules for the ternary connective $\circ$:

$$[\{\{a : p_1\}, \{b : p_2\}\}/\{a, b\} : \circ(p_1, p_2, p_3)]$$

$$[\{\{a : p_2, c : p_3\}\}/\{c\} : \circ(p_1, p_2, p_3)]$$

Clearly, the set $\{\{a : p_1\}, \{b : p_2\}, \{a : p_2, c : p_3\}\}$ is satisfiable, thus G_2 is not coherent.

Next we define some notions of cut-elimination in signed calculi:

Definition 3.3.15. Let G be a canonical signed calculus and let Θ be some set of sequents.

1. A cut is called a Θ-*cut* if the cut formula occurs in Θ. We say that a proof is Θ-cut-free if the only cuts in it are Θ-cuts.

2. A cut is called Θ-*analytic* if the cut formula is a subformula of some formula occurring in Θ. A proof is called Θ-analytic[6] if all cuts in it are Θ-analytic. We say that a sequent Ω has a *proper proof from* Θ *in* G whenever Ω has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G.

3. We say that a canonical calculus G admits *(standard) cut-elimination* if whenever $\vdash_G \Omega$, Ω has a cut-free proof in G. G admits *strong cut-elimination* if whenever $\Theta \vdash_G \Omega$, Ω has in G a Θ-cut-free proof from Θ.

4. G admits *strong analytic cut-elimination* if whenever $\Theta \vdash_G \Omega$, Ω has in G a $\Theta \cup \{\Omega\}$-analytic proof from Θ. G admits *analytic cut-elimination* if whenever $\vdash_G \Omega$, Ω has in G a $\{\Omega\}$-analytic proof.

Remark 3.3.16. Note that in a calculus G which allows strong analytic cut-elimination, whenever Ω is derivable from Θ in G, it also has a proper proof from Θ in G.

Example 3.3.17. Consider the following calculus G' for a language with a binary connective $\circ$ and $\mathcal{V} = \{a, b, c\}$. The rules of G' are as follows:

$$R_1 = [\{\{a : p_1\}\}/\{a, b\} : p_1 \circ p_2]\} \qquad R_2 = [\{\{a : p_1\}\}/\{b, c\} : p_1 \circ p_2]\}$$

In the following proof in G', the cut in the final step is analytic:

$$\dfrac{\dfrac{\{a : p_1, b : p_1, c : p_1\}}{\{b : p_1, c : p_1, b : (p_1 \circ p_2), c : (p_1 \circ p_2)\}} \qquad \dfrac{\{a : p_1, b : p_1, c : p_1\}}{\{b : p_1, c : p_1, a : (p_1 \circ p_2), b : (p_1 \circ p_2)\}}}{\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}}$$

[6] This is a generalization of the notion of analytic cut (see e.g. [43]).

3.3.1 Modular Semantics for Canonical Signed Calculi

Below we present a general method for providing finite non-deterministic semantics for canonical signed calculi in a modular way. We start by defining semantics for the simplest canonical calculus: the one without any canonical rules. Later we will see that the semantic effect of adding an arbitrary canonical rule corresponds to a certain simple refinement of the basic Nmatrix (i.e, leads to a reduction of the level of non-determinism in the basic Nmatrix).

Definition 3.3.18. $G_0^{(\mathcal{L},\mathcal{V})}$ *is the canonical calculus over a language* $\mathcal{L}$ *and a set of signs* $\mathcal{V}$*, whose set of canonical rules is empty.*

In the rest of this section we assume that our language $\mathcal{L}$, the set of signs $\mathcal{V}$ and the set of designated signs $\mathcal{D}$ are fixed. Accordingly, we shall write G_0 instead of $G_0^{(\mathcal{L},\mathcal{V})}$. It is obvious that G_0 is (trivially) coherent. We now define a strongly characteristic Nmatrix for G_0. Note that it has the maximal degree of non-determinism in interpreting the connectives of $\mathcal{L}$.

Definition 3.3.19. $\mathcal{M}_0 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is the Nmatrix in which* $\tilde{\diamond}(a_1, ..., a_n) = \mathcal{V}$ *for every* n*-ary connective* $\diamond$ *of* $\mathcal{L}$ *and* $a_1, ..., a_n \in \mathcal{V}$*.*

Theorem 3.3.20. $\mathcal{M}_0$ *is strongly characteristic for* G_0*.*

The proof is a simplified version of the proof of Theorem 3.3.30 in the sequel.

Next we handle the modular effect of a given canonical rule. The idea is that each rule which is added to G_0 imposes a certain semantic condition leading to some refinement of $\mathcal{M}_0$, while coherence guarantees that these semantic conditions are not contradictory.

The following notion extends Notation 3.2.7 for the two-signed case:

Definition 3.3.21. For $\langle a_1, ..., a_n \rangle \in \mathcal{V}^n$, the n-canonical set of clauses $\mathsf{C}_{\langle a_1,...,a_n \rangle}$ is defined as follows:

$$\mathsf{C}_{\langle a_1,...,a_n \rangle} = \{\{a_1 : p_1\}, \{a_2 : p_2\}, ..., \{a_n : p_n\}\}$$

The following lemmas are immediate by the definition of $\mathsf{C}_{\langle a_1,...,a_n \rangle}$:

Lemma 3.3.22. *Let* $\Theta_1, \Theta_2, \ldots, \Theta_m$ *be some* n*-canonical clauses. If the sets of clauses* $\mathsf{C}_{\langle a_1,...,a_n \rangle} \cup \Theta_1, ..., \mathsf{C}_{\langle a_1,...,a_n \rangle} \cup \Theta_m$ *are satisfiable, then so is the set* $\Theta_1 \cup \Theta_2 \ldots \cup \Theta_m \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$*.*

Lemma 3.3.23. *Let Θ be an n-canonical clause. $\Theta \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent iff for every $\Omega \in \Theta$ there is some $1 \leq i \leq n$, such that $a_i : p_i \in \Omega$.*

We are now ready to define the semantic condition that a canonical rule imposes on $\mathcal{M}_0$.

Definition 3.3.24. Let R be a canonical rule of the form $[\Theta/S : \diamond(p_1, ..., p_n)]$. $\mathsf{C}(\mathsf{R})$, the refining condition induced by R, is defined as follows:

$\mathsf{C}(\mathsf{R})$: For $a_1, ..., a_n \in \mathcal{V}$, if $\mathsf{C}_{\langle a_1,\ldots,a_n\rangle} \cup \Theta$ is consistent, then $\tilde{\diamond}(a_1, ..., a_n) \subseteq S$.

Intuitively, if $\Theta \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent, then a rule $[\Theta/S : \diamond(p_1, \ldots, p_n)]$ leads to the deletion from $\tilde{\diamond}(a_1, ..., a_n)$ of all the truth-values which are not in S. If some rules $[\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)], ..., [\Theta_m/S_2 : \diamond(p_1, \ldots, p_n)]$ "overlap" on the same $\langle a_1, ..., a_n\rangle$, their overall effect leads to $\tilde{\diamond}(a_1, ..., a_n) = S_1 \cap ... \cap S_m$ (the coherence of a calculus guarantees that $S_1 \cap ... \cap S_m$ is not empty in such a case).

Definition 3.3.25. Let G be a canonical calculus for $(\mathcal{L}, \mathcal{V})$.

1. Define an application of a rule $[\Theta/S : \diamond(p_1, ..., p_n)]$ of G on $a_1, ..., a_n \in \mathcal{V}$ as follows:

$$[\Theta/S : \diamond(p_1, ..., p_n)](a_1, ..., a_n) = \begin{cases} S & \text{if } \Theta \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle} \text{ is consistent} \\ \mathcal{V} & \textit{otherwise} \end{cases}$$

2. $\mathcal{M}_G = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is any Nmatrix, such that for every n-ary connective $\diamond$ and every $a_1, ..., a_n \in \mathcal{V}$:

$$\tilde{\diamond}_{\mathcal{M}_G}(a_1, ..., a_n) = \bigcap\{[\Theta/S : \diamond(p_1, ..., p_n)](a_1, ..., a_n) \mid [\Theta/S : \diamond(p_1, ..., p_n)] \in G\}$$

Proposition 3.3.26. *If G is coherent, then $\mathcal{M}_G$ is well-defined.*

Proof. It suffices to check that for every n-ary connective $\diamond$ and every $a_1, ..., a_n \in \mathcal{V}$, $\tilde{\diamond}_{\mathcal{M}_G}(a_1, ..., a_n)$ is not empty. Suppose by contradiction that for some n-ary connective $\diamond$ and some $a_1, ..., a_n \in \mathcal{V}$, $\tilde{\diamond}(a_1, ..., a_n) = \emptyset$. But then there are some rules of the forms $[\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)], ..., [\Theta_m/S_m : \diamond(p_1, \ldots, p_n)]$, for which it holds that $S_1 \cap ... \cap S_m = \emptyset$ and $\Theta_1 \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}, ..., \Theta_m \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ are consistent. By Lemma 3.3.22, $\Theta_1 \cup ...\Theta_m \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent, and so is $\Theta_1 \cup ... \cup \Theta_m$, in contradiction to our assumption about the coherence of G. □

Lemma 3.3.27. *Let G be a coherent calculus with a rule $R = [\Theta/S_r : \diamond(p_1, \ldots, p_n)]$. If $\Theta \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent, then $\tilde{\diamond}(a_1, \ldots, a_n) \subseteq S_r$.*

Proof. For $\overline{a} \in \mathcal{V}^n$, $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) = \bigcap\{[\Theta/S : \diamond(p_1, ..., p_n)](\overline{a}) \mid [\Theta/S : \diamond(p_1, ..., p_n)] \in G\}$. Thus $\tilde{\diamond}_{\mathcal{M}_G}(\overline{a}) \subseteq R(\overline{a}) = S_r$.

□

Example 3.3.28. Consider a calculus G with the following canonical rules for a unary connective $\bullet$ for $\mathcal{V} = \{t, f, \top, \bot\}$:

$$[\{t : p_1\}/\{t\} : \bullet p_1] \quad [\{f : p_1\}/\{f, \bot\} : \bullet p_1]$$

$$[\{f : p_1, \bot : p_1\}/\{t, \bot\} : \bullet]$$

and the following rule for conjunction:

$$[\{\{f : p_1, f : p_2\}\}/\{f\} : p_1 \wedge p_2]$$

Then the interpretations of $\wedge$ and $\bullet$ in $\mathcal{M}_G$ are as follows:

$\wedge$	t	f	$\top$	$\bot$
t	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$
f	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$
$\bot$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$

$\bullet$	
t	$\{t\}$
f	$\{\bot\}$
$\top$	$\mathcal{V}$
$\bot$	$\{t, \bot\}$

Let us explain how these truth-tables are obtained. We start with the basic Nmatrix $\mathcal{M}_0$, for which $\tilde{\bullet}_{\mathcal{M}_0}(x) = \mathcal{V}$ and $\tilde{\wedge}_{\mathcal{M}_0}(x, y) = \mathcal{V}$ for every $x, y \in \mathcal{V}$. Consider the first rule for $\bullet$. Since $\{\{t : p_1\}\}$ is only consistent with $\mathsf{C}_{\langle t \rangle}$, this rule affects $\tilde{\bullet}_{\mathcal{M}_G}(t)$ by deleting the truth-values $f, \top, \bot$ from $\tilde{\bullet}_{\mathcal{M}_0}(t)$, and so $\tilde{\bullet}_{\mathcal{M}_G}(t) = \{t\}$. The second and the third rules both affect the set $\tilde{\bullet}_{\mathcal{M}_G}(f)$ (since the sets $\{\{f : p_1\}\}$ and $\{\{f : p_1, \bot : p_1\}\}$ are both consistent with $\mathsf{C}_{\langle f \rangle}$): the second rule deletes the truth-values $t, \top$, while the third deletes $\top, f$ from $\tilde{\bullet}_{\mathcal{M}_0}$. Thus we are left with $\tilde{\bullet}_{\mathcal{M}_G}(f) = \{\bot\}$. The third rule also dictates $\tilde{\bullet}_{\mathcal{M}_G}(\bot) = \{t, \bot\}$. Finally, as we have underspecification concerning $\tilde{\bullet}_{\mathcal{M}_G}(\top)$, in this case $\tilde{\bullet}_{\mathcal{M}_G}(\top) = \{t, f, \top, \bot\}$. As for the the rule for $\wedge$, the set $\{\{f : p_1, f : p_2\}\}$ is consistent with $\mathsf{C}_{\langle x,y \rangle}$ whenever at least one of $x, y \in \mathcal{V}$ is 'f', and so the rule deletes $t, \top, \bot$ from $\tilde{\wedge}_{\mathcal{M}_0}(x, y)$ for every such x, y.

Suppose we now obtain G' by adding the following rule for $\wedge$ to G (clearly, the new calculus G' is still coherent):

$$[\{\{t : p_1, \top : p_1\}, \{\bot : p_2, f : p_2\}\}/\{f, \bot\} : p_1 \wedge p_2]$$

This rule deletes the truth-values $t, \top$ from $\tilde{\wedge}_{\mathcal{M}_G}(x, y)$ for every $x \in \{t, \top\}$ and $y \in \{f, \perp\}$. Thus the truth-table for $\wedge$ in $\mathcal{M}_{G'}$ is now modified as follows:

$\wedge$	t	f	$\top$	$\perp$
t	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\{f, \perp\}$
f	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\top$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\{f, \perp\}$
$\perp$	$\mathcal{V}$	$\{f\}$	$\mathcal{V}$	$\mathcal{V}$

Remark 3.3.29. It is easy to see that for a coherent calculus G, $\mathcal{M}_G$ is the weakest simple refinement of $\mathcal{M}_0$, in which all the conditions induced by the rules of G are satisfied. Thus if G' is a coherent calculus obtained from G by adding a new canonical rule, $\mathcal{M}'_G$ can be straightforwardly obtained from $\mathcal{M}_G$ by some deletions of options as dictated by the condition which corresponds to the new rule.

Theorem 3.3.30. *For every coherent canonical calculus G, $\mathcal{M}_G$ is a strongly characteristic Nmatrix for G.*

Proof. Strong soundness: Suppose that $\Theta \vdash_G \Omega$. We prove that $\Theta \vdash_{\mathcal{M}_G} \Omega$. The axioms and the structural rules are clearly sound. It remains to show the soundness of the canonical rules. Let $R = [\{\Sigma_1, \dots, \Sigma_m\}/S : \diamond(p_1, \dots, p_n)]$ and consider an application of the form:

$$\frac{\Omega \cup \Sigma_1^* \quad \dots \quad \Omega \cup \Sigma_m^*}{\Omega \cup S : \diamond(\psi_1, \dots, \psi_n)}$$

where for all $1 \leq j \leq m$, Σ_j^* is obtained from Σ_j by substituting ψ_i for p_i for all $1 \leq i \leq n$. Let v be some $\mathcal{M}_G$-legal valuation which satisfies the premises of the above application. We show that v also satisfies the conclusion. If v satisfies Ω, we are done. Otherwise, since v satisfies every premise of the application above, v satisfies Σ_j^* for all $1 \leq i \leq j$. Thus for every such j there is some $1 \leq i_j \leq n$, such that $v(\psi_{i_j}) : \psi_{i_j} \in \Sigma_j^*$, and so $v(\psi_{i_j}) : p_{i_j} \in \Sigma_j$. By Lemma 3.3.23, $\Sigma_j \cup \mathsf{C}_{\langle v(\psi_1),\dots,v(\psi_n)\rangle}$ is consistent for every $1 \leq j \leq n$, and so $\Sigma_1^* \cup \dots \Sigma_m^* \cup \mathsf{C}_{\langle v(\psi_1),\dots,v(\psi_n)\rangle}$ is consistent. By Lemma 3.3.27, $\tilde{\diamond}_{\mathcal{M}_G}(v(\psi_1), \dots, v(\psi_n)) \subseteq S$. Since v is $\mathcal{M}_G$-legal, $v(\diamond(\psi_1, ..., \psi_n)) \in S$ and so v satisfies the conclusion.

Strong completeness: Suppose that Ω has no proper proof from Θ in G (recall Definition 3.3.15). We will show that this implies $\Theta \not\vdash_{\mathcal{M}_G} \Omega$. It is a standard matter to show that Ω can be extended to a maximal set Ω^*, such that (i) no $\Omega' \subseteq \Omega^*$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G, and (ii) all formulas occurring in Ω^* are subformulas of formulas from $\Theta \cup \{\Omega\}$. We now show that Ω^* has the following properties:

1. If $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$ and $b_1 : \diamond(\psi_1, ..., \psi_n), ..., b_k : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$, then $a_i : \psi_i \in \Omega^*$ for some $1 \leq i \leq n$.

2. For every formula ψ which is a subformula of some formula from Θ, there is exactly one $l \in \mathcal{V}$, such that $l : \psi \notin \Omega^*$.

Let us prove the first property. Suppose by contradiction that for some $a_1, ..., a_n \in \mathcal{V}$, $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$ and $b_1 : \diamond(\psi_1, ..., \psi_n), ..., b_k : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$, but for every $1 \leq i \leq n$, $a_i : \psi_i \notin \Omega^*$. By the maximality of Ω^*, for every $1 \leq i \leq n$ there is some $\Omega'_i \subseteq \Omega^*$, such that $\Omega'_i \cup \{a_i : \psi_i\}$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G. First observe that $\{b_1, ..., b_k\} \neq \mathcal{V}$ (otherwise Ω^* would contain a logical axiom, in contradiction to property (i) of Ω^*). Then by definition of $\mathcal{M}_G$ there are some rules in G of the form $R_1 = [\Xi_1/S_1 : \diamond(p_1, ..., p_n)], ..., R_m = [\Xi_m/S_m : \diamond(p_1, ..., p_n)]$, such that $\Xi_1 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}, ..., \Xi_m \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ are consistent and $S_1 \cap ... \cap S_m = \{b_1, ..., b_k\}$. Now let $1 \leq j \leq m$ and $\Sigma \in \Xi_j$. By Lemma 3.3.23, there is some $1 \leq k_\Sigma \leq n$, such that $(a_{k_\Sigma} : p_{k_\Sigma}) \in \Sigma$ (since $\Xi_j \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent). Now by our assumption, $\Omega'_{k_\Sigma} \cup \{a_{k_\Sigma} : \psi_{k_\Sigma}\}$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G. By applying weakening we get a $\Theta \cup \{\Omega\}$-analytic proof of $\Omega'_{k_\Sigma} \cup \Sigma^*$ from Θ in G for every $\Sigma \in \Xi_j$, where Σ^* is obtained from Σ by replacing p_r by ψ_r for all $1 \leq r \leq n$. By applying weakening and the canonical rule R_j, we get a $\Theta \cup \{\Omega\}$-analytic proof of $\bigcup_{\Sigma \in \Xi_j} \Omega'_{k_\Sigma} \cup S_j : \diamond(\psi_1, ..., \psi_n)$ from Θ in G. Thus for all $1 \leq j \leq n$, there is some $\Omega_j \subseteq \Omega$, such that $\Omega_j \cup S_j : \diamond(\psi_1, ..., \psi_n)$ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G. Now by applying $\Theta \cup \{\Omega\}$-analytic cuts (recall that we assumed that $b_1 : \diamond(\psi_1, ..., \psi_n), \ldots, b_k : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$ and so $\diamond(\psi_1, ..., \psi_n)$ is a subformula of some formula from $\Theta \cup \{\Omega\}$), we get a $\Theta \cup \{\Omega\}$-analytic proof of $\Omega_1 \cup ... \cup \Omega_m \cup (S_1 \cap ... \cap S_m) : \diamond(\psi_1, ..., \psi_n) = \Omega_1 \cup ... \cup \Omega_m$ from Θ in G , in contradiction to property (i) of Ω^*.
Now we prove the second property. Let ψ be a subformula of some formula from Θ. Then there must be some $l \in \mathcal{V}$, such that $l : \psi \notin \Omega^*$ (otherwise Ω^* contains a logical axiom). Suppose by contradiction that there are some $l_1 \neq l_2$, such that both $l_1 : \psi$ and $l_2 : \psi$ are not in Ω^*. By the maximality of Ω^*, there are some $\Omega'_1, \Omega'_2 \subseteq \Omega^*$, such that $\Omega'_1 \cup \{l_1 : \psi\}$ and $\Omega'_2 \cup \{l_2 : \psi\}$ have $\Theta \cup \{\Omega\}$-analytic proofs from Θ in G. By applying cuts , we get a $\Theta \cup \{\Omega\}$-analytic proof of $\Omega'_1 \cup \Omega'_2 \subseteq \Omega^*$ from Θ in G, in contradiction to property (i) of Ω^*.

Next we define a partial valuation v on the subformulas of $\Theta \cup \{\Omega\}$ by induction on complexity of formulas. According to our goal, v is defined so that $v(\psi) \neq s$ for every $(s : \psi) \in \Omega^*$. First, let p be an atomic formula. As Ω^* cannot contain a logical axiom, there must be some $s_0 \in \mathcal{V}$, such that $(s_0 : p) \notin \Omega^*$. Define $v(p) = s_0$. Suppose we have defined v for formulas with complexity up to l, and let $\psi = \diamond(\psi_1, ..., \psi_n)$, where each ψ_i

is of complexity at most l. Hence $v(\psi_i)$ is already defined for each i. Now suppose that for every $1 \leq i \leq n$: $v(\psi_i) = a_i$ and $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$. Then there must be some $b \in \{b_1, ..., b_k\}$, such that $(b : \psi) \notin \Omega^*$ (otherwise by property 1 there would be some j, such that $(a_j : \psi_j) \in \Omega^*$, contradicting the induction hypothesis). Pick one such b and define $v(\psi) = b$. By the above construction, v is $\mathcal{M}_G$-legal and $v \not\models_{\mathcal{M}_G} \Omega^*$. Now let $\Sigma \in \Theta$. Then there must be some $a : \psi \in \Sigma$, such that $a : \psi \notin \Omega^*$ (otherwise $\Sigma \subseteq \Omega^*$, while Σ has a $\Theta \cup \{\Omega\}$-analytic proof from Θ in G, which is a contradiction to property (i) of Ω^*). By property 2, for every $l \in \mathcal{V} \setminus \{a\}$, $(l : \psi) \in \Omega^*$. By the property of v proven above, $v(\psi) \neq l$ for every $l \in \mathcal{V} \setminus \{a\}$. Thus $v(\psi) = a$, and so $v \models_{\mathcal{M}_G} \Sigma$. By Proposition 2.2.15, the partial valuation v can be extended to a full $\mathcal{M}_G$-legal valuation v_f. Thus we have constructed an $\mathcal{M}_G$-legal valuation v_f, such that $v_f \models_{\mathcal{M}_G} \Theta$, but $v_f \not\models_{\mathcal{M}_G} \Omega$. Hence, $\Theta \not\vdash_{\mathcal{M}_G} \Omega$.

□

From the proof of Theorem 3.3.30 we also have the following corollary:

Corollary 3.3.31. (Analytic cut-elimination) *Any coherent canonical calculus admits strong analytic cut-elimination.*

Remark 3.3.32. [27] provides a full axiomatization of finite Nmatrices: a canonical coherent signed calculus is constructed there for every finite Nmatrix. Theorem 3.3.30 provides the complementary link between canonical calculi and Nmatrices: every canonical coherent signed calculus has a corresponding characteristic finite Nmatrix.

3.3.2 Cut-elimination in Canonical Signed Calculi

In this section we provide a characterization of the notions of cut-elimination from Definition 3.3.15. We start with the following theorem, which establishes an *exact correspondence* between coherence of canonical calculi, non-deterministic matrices and analytic cut-elimination:

Theorem 3.3.33. *Let G be a canonical calculus. The following statements concerning G are equivalent.*

1. *G is coherent.*

2. *G has a strongly characteristic Nmatrix.*

3. *G admits strong analytic cut-elimination.*

4. *G admits analytic cut-elimination.*

Proof. (1) $\Rightarrow$ (2) follows by Theorem 3.3.30.
(1) $\Rightarrow$ (3) follows by Corollary 3.3.31.
(3) $\Rightarrow$ (4) follows by definition of strong analytic cut-elimination (Defn. 3.3.15).
Next we prove (2) $\Rightarrow$ (1). Suppose that G has a strongly characteristic Nmatrix $\mathcal{M}$ and suppose for contradiction that G is not coherent. Then there are some rules of the forms $R_1 = [\Theta_1 : /S_1 : \diamond(p_1, ..., p_n)], ..., R_m = [\Theta_m : /S_m : \diamond(p_1, ..., p_n)]$ in G, such that $\Theta = \Theta_1 \cup ... \cup \Theta_m$ is consistent and $S_1 \cap ... \cap S_m = \emptyset$. By applying the rule R_j on Θ_j for all $1 \leq j \leq m$, we get a proof of $S_j : \diamond(p_1, ..., p_n)$. Then by applying cuts we derive the empty sequent from $\Theta_1 \cup ... \cup \Theta_m$, in contradiction to the consistency of Θ (recall Corollary 3.3.11).
Finally, we prove (4) $\Rightarrow$ (1). Suppose that G admits analytic cut-elimination but is not coherent. Then there are rules $[\Theta_1 : /S_1 : \diamond(p_1, ..., p_n)], ..., [\Theta_m : /S_m : \diamond(p_1, ..., p_n)]$ in G, such that $\Theta = \Theta_1 \cup ... \cup \Theta_m$ is consistent and $S_1 \cap \ldots \cap S_m = \emptyset$. Let v be some atomic valuation which satisfies Θ (such valuation exists by Corollary 3.3.11). Let Π be the set of all signed formulas $a : p_i$ (for $1 \leq i \leq n$), such that $v(p_i) \neq a$. Then for every $\Omega \in \Theta$: $\Pi \cup \Omega$ is a logical axiom (indeed, since v satisfies Ω there is some $1 \leq j \leq n$, such that $v(p_j) : p_j \in \Omega$). Thus $a : p_j \in \Pi$ for every $a \in \mathcal{V} \setminus \{v(p_j)\}$). By applying the above canonical rules and then cuts, Π is provable in G:

$$\dfrac{\dfrac{\Pi \cup \Omega^1_1 \quad ... \quad \Pi \cup \Omega^1_{k_1}}{\Pi \cup S_1 : \diamond(p_1, ..., p_n)} \quad ... \quad \dfrac{\Pi \cup \Omega^m_1 \quad ... \quad \Pi \cup \Omega^m_{k_m}}{\Pi \cup S_m : \diamond(p_1, ..., p_n)}}{\Pi}$$

where for all $1 \leq j \leq m$: $\Theta_j = \{\Omega^j_1, ..., \Omega^j_{k_j}\}$. Π consists of atomic formulas only and does not contain a logical axiom, and so it has no proper proof in G (from $\emptyset$), in contradiction to our assumption that G admits analytic cut-elimination.

□

What about full (strong) cut-elimination? The next example shows that coherence is *not* a sufficient condition for it. Therefore a stronger condition is provided in the definition that follows.

Example 3.3.34. Consider the calculus G' from Example 3.3.17. G' is obviously coherent. A proof of the sequent $\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}$ is given in that example. However, this sequent clearly has no cut-free proof in G'.

Definition 3.3.35. A canonical calculus G is *dense* if for every $a_1, ..., a_n \in \mathcal{V}$ and every two rules of G of the forms $[\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)]$ and $[\Theta_2/S_2 : \diamond(p_1, \ldots, p_n)]$, such that $\Theta_1 \cup \Theta_2 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent, there is some rule $[\Theta/S : \diamond(p_1, \ldots, p_n)]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S \subseteq S_1 \cap S_2$.

To show that density implies coherence, we shall need the following lemma:

Lemma 3.3.36. *Let G be a dense canonical calculus. Let $[\Theta_1/S_1 : \diamond(p_1, ..., p_n)]$, . . . , $[\Theta_m/S_m : \diamond(p_1, ..., p_n)]$ be some rules of G, such that $\Theta_1 \cup ... \cup \Theta_m$ is consistent. Then for every $a_1, ..., a_n \in \mathcal{V}$, such that $\Theta_1 \cup \ldots \cup \Theta_m \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent, there is some rule $[\Theta/S : \diamond(p_1, \ldots, p_n)]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S \subseteq S_1 \cap \ldots \cap S_m$.*

Proof. We prove by induction on m. For $m = 2$ the claim follows by definition of density. Now suppose that the claim is true for any $m \leq k$, and let $[\Theta_1/S_1 : \diamond(p_1, ..., p_n)]$ . . . $[\Theta_{k+1}/S_{k+1} : \diamond(p_1, ..., p_n)]$ be some rules of G, such that $\Theta_1 \cup ... \cup \Theta_{k+1}$ is consistent. Now let $a_1, ..., a_n \in \mathcal{V}$, such that $\Theta_1 \cup \ldots \cup \Theta_{k+1} \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent. Then $\Theta_1 \cup \ldots \cup \Theta_k \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent. By the induction hypothesis, there is some rule of the form $[\Theta_0/S_0 : \diamond(p_1, \ldots, p_n)]$ in G, such that $\Theta_0 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S_0 \subseteq S_1 \cap \ldots \cap S_k$. By Lemma 3.3.22, $\Theta_0 \cup \Theta_{k+1} \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent (since both $\Theta_{k+1} \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ and $\Theta_0 \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ are consistent). By the density of G, there is some rule $[\Theta/S : \diamond(p_1, \ldots, p_n)]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n\rangle}$ is consistent and $S \subseteq S_0 \cap S_{k+1}$. But since $S_0 \subseteq S_1 \cap \ldots \cap S_k$, also $S \subseteq S_1 \cap \ldots \cap S_k \cap S_{k+1}$.

□

Proposition 3.3.37. *Every dense canonical calculus is coherent.*

Proof. Let G be a dense canonical calculus. Suppose that $[\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)]$, . . . , $[\Theta_m/S_m : \diamond(p_1, \ldots, p_n)]$ are rules of G, such that $S_1 \cap \ldots \cap S_m = \emptyset$. Suppose by contradiction that $\Theta_1 \cup \ldots \cup \Theta_m$ is consistent. By Lemma 3.3.36, there is some canonical rule $[\Theta/S : \diamond(p_1, \ldots, p_n)]$ in G, such that $S \subseteq S_1 \cap \ldots \cap S_m$. By definition of a canonical rule (recall Defn. 3.3.6) S is non-empty, in contradiction to our assumption. Thus G is coherent.

□

To provide an exact characterization of canonical systems which admit standard and strong cut-elimination, we will first need the following proposition:

Proposition 3.3.38. *Let G be a dense calculus. If Ω has no cut-free proof from Θ in G, then $\Theta \not\vdash_{\mathcal{M}_G} \Omega$.*

Proof. Like in the proof of Theorem 3.3.30, we extend Ω to a maximal set Ω^*, such that (i) no $\Omega' \subseteq \Omega^*$ has a Θ-cut-free proof from Θ in G, and (ii) all formulas occurring in Ω^* are subformulas of formulas from $\Theta \cup \{\Omega\}$. Let us show that Ω^* satisfies property (1) (from the proof of Theorem 3.3.30), namely that if $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$ and $b_1 : \diamond(\psi_1, ..., \psi_n), ..., b_k : \diamond(\psi_1, ..., \psi_n) \in \Omega^*$, then $a_i : \psi_i \in \Omega^*$ for some $1 \leq i \leq n$. Suppose by contradiction that $\tilde{\diamond}(a_1, ..., a_n) = \{b_1, ..., b_k\}$ and $b_1 : \diamond(\psi_1, ..., \psi_n), ..., b_k : \diamond(\psi_1, ..., \psi_n) \in$

Ω^*, but for every $1 \leq i \leq n$: $a_i : \psi_i \notin \Omega^*$. Then there is some $\Omega_i \subseteq \Omega^*$, s.t. $\Omega_i \cup \{a_i : \psi_i\}$ has a Θ-cut-free proof from Θ in G. First observe that $\{b_1, ..., b_k\} \neq \mathcal{V}$ (otherwise Ω^* would contain a logical axiom, in contradiction to property (i)). Then by definition of $\mathcal{M}_G$ there are some rules $R_1 = [\Xi_1/S_1 : \diamond(p_1, \ldots, p_n)], ..., R_m = [\Xi_m/S_m : \diamond(p_1, \ldots, p_n)]$ in G, such that $\Xi_1 \cup ... \cup \Xi_m \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent and $S_1 \cap ... \cap S_m = \{b_1, ..., b_k\}$. Since G is dense, by Lemma 3.3.36 there is some rule $R = [\Xi/S : \diamond(p_1, \ldots, p_n)]$, such that $\Xi \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent and $S \subseteq S_1 \cap S_2 ... \cap S_m$. By Lemma 3.3.23, there is some $1 \leq j_\Delta \leq n$, such that $(a_{j_\Delta} : p_{j_\Delta}) \in \Delta$ for every $\Delta \in \Sigma$ (since $\Xi \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent). Recall that $\Omega_i \cup \{a_{j_\Delta} : \psi_{j_\Delta}\}$ has a Θ-cut-free proof from Θ in G. Now let $\Xi = \{\Delta_1, ..., \Delta_l\}$. By applying weakening, the rule R and again weakening, $\Omega \cup (S_1 \cap ... \cap S_m) : \diamond(\psi_1, ..., \psi_n)$ has a Θ-cut-free proof from Θ in G:

$$\dfrac{\dfrac{\dfrac{\Omega_{j_{\Delta_1}} \cup \{a_{j_{\Delta_1}} : \psi_{j_{\Delta_1}}\}}{\Omega_{j_{\Delta_1}} \cup \ldots \cup \Omega_{j_{\Delta_l}} \cup \Delta_1} \quad \ldots \quad \dfrac{\Omega_{j_{\Delta_l}} \cup \{a_{j_{\Delta_l}} : \psi_{j_{\Delta_l}}\}}{\Omega_{j_{\Delta_1}} \cup \ldots \cup \Omega_{j_{\Delta_l}} \cup \Delta_l}}{\Omega_{j_{\Delta_1}} \cup \ldots \cup \Omega_{j_{\Delta_l}} \cup S : \diamond(\psi_1, ..., \psi_n)}}{\Omega_{j_{\Delta_1}} \cup \ldots \cup \Omega_{j_{\Delta_l}} \cup (S_1 \cap ... \cap S_m) : \diamond(\psi_1, ..., \psi_n)}$$

Recall that $S_1 \cap ... \cap S_m = \{b_1, ..., b_k\}$ and so there is some $\Omega' \subseteq \Omega$, which has a Θ-cut-free proof from Θ in G, in contradiction to our assumption.
The rest of the proof proceeds similarly to the proof of Theorem 3.3.30. □

Theorem 3.3.39. *Let G be a canonical calculus. Then the following statements concerning G are equivalent:*

1. *G is dense.*
2. *G admits cut-elimination.*
3. *G admits strong cut-elimination.*

Proof. $(1 \Rightarrow 3)$: Let G be a dense calculus. Then by Proposition 3.3.37, it is also coherent and so $\mathcal{M}_G$ is well-defined. If $\Theta \vdash_G \Omega$, then $\Theta \vdash_{\mathcal{M}_G} \Omega$. Thus by Proposition 3.3.38, Ω has a cut-free proof from Θ. Clearly, also $(3 \Rightarrow 2)$ holds. It remains to show that $(2 \Rightarrow 1)$. Suppose that G admits cut-elimination and assume by contradiction that G is not dense. Then there are some $a_1, ..., a_n \in \mathcal{V}$ and some rules $R_1 = [\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)]$ and $R_2 = [\Theta_2/S_2 : \diamond(p_1, \ldots, p_n)]$, such that $\Theta_1 \cup \Theta_2 \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent and $S_1 \cap S_2 \neq \emptyset$, but there is no rule $[\Theta/S : \diamond(p_1, \ldots, p_n)]$ in G, such that $\Theta \cup \mathsf{C}_{\langle a_1,\ldots,a_n\rangle}$ is consistent and $S \subseteq S_1 \cap S_2$. Now let $\Omega_0 = \bigcup_{1 \leq i \leq n} \{\mathcal{V} \setminus \{a_i\} : p_i\}$. By Lemma 3.3.23, for every $\Omega \in \Theta_1 \cup \Theta_2$, there is some $1 \leq i \leq n$, such that $a_i : p_i \in \Omega$. Thus for every $\Omega \in \Theta_1 \cup \Theta_2$, $\Omega \cup \Omega_0$ is a logical axiom. Let $\Theta_1 = \{\Omega^1_1, ..., \Omega^1_k\}$ and $\Theta_2 = \{\Omega^2_1, ..., \Omega^2_m\}$. By applying the canonical rules R_1 and R_2, and then cuts we get a proof of $\Omega_0 \cup S_1 \cap S_2 : \diamond(p_1, ..., p_n)$ in G[7]:

$$\dfrac{\dfrac{\Omega_1^1 \cup \Omega_0 \quad ... \quad \Omega_k^1 \cup \Omega_0}{\Omega_0 \cup S_1 : \diamond(p_1, ..., p_n)}\ R_1 \quad \dfrac{\Omega_1^2 \cup \Omega_0 \quad ... \quad \Omega_m^2 \cup \Omega_0}{\Omega_0 \cup S_2 : \diamond(p_1, ..., p_n)}\ R_2}{\Omega_0 \cup (S_1 \cap S_2) : \diamond(p_1, ..., p_n)}$$

However, since the axioms are atomic, it is easy to see that $\Omega_0 \cup (S_1 \cap S_2) : \diamond(p_1, ..., p_n)$ has no cut-free proof in G, in contradiction to our assumption. □

Proposition 3.3.40. *Every coherent canonical calculus G has an equivalent dense canonical calculus.*

Proof. Let G be a coherent canonical calculus. Then by Theorem 3.3.30, $\mathcal{M}_G$ is strongly characteristic for G. In [27] (see Theorem 4.1) a sound and complete canonical calculus $SF^d_{\mathcal{M}}$ is provided for every finite Nmatrix $\mathcal{M}$. It is easy to verify that $SF^d_{\mathcal{M}}$ is dense for every finite Nmatrix $\mathcal{M}$. $SF^d_{\mathcal{M}_G}$ is equivalent to G, hence the claim holds. □

Corollary 3.3.41. *Every coherent canonical calculus has an equivalent calculus which admits strong cut-elimination.*

Proof. Follows directly from Proposition 3.3.40 and Theorem 3.3.39.

□

For the special case of two-signed canonical calculi, corresponding to the systems described in Section 3.1, the criterions of coherence and density coincide:

Proposition 3.3.42. *A canonical calculus with two signs is dense iff it is coherent.*

Proof. Let G be a coherent calculus. Let $\diamond$ be an n-ary connective and $a_1, ..., a_n \in \mathcal{V}$. Let $R_1 = [\Theta_1/S_1 : \diamond(p_1, \ldots, p_n)]$ and $R_2 = [\Theta_2/S_2 : \diamond(p_1, \ldots, p_n)]$ be two rules of G such that $\Theta_1 \cup \Theta_2 \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ is consistent. Since G is coherent, $S_1 \cap S_2 \neq \emptyset$ (then either $S_1 = S_2 = \{t\}$, or $S_1 = S_2 = \{f\}$, or one of them is $\{t, f\}$. Hence either $S_1 \subseteq S_1 \cap S_2$ or $S_2 \subseteq S_1 \cap S_2$ and so there is a rule $R = [\Theta/S : \diamond(p_1, \ldots, p_n)]$, such that $\Theta \cup \mathsf{C}_{\langle a_1,...,a_n \rangle}$ is consistent and $S \subseteq S_1 \cap S_2$ (R is either R_1 or R_2). Hence G is dense.

□

The following easy corollary is a generalization of Theorem 3.1.6:

Corollary 3.3.43. *The following statements concerning a two signed canonical calculus G are equivalent:*

[7] Note that this is a generalization of Example 3.3.34.

1. *G is coherent.*
2. *G is dense.*
3. *G has a strongly characteristic Nmatrix.*
4. *G admits strong analytic cut-elimination.*
5. *G admits analytic cut-elimination.*
6. *G admits strong cut-elimination.*
7. *G admits (standard) cut-elimination.*

Chapter 4

Application: Nmatrices with Distance-based Reasoning

The logics that we have discussed so far were *monotonic* (i.e., it holds that if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then also $\Gamma' \vdash \psi$). However, in every day life it is often the case that previous conclusions are retracted in the presence of new information. To capture this property of commonsense reasoning, many *non-monotonic* formalisms have been developed (see, e.g. [107, 110, 76, 97]). [122] introduced the notion of *preferential semantics* (see also [104]), according to which an order relation, reflecting some condition or preference criteria, is defined on a set of valuations, and only the valuations that are minimal with respect to this order are relevant for making inferences from a given theory. Following this idea, in [3, 4] non-monotonic entailment relations are defined, which are based on distance-minimization as a primary preference criteria. Distance-minimization is a cornerstone behind many paradigms of handling uncertainty, such as belief revision (see, e.g. [48, 83, 101]), database integration systems (e.g. [2, 7, 66]), and formalisms for commonsense reasoning in the context of social choice theory (e.g. [98, 112]).

The distance-based framework of [3, 4] can be applied to reason in the presence of inconsistent information. However, since this framework is based on classical logic, it cannot capture inherently non-deterministic phenomena, like unpredictable circuit behavior or unknown computation models. Below we show that combining distance-based considerations with the framework of Nmatrices is especially useful for reasoning about non-deterministic phenomena in the presence of inconsistent information. We investigate some basic properties of the entailment relations obtained in our framework, and demonstrate their applicability for reasoning under uncertainty by some examples.

The material in this chapter is mainly based on [8, 9, 10, 11, 12].

4.1 Distance-based Semantics

In this section we briefly summarize the main definitions of the distance-based framework of [3, 4].

Henceforth $\mathcal{L}_{cl}$ is the propositional language with the classical connectives and a *finite* set $\mathsf{Atoms} = \{p_1, \ldots, p_m\}$ of atomic formulas. A *finite* multiset of formulas in $\mathcal{L}_{cl}$ is called below a *theory*. For a theory Γ, we denote by $\mathsf{Atoms}(\Gamma)$ the set of atomic formulas that occur in Γ.

Definition 4.1.1. A classical valuation for $\mathcal{L}_{cl}$ is any function $\nu : \mathsf{Atoms} \to \{\mathbf{t}, \mathbf{f}\}$. We shall denote ν by the tuple $\langle p_1 : \nu(p_1), \ldots, p_m : \nu(p_m)\rangle$. Valuations are extended to the formulas of $\mathcal{L}_{cl}$ in the standard way (i.e, respecting the classical interpretations of the connectives). We denote the set of all the classical valuations for $\mathcal{L}_{cl}$ by Λ_{cl}. The set of models of a formula ψ (a theory Γ) is denoted by $\mathsf{mod}(\psi)$ ($\mathsf{mod}(\Gamma)$).

The key notion of distance-based semantics is that of a *distance function*:

Definition 4.1.2. A *pseudo-distance* on a set U is a function $d : U \times U \to \mathbb{R}^+$, satisfying the following conditions:

- *symmetry:* for all $\nu, \mu \in U$ $d(\nu, \mu) = d(\mu, \nu)$,
- *identity preservation:* for all $\nu, \mu \in U$ $d(\nu, \mu) = 0$ iff $\nu = \mu$.

A pseudo-distance d is a *distance* function on U if it has the following property:

- *triangular inequality:* for all $\nu, \mu, \sigma \in U$ $d(\nu, \sigma) \leq d(\nu, \mu) + d(\mu, \sigma)$.

Example 4.1.3. Consider the following well-known distances on Λ_{cl}:

- *The drastic distance*[1]: $d_U(\nu, \mu) = 0$ if $\nu = \mu$ and $d_U(\nu, \mu) = 1$ otherwise.
- *The Hamming distance*: $d_H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$.

Definition 4.1.4. A *numeric aggregation function* is function f whose argument is a multiset of real numbers and whose values are real numbers, such that: (i) f is non-decreasing in the value of its argument, (ii) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$, and (iii) $f(\{x\}) = x$ for every $x \in \mathbb{R}$.

[1] Also known as the *discrete metric*.

Examples of aggregation functions are: Σ, average, maximum, etc.

Next a "distance" between a valuation and a formula, and between a valuation and a theory is defined. Intuitively, this "distance" measures how close a valuation is to satisfying a formula and a theory.

Definition 4.1.5. Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, a valuation $\nu \in \Lambda_{cl}$, a pseudo-distance d, and an aggregation function f, define:

- $d(\nu, \psi_i) = \begin{cases} \min\{d(\nu, \mu) \mid \mu \in \mathsf{mod}(\psi_i)\} & \text{if } \mathsf{mod}(\psi_i) \neq \emptyset, \\ 1 + \max\{d(\nu_1, \nu_2) \mid \nu_1, \nu_2 \in \Lambda\} & \text{otherwise.} \end{cases}$

- $\delta_{d,f}(\nu, \Gamma) = f(\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\})$.

Remark 4.1.6. If ψ is a tautology, then all the valuations are equally close to ψ and their distance from it is zero. If ψ is a contradiction, then all valuations are equally distant from ψ by the maximal number $1 + \max\{d(\nu_1, \nu_2) \mid \nu_1, \nu_2 \in \Lambda_{cl}\}$. In the other cases, the valuations that are "closest" to ψ are its models and their distance to ψ is zero. This also implies that $\delta_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in \mathsf{mod}(\Gamma)$.

The next definition captures the intuition that the relevant valuations that should be used to make inferences from a theory Γ are those that are "closest" to Γ in the following sense:

Definition 4.1.7. The *most plausible* valuations of Γ, with respect to a pseudo distance d and an aggregation function f on Λ, are defined as follows:

$$\Delta_{d,f}(\Gamma) = \begin{cases} \{\nu \in \Lambda \mid \forall \mu \in \Lambda \ \delta_{d,f}(\nu, \Gamma) \leq \delta_{d,f}(\mu, \Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda_{cl} & \text{otherwise.} \end{cases}$$

Definition 4.1.8. For a pseudo distance d and an aggregation function f, define $\Gamma \models_{d,f} \psi$ if $\Delta_{d,f}(\Gamma) \subseteq \mathsf{mod}(\psi)$.

In other words, any conclusion from a theory Γ should follow from *all* of the most plausible valuations of Γ.

Example 4.1.9. Let $\Gamma = \{p, \neg p, q\}$. This theory is not (classically) satisfiable, and everything follows from it in the classical consequence relation. Yet, as q is not related to the contradiction in Γ, it seems counterintuitive to infer $\neg q$ in this case, while one would still expect to infer q. Moreover, since is not clear which of the formulas $p, \neg p$ is true, one may want to be able to infer neither $\neg p$ nor p. To see how this is captured in the framework defined above, consider for instance the drastic distance d_U and the summation function Σ. Then:

$$\delta_{d_U,\Sigma}(\{p\!:\!\mathbf{t}, q\!:\!\mathbf{t}\}, \Gamma) = 1, \quad \delta_{d_U,\Sigma}(\{p\!:\!\mathbf{f}, q\!:\!\mathbf{t}\}, \Gamma) = 1,$$

$$\delta_{d_U,\Sigma}(\{p\!:\!\mathbf{t}, q\!:\!\mathbf{f}\}, \Gamma) = 2, \quad \delta_{d_U,\Sigma}(\{p\!:\!\mathbf{f}, q\!:\!\mathbf{f}\}, \Gamma) = 2,$$

Thus valuations in which q is assigned $\mathbf{f}$ are more distant from Γ than valuations in which q is assigned $\mathbf{t}$. It follows that

$$\Gamma \models_{d_U,\Sigma} q, \quad \Gamma \not\models_{d_U,\Sigma} \neg q, \quad \Gamma \not\models_{d_U,\Sigma} p, \quad \Gamma \not\models_{d_U,\Sigma} \neg p,$$

as intuitively expected. Similar results are obtained, e.g., for $\models_{d_H,\Sigma}$.

4.2 Combining Distance-based Semantics with Nmatrices

4.2.1 Motivation

The underlying logic in the distance-based framework presented in the previous section is classical logic. However, examples from Section 1.1 provide various motivations for situations in which logics based on Nmatrices are more appropriate. It should be also noted that all the non-deterministic logics that we considered were induced by tcrs (scrs) and are thus monotonic. Moreover, they are inconsistency-intolerant in the sense that for any Nmatrix $\mathcal{M}$, if a theory Γ is not $\mathcal{M}$-satisfiable, everything follows from it. To overcome this, logics based on Nmatrices can be refined using distance-based considerations. In what follows, Nmatrices can be used as an underlying framework for the distance-based framework presented in the previous section. Our focus will be on two-valued Nmatrices, although the proposed method can be easily extended to Nmatrices with more than two values.

For a motivation for incorporating distance-based considerations into the framework of Nmatrices, consider the following example.

Example 4.2.1. Suppose that only partial information is known about the logical circuit below, according to one of the following scenarios:

Scenario A: the gates marked with '?' are functioning properly, but it is unknown whether they are XOR or OR gates.
Scenario B: the gates marked with '?' are faulty and behave unpredictably whenever their inputs are both on (i.e, have the value $\mathbf{t}$).

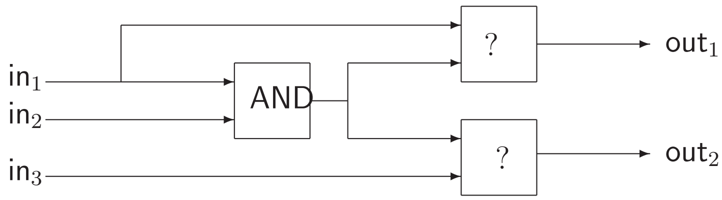

Figure 4.1: The circuit of Example 4.2.1.

In both of these scenarios the functionality of the '?'-gates can be captured using the following non-deterministic truth-table:

		$\diamond$
t	**t**	$\{\mathbf{t}, \mathbf{f}\}$
t	**f**	$\{\mathbf{t}\}$
f	**t**	$\{\mathbf{t}\}$
f	**f**	$\{\mathbf{f}\}$

However, the static semantics is more appropriate for scenario A, while the dynamic semantics fits scenario B. This is due to the fact that the gates in scenario A have deterministic (but unknown) behaviour, that is a given input always results in the same output. In scenario B, however, the gate reacts non-deterministically and may give different outputs for the same input at different times.
Despite the non-deterministic component in the circuit specification, we are still able to make some useful inferences about the circuit. Possible inferences that can be made are:

(CN1) When all the inputs are equal, the value of out$_1$ is equal to the value of out$_2$ (in scenario A).
(CN2) When all the input lines are off (i.e., have the value **f**), then so are both of the output lines of the circuit (in scenarios A and B).

Next suppose that we receive a new indication about the circuit's behavior, which contradicts our previous knowledge about the circuit. Specifically, we learn that when in$_2$ and in$_3$ are turned off, out$_2$ is turned on. This is not possible in both scenarios, and the obtained set of premises is now inconsistent. Using distance-based considerations will enable us to draw rational conclusions about the circuit in such situations. For instance, in scenario A we may retain the conclusion (CN1), as this fact should not be affected by the contradictory evidence about the circuit. On the other hand, conclusion (CN2) is most likely to be withdrawn in the presence of the new information.

4.2.2 Extending the Distance-based Framework

In what follows $\mathcal{L}$ is any propositional language. *Note that we no longer assume a finite number of atoms in* $\mathcal{L}$ (for reasons that will become clear in the sequel).
Next we observe that when moving from classical logic to Nmatrices, it is not enough to consider valuations defined on atomic formulas of $\mathcal{L}$, as the truth-values assigned to atomic formulas do not uniquely determine the truth-values assigned to complex formulas of $\mathcal{L}$. It follows that there are infinitely many complex formulas to consider when comparing between two valuations in a given Nmatrix. Hence we need to introduce further restrictions on the computation of a distance between valuations. Namely, we make the distance computations *context dependent*, that is restricted to a certain set of relevant formulas closed under subformulas.

Notation 4.2.2. Let $\mathcal{M}$ be an Nmatrix. We denote by $\Lambda^{\mathsf{s}}_{\mathcal{M}}$ ($\Lambda^{\mathsf{d}}_{\mathcal{M}}$) the set of all the static (dynamic) $\mathcal{M}$-valuations of $\mathcal{M}$. We denote by $mod^{\mathsf{s}}_{\mathcal{M}}(\psi)$ ($mod^{\mathsf{d}}_{\mathcal{M}}(\psi)$) the set of all the static (dynamic) $\mathcal{M}$-valuations which are models of ψ. We write $\Lambda_{\mathcal{M}}$ and $mod_{\mathcal{M}}$ instead of $\Lambda^{x}_{\mathcal{M}}$ and $mod^{x}_{\mathcal{M}}$ when x is immaterial.

Definition 4.2.3. A *context* C is a finite set of $\mathcal{L}$-formulas closed under subformulas. The *restriction to* C of a valuation $\nu \in \Lambda_{\mathcal{M}}$ is the partial valuation $\nu^{\downarrow\mathsf{C}}$ on C, such that $\nu^{\downarrow\mathsf{C}}(\psi) = \nu(\psi)$ for every ψ in C. The restriction to C of $\Lambda_{\mathcal{M}}$ is the set $\Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}} = \{\nu^{\downarrow\mathsf{C}} \mid \nu \in \Lambda_{\mathcal{M}}\}$.

Example 4.2.4. It is easy to verify that for every Nmartrix $\mathcal{M}$ and every context C, the following functions are distances on $\Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}}$:

- *The drastic (uniform) distance:* $d^{\downarrow\mathsf{C}}_{U}(\nu,\mu) = 0$ if $\nu = \mu$ and $d_U(\nu,\mu) = 1$ otherwise.

- *The Hamming distance:* $d^{\downarrow\mathsf{C}}_{H}(\nu,\mu) = |\{\psi \in \mathsf{C} \mid \nu(\psi) \neq \mu(\psi)\}\,|$.

More generally, "distances" between valuations are now defined as follows:

Definition 4.2.5. Let $\mathcal{M}$ be an Nmatrix and d a function from $\bigcup_{\{\mathsf{C}=\mathsf{SF}(\Gamma)\mid\Gamma\subseteq Frm_{\mathcal{L}}\}} \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}} \times \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}}$ to $\mathbb{R}^{+}$.

- The *restriction* of d to a context C is the function $d^{\downarrow\mathsf{C}} : \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}} \times \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}} \to \mathbb{R}^{+}$, such that for every $\nu,\mu \in \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}}$: $d^{\downarrow\mathsf{C}}(\nu,\mu) = d(\nu,\mu)$.

- We say that d is a *generic (pseudo) distance* on $\Lambda_{\mathcal{M}}$ if for every context C, $d^{\downarrow\mathsf{C}}$ is a (pseudo) distance on $\Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}}$.

Example 4.2.6. Let $\mathcal{M}$ be an Nmatrix. Denote $\Psi = \bigcup_{\{\mathsf{C}=\mathsf{SF}(\Gamma)\mid\Gamma\subseteq Frm_{\mathcal{L}}\}} \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}} \times \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}}$. Define the following functions for every context C and every $\mu, \nu \in \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}}$:

$$\mathbf{d_U}(\mu,\nu) = d_U^{\downarrow\mathsf{C}}(\mu,\nu)$$

$$\mathbf{d_H}(\mu,\nu) = d_H^{\downarrow\mathsf{C}}(\mu,\nu)$$

where the functions $d_U^{\downarrow\mathsf{C}}$ and $d_H^{\downarrow\mathsf{C}}$ are defined in Example 4.2.4. It is clear that $\mathbf{d_U}$ and $\mathbf{d_H}$ are generic distances on $\Lambda_{\mathcal{M}}$. Note the difference between the generic distances $\mathbf{d_U}, \mathbf{d_H}$ and distances d_U, d_H from Example 4.1.3.

General Constructions of Generic Distances

The construction of the generic distances $\mathbf{d_U}, \mathbf{d_H}$ is easy because their restrictions (Example 4.2.4) are already known. Below we describe a general method for constructing generic distances and show that $\mathbf{d_U}, \mathbf{d_H}$ are particular instances that can be obtained by our method.

Definition 4.2.7. Let $\mathcal{M}$ be an Nmatrix and C a context.

1. Define the function $\nabla : \{\mathbf{t}, \mathbf{f}\} \to \{0,1\}$ as follows: $\nabla(a_1, a_2) = 0$ if $a_1 = a_2$ and $\nabla(v_1, v_2) = 1$ otherwise.

2. For every $\psi \in \mathsf{C}$, define the function $\bowtie^{\psi}$: $\Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}} \times \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}} \to \{0,1\}$ by induction as follows:

   - for an atomic formula p, let $\bowtie^{p}(\nu,\mu) = \nabla(\nu(p), \mu(p))$
   - for a formula $\psi = \diamond(\psi_1, \ldots, \psi_n)$, define
   $$\bowtie^{\psi}(\nu,\mu) = \begin{cases} 1 & \text{if } \nu(\psi) \neq \mu(\psi) \text{ but } \forall i\ \nu(\psi_i) = \mu(\psi_i), \\ 0 & \text{otherwise.} \end{cases}$$

3. Let g be an aggregation function. Define the following functions from $\Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}} \times \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}}$ to $\mathbb{R}^+$:

   - $d_{\nabla,g}^{\downarrow\mathsf{C}}(\nu,\mu) = g\big(\{\nabla(\nu(\psi),\mu(\psi)) \mid \psi \in \mathsf{C}\}\big)$,
   - $d_{\bowtie,g}^{\downarrow\mathsf{C}}(\nu,\mu) = g\big(\{\bowtie^{\psi}(\nu,\mu) \mid \psi \in \mathsf{C}\}\big)$.

The difference between $d_{\nabla,g}^{\downarrow\mathsf{C}}$ and $d_{\bowtie,g}^{\downarrow\mathsf{C}}$ is in the treatment of the non-deterministic choices made by the valuations. This is demonstrated in the following example.

Example 4.2.8. Consider an Nmatrix $\mathcal{M}$ with $\tilde{\neg}_{\mathcal{M}}(\mathbf{t}) = \{\mathbf{t}, \mathbf{f}\}$ and $\tilde{\neg}_{\mathcal{M}}(\mathbf{f}) = \{\mathbf{t}\}$, and the following valuations in $\Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}}$ for $\mathsf{C} = \{p, \neg p, \neg\neg p\}$:

$$\begin{aligned} &\nu_1(p) = \mathbf{t}, \quad \nu_1(\neg p) = \mathbf{f}, \quad \nu_1(\neg\neg p) = \mathbf{t} \\ &\nu_2(p) = \mathbf{t}, \quad \nu_2(\neg p) = \mathbf{t}, \quad \nu_2(\neg\neg p) = \mathbf{f} \\ &\nu_3(p) = \mathbf{f}, \quad \nu_3(\neg p) = \mathbf{t}, \quad \nu_3(\neg\neg p) = \mathbf{t} \end{aligned}$$

Using $d_{\nabla,\Sigma}^{\downarrow\mathsf{C}}$, all the valuations are equally distant from each other, as they differ on exactly two formulas in C:

$$d_{\nabla,\Sigma}^{\downarrow\mathsf{C}}(\nu_1, \nu_2) = d_{\nabla,\Sigma}^{\downarrow\mathsf{C}}(\nu_1, \nu_3) = d_{\nabla,\Sigma}^{\downarrow\mathsf{C}}(\nu_2, \nu_3) = 2.$$

Using $d_{\bowtie,\Sigma}^{\downarrow\mathsf{C}}$, however, the situation is different, as

$$d_{\bowtie,\Sigma}^{\downarrow\mathsf{C}}(\nu_1, \nu_2) = d_{\bowtie,\Sigma}^{\downarrow\mathsf{C}}(\nu_1, \nu_3) = 1, \text{ but } d_{\bowtie,\Sigma}^{\downarrow\mathsf{C}}(\nu_2, \nu_3) = 2.$$

This may be explained by the fact that ν_1 and ν_2 make one different choice (in the transition from p to $\neg p$) and so are ν_1 and ν_3 (in the initial value of p), while ν_2 and ν_3 make two different choices (in the initial value of p and in the transition from $\neg p$ to $\neg\neg p$). So, while $d_{\nabla,g}$ compares *truth assignments*, $d_{\bowtie,g}$ compares (initial and non-deterministic) *choices*.

Remark 4.2.9. It is easy to verify that the distances from Examples 4.1.3 and 4.2.4 are specific instances of the functions obtained by the methods above:

- *Drastic distance:* For classical valuations $\nu, \mu \in \Lambda_{cl}$ (assuming that the set of atoms is *finite*):

  $$d_U(\nu, \mu) = d_{\nabla,\max}^{\downarrow\mathsf{Atoms}}(\nu, \mu) = d_{\bowtie,\max}^{\downarrow\mathsf{Atoms}}(\nu, \mu).$$

  For any Nmatrix $\mathcal{M}$, any context C and any $\nu, \mu \in \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}}$ it holds that:

  $$d_U^{\downarrow\mathsf{C}}(\nu, \mu) = d_{\nabla,\max}^{\downarrow\mathsf{C}}(\nu, \mu) = d_{\bowtie,\max}^{\downarrow\mathsf{C}}(\nu, \mu)$$

- *Hamming distance*: For classical valuations $\nu, \mu \in \Lambda_{cl}$ (assuming that the set of atoms is *finite*):

  $$d_H(\nu, \mu) = d_{\nabla,\Sigma}^{\downarrow\mathsf{Atoms}}(\nu, \mu) = d_{\bowtie,\Sigma}^{\downarrow\mathsf{Atoms}}(\nu, \mu).$$

  For any Nmatrix $\mathcal{M}$, any context C and any $\nu, \mu \in \Lambda_{\mathcal{M}}^{\downarrow\mathsf{C}}$ it holds that:

  $$d_H^{\downarrow\mathsf{C}}(\nu, \mu) = d_{\nabla,\Sigma}^{\downarrow\mathsf{C}}(\nu, \mu)$$

The following easy proposition follows:

Proposition 4.2.10. *For every structure* $\mathcal{M}$*, context* C*, and aggregation function* g*,* $d^{\downarrow\mathsf{C}}_{\nabla,g}$ *and* $d^{\downarrow\mathsf{C}}_{\bowtie,g}$ *are pseudo-distances on* $\Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}}$*.*

Proposition 4.2.10 provides a general method for constructing generic pseudo distances:

Corollary 4.2.11. *Let* $\Psi = \bigcup_{\{\mathsf{C}=\mathsf{SF}(\Gamma)|\Gamma\subseteq Frm_{\mathcal{L}}\}} \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}} \times \Lambda^{\downarrow\mathsf{C}}_{\mathcal{M}}$ *for an Nmatrix* $\mathcal{M}$*. Let* g *be an aggregation function. Define the functions* $d_{\nabla,g}, d_{\bowtie,g} : \Psi \to \mathbb{R}^{+}$ *as follows for every context* C*:*

$$d_{\nabla,g}(\nu,\mu) = d^{\downarrow\mathsf{C}}_{\nabla,g}(\nu,\mu)$$

$$d_{\bowtie,g}(\nu,\mu) = d^{\downarrow\mathsf{C}}_{\bowtie,g}(\nu^{\downarrow\mathsf{C}},\mu^{\downarrow\mathsf{C}})$$

Then $d_{\nabla,g}$ *and* $d_{\bowtie,g}$ *are generic pseudo-distances on* $\Lambda_{\mathcal{M}}$*.*

By Remark 4.2.9, the generic distances $\mathbf{d_U}, \mathbf{d_H}$ (Example 4.2.6) are the functions $d_{\nabla,\max}$ and $d_{\nabla,\Sigma}$ respectively.

Distance-based Entailments

Definition 4.2.12. A *setting* for a language $\mathcal{L}$ is a tuple $\mathcal{S} = \langle \mathcal{M}, (d,x), f\rangle$, where $\mathcal{M}$ is a an Nmatrix, d is a generic pseudo-distance on $\Lambda^{x}_{\mathcal{M}}$ for $x \in \{\mathsf{d},\mathsf{s}\}$, and f is an aggregation function.

A setting identifies the underlying logic of the framework and can be used for measuring the "distance" between valuations and formulas, and between valuations and theories.

Definition 4.2.13. Given a setting $\mathcal{S} = \langle \mathcal{M}, (d,x), f\rangle$ for a language $\mathcal{L}$, define for every valuation $\nu \in \Lambda^{x}_{\mathcal{M}}$, theory $\Gamma = \{\psi_1,\ldots,\psi_n\}$ in $\mathcal{L}$, and context C,

- $d^{\downarrow\mathsf{C}}(\nu,\psi_i) = \begin{cases} \min\{d^{\downarrow\mathsf{C}}(\nu^{\downarrow\mathsf{C}},\mu^{\downarrow\mathsf{C}}) \mid \mu \in mod^{x}_{\mathcal{M}}(\psi_i)\} & \text{if } mod^{x}_{\mathcal{M}}(\psi_i) \neq \emptyset, \\ 1+\max\{d^{\downarrow\mathsf{C}}(\mu_1^{\downarrow\mathsf{C}},\mu_2^{\downarrow\mathsf{C}}) \mid \mu_1,\mu_2 \in \Lambda^{x}_{\mathcal{M}}\} & \text{otherwise.} \end{cases}$

- $\delta^{\downarrow\mathsf{C}}_{d,f}(\nu,\Gamma) = f(\{d^{\downarrow\mathsf{C}}(\nu,\psi_1),\ldots,d^{\downarrow\mathsf{C}}(\nu,\psi_n)\})$.

Note that Remark 4.1.6 can be extended to the non-deterministic case as well. Another property which follows directly from the definition above is that the above "distances" are not affected by "irrelevant" formulas (i.e., formulas that are not part of the relevant context)[2]:

[2] This property was called *unbiasedness* in [3, 4, 10].

Proposition 4.2.14. *Let $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ be a setting and Γ a theory. Then for every context C, valuations $\nu_1, \nu_2 \in \Lambda^x_{\mathcal{M}}$, and formula $\psi \in \Gamma$, if $\nu_1^{\downarrow \mathsf{C}} = \nu_2^{\downarrow \mathsf{C}}$ then $d^{\downarrow \mathsf{C}}(\nu_1, \psi) = d^{\downarrow \mathsf{C}}(\nu_2, \psi)$ and $\delta^{\downarrow \mathsf{C}}_{d,f}(\nu_1, \Gamma) = \delta^{\downarrow \mathsf{C}}_{d,f}(\nu_2, \Gamma)$.*

Which context C should be used to measure distances? Since the intuition behind Definition 4.2.13 is to measure how "close" a valuation is to satisfying a formula and a theory, we expect the "distance" between a formula ψ and and a valuation ν to be zero iff ν is a model of ψ in a given Nmatrix. Hence, we are interested only in contexts for which this property is satisfied:

Proposition 4.2.15. *Let $\mathcal{M}$ be an Nmatrix, C a context, and $x \in \{\mathsf{d}, \mathsf{s}\}$. If $\mathsf{SF}(\psi) \subseteq \mathsf{C}$, then for all $\nu \in \Lambda^x_{\mathcal{M}}$: $d^{\downarrow \mathsf{C}}(\nu, \psi) = 0$ iff $\nu \in mod^x_{\mathcal{M}}(\psi)$.*

Proof. One direction is trivial. For the other direction, let $\nu \in \Lambda^x_{\mathcal{M}}$ such that $d^{\downarrow \mathsf{C}}(\nu, \psi) = 0$. Then there is some $\mu \in mod^x_{\mathcal{M}}(\psi)$ such that $d^{\downarrow \mathsf{C}}(\nu^{\downarrow \mathsf{C}}, \mu^{\downarrow \mathsf{C}}) = 0$. Since $d^{\downarrow \mathsf{C}}$ is a pseudo-distance on $\Lambda^x_{\mathcal{M}}$, necessarily $\nu^{\downarrow \mathsf{C}} = \mu^{\downarrow \mathsf{C}}$. As $\psi \in \mathsf{C}$, $\nu(\psi) = \mu(\psi)$, and so $\nu \in mod^x_{\mathcal{M}}(\psi)$. $\square$

Corollary 4.2.16. *Let $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ be a setting and C a context. For every theory $\Gamma \subseteq \mathsf{C}$ and for all $\nu \in \Lambda^x_{\mathcal{M}}$: $\delta^{\downarrow \mathsf{C}}_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in mod^x_{\mathcal{M}}(\Gamma)$.*

As contexts are closed under subformulas, the last corollary implies that the most appropriate contexts to use are those that include all the subformulas of the premises, that is for a set Γ we evaluate distance with respect to the context $\mathsf{C} = SF(\Gamma)$.

Definition 4.2.17. The *most plausible valuations* of Γ with respect to a setting $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ are:

$$\Delta_{\mathcal{S}}(\Gamma) = \begin{cases} \{\nu \in \Lambda^x_{\mathcal{M}} \mid \forall \mu \in \Lambda^x_{\mathcal{M}}\ \delta^{\downarrow \mathsf{SF}(\Gamma)}_{d,f}(\nu, \Gamma) \leq \delta^{\downarrow \mathsf{SF}(\Gamma)}_{d,f}(\mu, \Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda^x_{\mathcal{M}} & \text{otherwise.} \end{cases}$$

The following easy proposition extends a similar proposition from [3, 4]:

Proposition 4.2.18. *Let $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ be a setting and Γ a theory.*

1. *The set $\Delta_{\mathcal{S}}(\Gamma)$ is non-empty.*
2. *Γ is $\mathcal{M}$-satisfiable iff $\Delta_{\mathcal{S}}(\Gamma) = mod^x_{\mathcal{M}}(\Gamma)$.*

Example 4.2.19. Consider a setting $\mathcal{S}_1 = \langle \mathcal{M}, (d_{\nabla,\Sigma}, \mathsf{d}), \Sigma \rangle$ for $\mathcal{L} = \{\neg, \diamond\}$, where $d_{\nabla,\Sigma}$ is the generic distance from Corollary 4.2.11. $\mathcal{M}$ is the Nmatrix with the classical interpretation of negation and the following interpretation of $\diamond$:

$\tilde{\diamond}$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{f}\}$

Let $\Gamma = \{p, q, \neg(p \diamond q)\}$. This theory is not satisfiable by any dynamic $\mathcal{M}$-valuation. Denote $\mathsf{C} = \mathsf{SF}(\Gamma)$. Let us compute the set of its most plausible valuations, $\Delta_{\mathcal{S}_1}(\Gamma)$:

	p	q	$p \diamond q$	$\neg(p \diamond q)$	$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_i, p)$	$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_i, q)$	$d^{\downarrow\mathsf{C}}_{\nabla,\Sigma}(\nu_i, \neg(p \diamond q))$	$\delta^{\downarrow\mathsf{C}}_{\mathcal{S}_1}(\nu_i, \Gamma)$
ν_1	**t**	**t**	**t**	**f**	0	0	3	3
ν_2	**t**	**f**	**t**	**f**	0	1	2	3
ν_3	**t**	**f**	**f**	**t**	0	1	0	**1**
ν_4	**f**	**t**	**t**	**f**	1	0	2	3
ν_5	**f**	**t**	**f**	**t**	1	0	0	**1**
ν_6	**f**	**f**	**f**	**t**	1	1	0	2

It follows that $\Delta_{\mathcal{S}_1}(\Gamma) = \{\nu_3, \nu_5\}$.

Consider now $\mathcal{S}_2 = \langle \mathcal{M}, (d_{\bowtie,\Sigma}, \mathsf{d}), \Sigma \rangle$, where $d_{\bowtie,\Sigma}$ is the generic distance from Corollary 4.2.11. For the theory Γ we now have:

	p	q	$p \curlywedge q$	$\neg(p \diamond q)$	$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_i, p)$	$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_i, q)$	$d^{\downarrow\mathsf{C}}_{\bowtie,\Sigma}(\nu_i, \neg(p \diamond q))$	$\delta^{\downarrow\mathsf{C}}_{\mathcal{S}_2}(\nu_i, \Gamma)$
ν_1	**t**	**t**	**t**	**f**	0	0	1	**1**
ν_2	**t**	**f**	**t**	**f**	0	1	1	2
ν_3	**t**	**f**	**f**	**t**	0	1	0	**1**
ν_4	**f**	**t**	**t**	**f**	1	0	1	2
ν_5	**f**	**t**	**f**	**t**	1	0	0	**1**
ν_6	**f**	**f**	**f**	**t**	1	1	0	2

So this time $\Delta_{\mathcal{S}_2}(\Gamma) = \{\nu_1, \nu_3, \nu_5\}$.

Now we are ready to define entailment relations based on distance minimization.

Definition 4.2.20. Let $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ be a setting. $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ if $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod^x_{\mathcal{M}}(\psi)$ or [3] $\Gamma = \{\psi\}$.

Example 4.2.21. Extend the setting from Example 4.2.19 by including classical disjunction. Then for $\Gamma = \{p, q, \neg(p \diamond q)\}$ it holds that $\Gamma \mid\!\sim_{\mathcal{S}_1} \neg p \vee \neg q$ while $\Gamma \not\mid\!\sim_{\mathcal{S}_2} \neg p \vee \neg q$.

Example 4.2.22. Consider the circuit given in Figure 4.2.
Suppose that we receive information from some source that G_1 and G_2 are two faulty AND gates which behave unpredictably when both of their inputs are on. Such behavior can be captured (using the dynamic approach) by the following non-deterministic truth-table:

[3] The purpose of this addition is to preserve cautious reflexivity of $\mid\!\sim_{\mathcal{S}}$, see Definition 4.2.28 below.

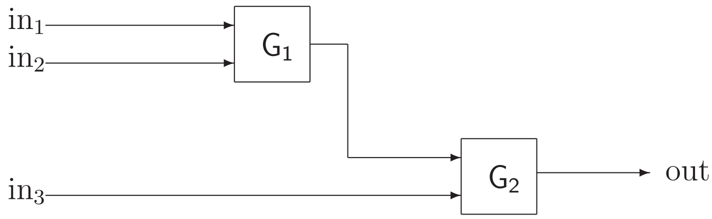

Figure 4.2: The circuit of Example 4.2.22.

$\curlywedge$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t},\mathbf{f}\}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$

We also use the connectives $\neg, \rightarrow, \vee$ with their corresponding classical interpretations. Denote by $\mathcal{M}$ the Nmatrix with such interpretations of $\curlywedge, \neg, \rightarrow, \vee$.

Furthermore, after experimenting with the circuit, we conclude that whenever one of the input lines of the circuit is on, then so is the output line. Hence our current knowledge can be represented by the following theory:

$$\Gamma = \Big\{ \ (in_1 \vee in_2 \vee in_3) \rightarrow out \ \Big\},$$

where out denotes the formula $((in_1 \curlywedge in_2) \curlywedge in_3)$. For convenience, we list the 11 possible partial valuations from $\Lambda_{\mathcal{M}}^{\downarrow \mathsf{SF}(\Gamma)}$ in Table 4.1. Two of these valuations are models of Γ. Thus for *every* setting of the form $\mathcal{S} = \langle \mathcal{M}, \langle d, \mathsf{d} \rangle, f \rangle$,

$$\Delta_{\mathcal{S}}(\Gamma) = mod_{\mathcal{M}}(\Gamma) = \left\{ \begin{array}{l} \nu_1 = \big\{in_1 \!:\! \mathbf{t},\ in_2 \!:\! \mathbf{t},\ in_3 \!:\! \mathbf{t},\ in_1 \curlywedge in_2 \!:\! \mathbf{t},\ out \!:\! \mathbf{t}\big\}, \\ \nu_{11} = \big\{in_1 \!:\! \mathbf{f},\ in_2 \!:\! \mathbf{f},\ in_3 \!:\! \mathbf{f},\ in_1 \curlywedge in_2 \!:\! \mathbf{f},\ out \!:\! \mathbf{f}\big\} \end{array} \right\},$$

Hence we can infer from Γ (using $\mid\!\sim_{\mathcal{S}}$):

(a) When all the input lines are off, so is the output line.

In fact, we can infer an even stronger conclusion:

(b) When all the input lines have the value b, the output line of the circuit is also b.

Suppose now that we receive a new piece of information from another source: the value of out is always different from the value of the output of G_1. This knowledge can be represented by the formula $\psi_2 = (in_1 \curlywedge in_2) \leftrightarrow \neg out$, and our current knowledge can be represented by the theory $\Gamma' = \Gamma \cup \big\{\psi_2\big\}$. It is easy to verify that the new information is inconsistent with our previous knowledge (i.e., Γ' is not $\mathcal{M}$-satisfiable). Of course,

	in_1	in_2	in_3	G_1	out	$\delta(\psi_1)$	$\delta(\psi_2)$	$\delta(\Gamma)$	$\delta(\Gamma')$
ν_1	**t**	**t**	**t**	**t**	**t**	0	1	**0**	**1**
ν_2	**t**	**t**	**t**	**t**	**f**	1	0	1	**1**
ν_3	**t**	**t**	**t**	**f**	**f**	1	1	1	2
ν_4	**t**	**t**	**f**	**t**	**f**	1	0	1	**1**
ν_5	**t**	**t**	**f**	**f**	**f**	1	1	1	2
ν_6	**t**	**f**	**t**	**f**	**f**	1	1	1	2
ν_7	**t**	**f**	**f**	**f**	**f**	1	1	1	2
ν_8	**f**	**t**	**t**	**f**	**f**	1	1	1	2
ν_9	**f**	**t**	**f**	**f**	**f**	1	1	1	2
ν_{10}	**f**	**f**	**t**	**f**	**f**	1	1	1	2
ν_{11}	**f**	**f**	**f**	**f**	**f**	0	1	**0**	**1**

Table 4.1: Distances to elements of $\Lambda_{\mathcal{M}}^{\downarrow\mathsf{SF}(\Gamma)}$ in Example 4.2.22. The following abbreviations are used: $\mathsf{G}_1 = (in_1 \curlywedge in_2)$, $\psi_1 = (in_1 \vee in_2 \vee in_3) \rightarrow out$, $out = ((in_1 \curlywedge in_2) \curlywedge in_3)$, $\psi_2 = (in_1 \curlywedge in_2) \leftrightarrow \neg out$. Also, $\delta(\cdot)$ abbreviates $\delta_{\mathbf{d_U},\Sigma}(\nu,\cdot)$ for the relevant valuation ν.

$\models_{\mathcal{M}}$ is trivialized in this case: everything can be inferred from Γ'. This, however, is not the case for $\mid\!\sim_{\mathcal{S}}$. For instance, for $\mathcal{S} = \langle \mathcal{M}, (\mathbf{d_U}, \mathsf{d}), \Sigma\rangle$, we have that in the notations of Table 4.1[4],

$$\Delta_{\mathcal{S}}(\Gamma') = \left\{ \begin{array}{l} \nu_1 = \{in_1{:}\mathbf{t},\ in_2{:}\mathbf{t},\ in_3{:}\mathbf{t},\ in_1 \curlywedge in_2{:}\mathbf{t},\ out{:}\mathbf{t}\}, \\ \nu_2 = \{in_1{:}\mathbf{t},\ in_2{:}\mathbf{t},\ in_3{:}\mathbf{t},\ in_1 \curlywedge in_2{:}\mathbf{t},\ out{:}\mathbf{f}\}, \\ \nu_4 = \{in_1{:}\mathbf{t},\ in_2{:}\mathbf{t},\ in_3{:}\mathbf{f},\ in_1 \curlywedge in_2{:}\mathbf{t},\ out{:}\mathbf{f}\}, \\ \nu_{11} = \{in_1{:}\mathbf{f},\ in_2{:}\mathbf{f},\ in_3{:}\mathbf{f},\ in_1 \curlywedge in_2{:}\mathbf{f},\ out{:}\mathbf{f}\} \end{array} \right\}.$$

Using $\mid\!\sim_{\mathcal{S}}$, we can still infer conclusion (a). This shows that unlike $\vdash_{\mathcal{M}}$, $\mid\!\sim_{\mathcal{S}}$ is inconsistency-tolerant. However, conclusion (b) is no longer valid. This shows that $\mid\!\sim_{\mathcal{S}}$ is non-monotonic (as will be discussed below).

Basic Properties of Distance-based Entailments

Below we consider some basic properties of the distance-based entailment relations defined above.

Proposition 4.2.23. *Let $\mathcal{S} = \langle \mathcal{M}, (d, x), f\rangle$ be a setting and suppose that Γ is satisfiable by an $\mathcal{M}$-valuation (a static one if $x = \mathsf{s}$ and a dynamic one if $x = \mathsf{d}$). Then for every*

[4] For simplicity of presentation, we write in Table 4.1 most but not all of the truth-values assigned to the subformulas of Γ'.

formula ψ*:* $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ *iff* $\Gamma \vdash^x_{\mathcal{M}} \psi$*.*

Proof. This is an immediate consequence of Proposition 4.2.18. □

Note that the above proposition does not imply that $\mid\!\sim_{\mathcal{S}}$ coincides with the $\vdash^x_{\mathcal{M}}$ relation. In fact, as we show below, $\mid\!\sim_{\mathcal{S}}$ is *not* even a consequence relation!

Definition 4.2.24. Two theories Γ_1 and Γ_2 are called *independent* if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.

Proposition 4.2.25. *Let* $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ *be a setting. For every* Γ *and every* ψ *such that* Γ *and* $\{\psi\}$ *are independent,* $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ *iff* ψ *is an* $\mathcal{M}$*-tautology.*

Proof. One direction is clear: if ψ is an $\mathcal{M}$-tautology, then for every $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, $\nu(\psi) = \mathbf{t}$ and so $\Gamma \mid\!\sim_{\mathcal{S}} \psi$. For the converse, suppose that ψ is not an $\mathcal{M}$-tautology. Then there is some $\mathcal{M}$-valuation ξ, such that $\xi(\psi) = \mathbf{f}$. Let $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. If $\nu(\psi) = \mathbf{f}$, we are done. Otherwise let μ be any $\mathcal{M}$-valuation, such that $\mu(\varphi) = \nu(\varphi)$ for every $\varphi \in \mathsf{SF}(\Gamma)$ and $\mu(\varphi) = \xi(\varphi)$ for $\varphi \in \mathsf{SF}(\psi)$. Note that such valuation exists by analycity of Nmatrices (Proposition 2.2.15) and the fact that Γ and $\{\psi\}$ are independent. By Proposition 4.2.14, $d^{\downarrow \mathsf{SF}(\Gamma)}(\nu, \varphi) = d^{\downarrow \mathsf{SF}(\Gamma)}(\mu, \varphi)$ for every $\varphi \in \Gamma$. Thus, $\delta^{\downarrow \mathsf{SF}(\Gamma)}_{d,f}(\nu, \Gamma) = \delta^{\downarrow \mathsf{SF}(\Gamma)}_{d,f}(\mu, \Gamma)$ and $\mu \in \Delta_{\mathcal{S}}(\Gamma)$. But $\mu(\psi) = \xi(\psi) = \mathbf{f}$ and so $\Gamma \not\mid\!\sim_{\mathcal{S}} \psi$. □

This leads us to the conclusion that the entailment relation $\mid\!\sim_{\mathcal{S}}$ is never trivialized:

Corollary 4.2.26. *For every (finite theory)* Γ *there is a formula* ψ*, such that* $\Gamma \not\mid\!\sim_{\mathcal{S}} \psi$*.*

Proof. Choose an atom $p \notin \mathsf{SF}(\Gamma)$. As Γ and $\{p\}$ are independent, by Proposition 4.2.25, $\Gamma \not\mid\!\sim_{\mathcal{S}} p$. □

An even stronger property can be established for settings with Nmatrices including classical negation:

Proposition 4.2.27. *Let* $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ *be a setting where* $\mathcal{M}$ *includes the classical negation. Then for every* Γ *and every* ψ*, if* $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ *then* $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg\psi$*.*

Proof. Suppose for contradiction that there is a formula ψ such that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ and $\Gamma \mid\!\sim_{\mathcal{S}} \neg\psi$. Then $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod^x_{\mathcal{M}}(\psi)$ and $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod^x_{\mathcal{M}}(\neg\psi)$. But $mod^x_{\mathcal{M}}(\psi) \cap mod^x_{\mathcal{M}}(\neg\psi) = \emptyset$, and so $\Delta_{\mathcal{S}}(\Gamma) = \emptyset$, in contradiction to the fact that $\Delta_{\mathcal{S}}(\Gamma) \neq \emptyset$ for every Γ (Proposition 4.2.18-1).

□

The above proposition implies that the entailment relation $\mid\!\sim_{\mathcal{S}}$ is in general non-monotonic. Indeed, consider any setting $\mathcal{S}$ satisfying the conditions of Proposition 4.2.27. By definition of $\mid\!\sim_{\mathcal{S}}$, it holds that $p \mid\!\sim_{\mathcal{S}} p$ and $\neg p \mid\!\sim_{\mathcal{S}} \neg p$. By Proposition 4.2.27 either $p, \neg p \not\mid\!\sim_{\mathcal{S}} p$ or $p, \neg p \not\mid\!\sim_{\mathcal{S}} \neg p$. Moreover, $\mid\!\sim_{\mathcal{S}}$ is not reflexive either. For instance, reflexivity does not hold for the setting $\mathcal{S}_1$ from Example 4.2.19: $\{p, q, \neg(p \diamond q)\} \not\mid\!\sim_{\mathcal{S}_1} q$. Thus the entailment relation $\mid\!\sim_{\mathcal{S}}$ is not a standard Tarskian consequence relation[5] (Definition 2.1.1). In the context of non-monotonic reasoning, however, it is usual to consider the following weaker notion of relation (see, e.g., [6, 97, 100, 104]):

Definition 4.2.28. A *cautious* consequence relation for $\mathcal{L}$ is a binary relation $\mid\!\sim$ between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas, satisfying the following conditions:

Cautious Reflexivity: $\psi \mid\!\sim \psi$.

Cautious Monotonicity [77]: if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$, then $\Gamma, \psi \mid\!\sim \phi$.

Cautious Transitivity [97]: if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$, then $\Gamma \mid\!\sim \phi$.

We show that for settings based on *hereditary* functions defined below, $\mid\!\sim_{\mathcal{S}}$ is indeed a cautious consequence relation.

Definition 4.2.29. We say that an aggregation function f is *hereditary* if for every $z_1, \ldots, z_m$ it holds that $f(\{x_1, \ldots, x_n\}) < f(\{y_1, \ldots, y_n\})$ implies $f(\{x_1, \ldots, x_n, z_1, \ldots, z_m\}) < f(\{y_1, \ldots, y_n, z_1, \ldots, z_m\})$.

For instance, summation is hereditary, while the maximum function is not.

Theorem 4.2.30. *Let $\mathcal{S} = \langle \mathcal{M}, (d, x), f \rangle$ be a setting where f is hereditary. Then $\mid\!\sim_{\mathcal{S}}$ is a cautious consequence relation.*

Proof. Cautious reflexivity follows directly from the definition of $\mid\!\sim_{\mathcal{S}}$. The proofs of the two other properties are a straightforward adaptation of the proofs from [4] for the classical case.

$\square$

The results presented above are only a first step towards developing a general framework combining distance-based considerations with Nmatrices. We have so far only focused on two-valued Nmatrices, while extending the framework to more than two values may lead to new ways of constructing useful distances along the lines of the general construction presented above. The computational aspects of this framework are also a question for further research. First steps towards investigating these aspects and analyzing several important special cases of distance-based entailments were made in [9, 13] for the deterministic case.

[5] It was shown in [4] that the properties of reflexivity, monotonicity and transitivity are violated already in distance-based entailments based on the classical matrix.

Chapter 5

Extending Nmatrices with Quantifiers

So far we have described the semantic framework of Nmatrices on the propositional level and presented a number of applications of this framework. The following part is devoted to extending the framework of Nmatrices to languages with quantifiers.

The simplest and most well-known quantifiers are of course the first-order quantifiers $\forall$ and $\exists$ (and they are discussed in Section 5.2.3 below). However, we will start by exploring a slightly more general notion of quantifiers. By a (unary) quantifier we mean a logical constant which (may) bind a variable when applied to a formula. In other words, if $\mathcal{Q}$ is a quantifier, x is a variable and ψ is a formula, then $\mathcal{Q}x\psi$ is a formula in which all occurrences of x are bound by $\mathcal{Q}$. We shall then further generalize this notion of quantifers to *multi-ary quantifiers*, which are logical constants that can be applied to more than one formula. If $\mathcal{Q}$ is an n-ary quantifier, x is a variable and $\psi_1, ..., \psi_n$ are formulas, then $\mathcal{Q}x(\psi_1, ..., \psi_n)$ is a formula in which all occurrences of x are bound by $\mathcal{Q}$.

5.1 Many-valued Matrices with Unary Quantifers

We start with a brief summary on ordinary (unary) quantifiers and their treatment in the framework of standard many-valued matrices. In what follows, L is a language, which includes a set of propositional connectives, a set of quantifiers, a countable set of variables, and a signature, consisting of a non-empty set of predicate symbols, a set of function symbols, and a set of constants. Frm_L is the set of (standardly defined) wffs of L, and Frm_L^{cl} is its set of closed wffs. Trm_L is the set of terms of L, and Trm_L^{cl} is its set of closed terms. In ordinary (deterministic) many-valued matrices (unary) quantifiers are standardly interpreted using the notion of *distributions*. This notion is due to Mostowski

([111]; the term 'distribution' was later coined in [53].

Definition 5.1.1. *Given a set of truth values $\mathcal{V}$, a* distribution *of a quantifier $\mathcal{Q}$ is a function $\lambda_{\mathcal{Q}} : (2^{\mathcal{V}} \setminus \{\emptyset\}) \to \mathcal{V}$.*

The following is a standard definition (see, e.g. [126]) of a deterministic matrix with distribution quantifiers:

Definition 5.1.2. A matrix for L is a tuple $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth-values,
- $\mathcal{D}$ is a non-empty proper set of $\mathcal{V}$,
- $\mathcal{O}$ includes a function $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective of L, and a function $\tilde{\mathcal{Q}} : 2^{\mathcal{V}} \setminus \{\emptyset\} \to \mathcal{V}$ for every quantifier of L.

Example 5.1.3. Consider the matrix $\mathcal{P} = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O} \rangle$ for a first-order language L, where $\mathcal{O}$ contains the following (standard) interpretations of $\forall$ and $\exists$:

$\mathbf{H}$	$\tilde{\forall}(\mathbf{H})$	$\tilde{\exists}(\mathbf{H})$
$\{\mathbf{t}\}$	$\mathbf{t}$	$\mathbf{t}$
$\{\mathbf{t}, \mathbf{f}\}$	$\mathbf{f}$	$\mathbf{t}$
$\{\mathbf{f}\}$	$\mathbf{f}$	$\mathbf{f}$

The notion of a structure is defined standardly:

Definition 5.1.4. Let $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for L. An L-structure S for $\mathcal{P}$ is a pair $\langle D, I \rangle$ where D is a (non-empty) domain and I is an interpretation of constants, predicate symbols and function symbols of L, which satisfies:

- For every constant c of L: $I(c) \in D$.
- For every n-ary predicate symbol p of L: $I(p) \in D^n \to \mathcal{V}$.
- For every n-ary function symbol f of L: $I(f) \in D^n \to D$.

There are two main approaches to interpreting quantified formulas: the objectual (referential) approach, which uses *assignments*, and the substitutional approach ([99]), which is based on *substitutions*. Below we shortly review these two approaches. In the better known objectual approach (used in most standard textbooks on classical first-order logic, like [108, 72, 127]), a variable is thought of as ranging over a set of objects from the domain, and assignments map variables to elements of the domain. In the context of many-valued deterministic matrices this is usually formalized as follows (see e.g. [126, 87]).

Definition 5.1.5. Given an L-structure $S = \langle D, I\rangle$, an *assignment* G in S is any function mapping the variables of L to D. For any $a \in D$ we denote by $G[x := a]$ the assignment which is similar to G, except that it assigns a to x. G is extended to L-terms as follows: $G(c) = I(c)$ for every constant c of L and $G(f(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(f)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$ for every n-ary function symbol f of L and $\mathbf{t}_1, ..., \mathbf{t}_n \in Trm_L$.

Definition 5.1.6. Let S be an L-structure for a matrix $\mathcal{P}$ and let G be an assignment in S. The valuation $v_{S,G} : Frm_L \to \mathcal{V}$ is defined as follows:

- $v_{S,G}(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$.
- $v_{S,G}(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v_{S,G}(\psi_1), ..., v_{S,G}(\psi_n))$.
- $v_{S,G}(\mathcal{Q}x\psi) = \tilde{\mathcal{Q}}(\{v_{S,G[x:=a]}(\psi) \mid a \in D\})$.

In the alternative substitutional approach to quantification (used e.g. for first-order classical logic in the classical textbook [120]) a variable is thought of as ranging over syntactical (closed) terms rather than over elements of the domain. Accordingly, the key notion in this approach is that of a *substitution instance* (rather than an assignment):

Definition 5.1.7. For any formula ψ, a *substitution L-instance of ψ* has the form $\psi\{\mathbf{t}_1/x_1, ..., \mathbf{t}_n/x_n\}$, where for all $1 \leq i \leq n$, $\mathbf{t}_i$ is an L-term free for x_i in ψ. A *substitution L-instance of* Γ is a set $\{\psi\{\mathbf{t}_1/x_1, ..., \mathbf{t}_n/x_n\} \mid \psi \in \Gamma\}$ for some $\mathbf{t}_1, ..., \mathbf{t}_n \in Trm_L$ which are free for $x_1, ..., x_n$ (respectively) in all the formulas of Γ.

The main idea of the substitutional approach is that a formula is interpreted in terms of its substitution instances. Thus a formula $\forall\psi x$ ($\exists x\psi$) is true if and only if each (at least one) of the closed substitution instances of ψ is true. To apply this approach, we need to assume that every element of the domain has a closed term referring to it. This condition can be satisfied by extending the language with individual constants:

Definition 5.1.8. For an L-structure $S = \langle D, I\rangle$ for a matrix $\mathcal{P}$, $L(D)$ is the language obtained from L by adding to it the set of *individual constants* $\{\overline{a} \mid a \in D\}$. The $L(D)$-structure which is induced by S is $\langle D, I'\rangle$, where I' is the unique extension of I to $L(D)$ such that $I'(\overline{a}) = a$. I' is extended to interpret closed terms of $L(D)$ as follows:

$$I'(f(\mathbf{t}_1, ..., \mathbf{t}_n)) = I'(f)(I'(\mathbf{t}_1), ..., I'(\mathbf{t}_n))$$

Henceforth we shall identify an L-structure S with the $L(D)$-structure which is induced by S.

Here is the substitutional counterpart of the notion of a valuation given in Definition 5.1.6:

Definition 5.1.9. Let $S = \langle D, I\rangle$ be an L-structure for a matrix $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$. The *valuation* $v_S : Frm^{\mathsf{cl}}_{L(D)} \to \mathcal{V}$ is defined as follows:

- $v_S(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$
- $v_S(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$
- $v_S(\mathcal{Q}x\psi) = \tilde{\mathcal{Q}}(\{v_S(\psi\{\overline{a}/x\}) \mid a \in D\})$

For reasons that will become clear in the sequel, in what follows we shall use the substitutional approach to define the consequence relations we are interested in, and not the objectual one.

Definition 5.1.10. Let $S = \langle D, I\rangle$ be an L-structure for a matrix $\mathcal{P} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$.

- The valuation v_S *satisfies* a sentence ψ (denoted by $v_S \models \psi$), if $v_S(\psi) \in \mathcal{D}$. v_S is a *model* of $\Gamma \subseteq Frm^{\mathsf{cl}}_{L(D)}$ (denoted by $v_S \models \Gamma$), if $v_S(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.
- v_S satisfies a formula $\varphi \in Frm_L$, denoted by $v_S \models \varphi$, if for every closed $L(D)$-instance φ' of φ, $v_S(\varphi') \in \mathcal{D}$. v_S satisfies a set of formulas $\Gamma \subseteq Frm_L$, denoted by $v_S \models \Gamma$, if for every closed $L(D)$-instance Γ' of Γ, $v_S \models \Gamma'$.

In contrast to the propositional case, there is more than one natural way of defining consequence relations induced by a given matrix when variables and quantifiers are involved. Two such relations which are usually associated with first-order logic are the truth and the validity consequence relations ([15]). Using the substitutional approach they can be generalized to the context of many-valued matrices as follows:

Definition 5.1.11.
- For sets of L-formulas Γ, Δ, we say that $\Gamma \vdash^t_{\mathcal{P}} \Delta$ if for every L-structure S and every closed $L(D)$-instance $\Gamma' \cup \Delta'$ of $\Gamma \cup \Delta$: $v_S \models \Gamma'$ implies that $v_S \models \psi$ for some $\psi \in \Delta'$.
- We say that $\Gamma \vdash^v_{\mathcal{P}} \Delta$ if for every L-structure S: $v_S \models \Gamma$ implies that $v_S \models \psi$ for some $\psi \in \Delta$.

To demonstrate the difference between the validity and the truth consequence relations, consider a matrix $\mathcal{P}$ for a first-order language L with the standard interpretations of the quantifiers $\forall$ and $\exists$ from Example 5.1.3. Then $p(x) \vdash^v_{\mathcal{P}} \forall x p(x)$, but $p(x) \nvdash^t_{\mathcal{P}} \forall x p(x)$. On the other hand, the classical deduction theorem holds for $\vdash^t_{\mathcal{P}}$, but not for $\vdash^v_{\mathcal{P}}$. However, the two consequence relations are identical from the point of view of theoremhood (i.e., $\vdash^t_{\mathcal{P}} \psi$ iff $\vdash^v_{\mathcal{P}} \psi$). This is a special case of the second part of the following well-known proposition:

Proposition 5.1.12. *Let $\mathcal{P}$ be a matrix for L.*

1. $\Gamma \vdash^t_{\mathcal{P}} \psi$ *implies* $\Gamma \vdash^v_{\mathcal{P}} \psi$.
2. *If* $\Gamma \subseteq Frm^{\mathsf{cl}}_L$ *(i.e,* Γ *consists of* sentences*), then* $\Gamma \vdash^t_{\mathcal{P}} \psi$ *iff* $\Gamma \vdash^v_{\mathcal{P}} \psi$.

5.2 Nmatrices with Unary Quantifiers

5.2.1 Basic Definitions

The extension of Nmatrices to languages with quantifiers is a natural generalization of Definition 5.1.2:

Definition 5.2.1. An Nmatrix for L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth-values,
- $\mathcal{D}$ is a non-empty proper set of $\mathcal{V}$,
- $\mathcal{O}$ includes a function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective of L, and a function $\tilde{\mathcal{Q}} : 2^{\mathcal{V}} \setminus \{\emptyset\} \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every quantifier of L.

Example 5.2.2. Consider the Nmatrix $\mathcal{M} = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O} \rangle$ for a first-order language L, where $\mathcal{O}$ contains the following (non-standard) interpretations of $\forall$ and $\exists$:

$\mathbf{H}$	$\tilde{\forall}(\mathbf{H})$	$\tilde{\exists}(\mathbf{H})$
$\{\mathbf{t}\}$	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{t}\}$
$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}, \mathbf{f}\}$
$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$

L-structures for Nmatrices are defined like in Definition 5.1.4. However, it seems difficult to apply the objectual approach to quantification in the context of Nmatrices. Intuitively, the difficulty is related to the fact that we refer to another (not always unique) valuation in the definition of a $\mathcal{M}$-legal valuation. Indeed, a naive generalization of the notion of a valuation $v_{S,G}$ from Definition 5.1.6 would be a valuation satisfying the following conditions:

1. $v_{S,G}(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(G(\mathbf{t}_1), ..., G(\mathbf{t}_n))$.
2. $v_{S,G}(\diamond(\psi_1, ..., \psi_n)) \in \tilde{\diamond}(v_{S,G}(\psi_1), ..., v_{S,G}(\psi_n))$.
3. $v_{S,G}(\mathcal{Q}x\psi) \in \tilde{\mathcal{Q}}[\{v_{S,G[x:=a]}(\psi) \mid a \in D\}]$.

However, the last condition is not well defined: a valuation $v_{S,G[x:=a]}$ is not necessarily unique, since, unlike in the deterministic case, an L-structure S and an assignment G *do not uniquely determine* the valuation. One possible alternative is to consider *all* such valuations, i.e. reformulating the last condition as follows:

$$v(\mathcal{Q}x\psi) \in \tilde{\mathcal{Q}}[\{v'(\psi) \mid a \in D \text{ and } v' \text{ is an } \mathcal{M}\text{-legal } S, G[x := a]\text{-valuation}\}].$$

But this is counter-intuitive, since all the choices of truth values made by v for the subsentences of ψ become irrelevant for the choice of $v(\mathcal{Q}x\psi)$. It is thus not clear which of the possible valuations should be chosen for computing $v_{S,G[x:=a]}(\mathcal{Q}x\psi)$, none of the alternatives seem to lead to a satisfactory solution.

The substitutional approach, in contrast, *is* suitable for the non-deterministic context.

Definition 5.2.3. Let $S = \langle D, I\rangle$ be an L-structure.

1. A set of sentences $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ is *closed under subsentences with respect to S* if (i) for every n-ary connective $\diamond$ of L: $\psi_1, ..., \psi_n \in W$ whenever $\diamond(\psi_1, ..., \psi_n) \in W$, and (ii) for every quantifier $\mathcal{Q}$ of L and every $a \in D$: if $\mathcal{Q}x\psi \in W$, then $\psi\{\overline{a}/x\} \in W$.

2. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S. We say that a partial S-valuation $v : W \to \mathcal{V}$ is *semi-legal in $\mathcal{M}$* if it satisfies the following conditions:

   - $v(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$
   - $v(\diamond(\psi_1, ..., \psi_n)) \in \tilde{\diamond}_{\mathcal{M}}(v(\psi_1), ..., v(\psi_n))$
   - $v(\mathcal{Q}x\psi) \in \tilde{\mathcal{Q}}(\{v(\psi\{\overline{a}/x\}) \mid a \in D\})$

   A partial S-valuation v in $\mathcal{M}$ is a (full) S-valuation if its domain is $Frm^{\mathsf{cl}}_{L(D)}$.

It is easy to see that the above notion of a valuation is now well-defined. This is due to the fact that the truth-value $v(\mathcal{Q}x\psi)$ depends on the truth-values assigned by v *itself* to the subsentences of $\mathcal{Q}x\psi$ (unlike in our previous attempt using objectual quantification, where $v_{S,G[x:=a]}$ was used in the definition of $v_{S,G}$).

Remark 5.2.4. It is important to stress the difference between our use of notation in the above definition and the one used in Definition 5.1.9. Given a (deterministic) matrix $\mathcal{P}$ and an L-structure S, the valuation v_S is uniquely determined by S and $\mathcal{P}$. However, this is not the case for non-deterministic valuations in an Nmatrix $\mathcal{M}$ (although S does determine the truth-values of the atomic sentences), and so we write "an S-valuation v" (compare to "the valuation v_S").

Definition 5.2.5. Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S, and let $v : W \to \mathcal{V}$ be a partial S-valuation.

- v *satisfies* a sentence $\psi \in W$ (denoted by $v \models \psi$), if $v(\psi) \in \mathcal{D}$. v is a *model* of $\Gamma \subseteq W$ (denoted by $v \models \Gamma$), if $v(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.
- v satisfies a formula $\varphi \in Frm_L$ (denoted by $v \models \varphi$), if for every closed $L(D)$-instance φ' of φ, ($v(\varphi')$ is defined and) $v(\varphi') \in \mathcal{D}$. v is a *model* of $\Gamma \subseteq Frm_L$ (denoted by $v \models \Gamma$), if for every closed $L(D)$-instance Γ' of Γ, $v \models \Gamma'$.

The following simple analycity property is analogous to that given in Proposition 2.2.15 for the propositional case:

Proposition 5.2.6. *Let $\mathcal{M}$ be an Nmatrix for L and S an L-structure for $\mathcal{M}$. Any partial S-valuation v, which is semi-legal in $\mathcal{M}$ can be extended to a full S-valuation, which is semi-legal in $\mathcal{M}$.*

5.2.2 The Principles of α-Equivalence and Identity

At this point we note two important problems concerning the above naive semantics, which do not arise on the propositional level. The first problem is related to the principle of α-equivalence, capturing the idea that the names of bound variables are immaterial. It is of course quite reasonable to expect that in any useful semantics two α-equivalent sentences are always assigned the same truth-value. However, this is not necessarily the case for valuations in Nmatrices as defined above. As an example, consider a language L_a with the unary connective $\neg$ and the quantifier $\forall$. Let $\mathcal{M}_a = \langle\{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O}\rangle$ be the Nmatrix for L_a with the standard (deterministic) interpretation of $\forall$ and the non-deterministic interpretation of $\neg$ given in Example 2.2.8. Let $S_a = \langle\{a\}, I_a\rangle$ be the simple L_a-structure, such that $I_a(c_a) = a$ and $I_a(p)(a) = \mathbf{f}$. Clearly, there is a $\mathcal{M}_a$-semi-legal S_a-valuation v, such that $v(\neg\forall x p(x)) = \mathbf{t}$ and $v(\neg\forall y p(y)) = \mathbf{f}$. Hence two α-equivalent formulas are not necessarily assigned the same truth-value by a $\mathcal{M}_a$-semi-legal S_a-valuation![1] The second problem is related to the nature of identity and becomes really crucial if equality is added to the language. Suppose we have two terms, denoting the same object. It is again reasonable to expect that we should be able to use these terms interchangeably, or substitute one term for another in any context. Returning to our example, suppose we add another constant d_a to the language L_a and extend

[1] Of course, two different occurrences of the same formula are still assigned the same truth-value, since a valuation is a mapping from *formulas* to truth-values.

the structure S_a to interpret it: $I(d_a) = a$. Thus the constants d_a and c_a refer to the same element a, but there is a $\mathcal{M}_a$-legal valuation v, such that $v(\neg p(c_a)) = \mathbf{t}$ and $v(\neg p(d_a)) = \mathbf{f}$.

These problems are directly related to introducing a new level of freedom by the non-deterministic choice of truth-values for quantified formulas. In view of these issues, further limitations need to be imposed on this choice. This can be done by introducing the following congruence relation, capturing these principles.

Definition 5.2.7. Let $S = \langle D, I \rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ between terms of $L(D)$ is defined as follows:

- $x \sim^S x$ for every variable x of L.
- If $\mathbf{t}, \mathbf{t}' \in Trm^{\mathsf{cl}}_{L(D)}$ and $I(\mathbf{t}) = I(\mathbf{t}')$, then $\mathbf{t} \sim^S \mathbf{t}'$.
- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $f(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S f(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.

The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:

- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, \mathbf{t}_2 \sim^S \mathbf{t}'_2, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $p(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S p(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.
- If $\psi_i \sim^S \varphi_i$ for all $1 \leq i \leq n$, then $\diamond(\psi_1, ..., \psi_n) \sim^S \diamond(\varphi_1, ..., \varphi_n)$ for every n-ary connective $\diamond$ of L.
- If $\psi\{z/x\} \sim^S \varphi\{z/y\}$, where x, y are distinct variables and z is a new[2] variable, then $\mathcal{Q}x\psi \sim^S \mathcal{Q}y\varphi$ for every quantifier $\mathcal{Q}$ of L.

The following lemmas can be easily proved:

Lemma 5.2.8. *Let $S = \langle D, I \rangle$ be an L-structure. For every two terms $\boldsymbol{s}_1, \boldsymbol{s}_2$ of $L(D)$, if $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$ then one of the following holds:*

- *$\boldsymbol{s}_1 = \boldsymbol{s}_2 = x$ for some variable x of L.*
- *$\boldsymbol{s}_1$, $\boldsymbol{s}_2 \in Trm^{\mathsf{cl}}_{L(D)}$ and $I(\boldsymbol{s}_1) = I(\boldsymbol{s}_2)$.*
- *$\boldsymbol{s}_1 = f(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$, $\boldsymbol{s}_2 = f(\boldsymbol{t}'_1, \ldots, \boldsymbol{t}'_n)$ and for all $1 \leq i \leq n$: $\boldsymbol{t}_i \sim^S \boldsymbol{t}'_i$.*

Lemma 5.2.9. *Let $S = \langle D, I \rangle$ be an L-structure. For every two formulas ψ, φ of $L(D)$, if $\psi \sim^S \varphi$ then one of the following holds:*

- *$\psi = p(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ and $\varphi = p(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$, where $\boldsymbol{t}_i \sim^S \boldsymbol{s}_i$ for all $1 \leq i \leq n$.*

[2] It is easy to check that the definition is independent of the choice of z.

- $\psi = \diamond(\psi_1, \dots, \psi_n)$ *and* $\varphi = \diamond(\varphi_1, \dots, \varphi_n)$ *for some* n*-ary connective* $\diamond$ *of* L*, and for all* $1 \le i \le n$*:* $\psi_i \sim^S \varphi_i$.
- $\psi = \mathcal{Q}x\psi_0$ *and* $\varphi = \mathcal{Q}y\varphi_0$ *for some quantifier* $\mathcal{Q}$ *of* L*, and for any fresh variable* z*:* $\psi_0\{z/x\} \sim^S \varphi_0\{z/x\}$.

Lemma 5.2.10. *Let* S *be an* L*-structure.*

1. *If* $\psi \sim^S \varphi$*, then* $Fv(\psi) = Fv(\varphi)$.
2. *If* $\boldsymbol{t}_1, \boldsymbol{t}_2 \in Trm^{\mathsf{cl}}_{L(D)}$*, then* $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$ *iff* $I(\boldsymbol{t}_1) = I(\boldsymbol{t}_2)$.

Lemma 5.2.11. *Let* S *be an* L*-structure. Let* $\boldsymbol{t}_1, \boldsymbol{t}_2$ *be closed terms of* $L(D)$ *such that* $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$*. Let* ψ_1, ψ_2 *be* $L(D)$*-formulas such that* $\psi_1 \sim^S \psi_2$*. Then for any variable* x*:* $\psi_1\{\boldsymbol{t}_1/x\} \sim^S \psi_2\{\boldsymbol{t}_2/x\}$.

Proof. First it is easy to prove that $(*)$ for every two $L(D)$-terms $\mathbf{s}_1, \mathbf{s}_2$, such that $\mathbf{s}_1 \sim^S \mathbf{s}_2$ it holds that $\mathbf{s}_1\{\mathbf{t}_1/x\} \sim^S \mathbf{s}_2\{\mathbf{t}_2/x\}$. The proof is by induction on the structure of $\mathbf{s}_1$ and $\mathbf{s}_2$. Next, suppose that $\psi_1 \sim^S \psi_2$. We prove the lemma by induction on the structure of ψ_1 and ψ_2:

- If ψ_1, ψ_2 are atomic formulas, then $\psi_1 = p(\mathbf{s}^1_1, \dots, \mathbf{s}^1_n)$ and $\psi_2 = p(\mathbf{s}^2_1, \dots, \mathbf{s}^2_n)$, where $\mathbf{s}^1_j \sim^S \mathbf{s}^2_j$ for all $1 \le i \le n$. The claim follows by $(*)$ above.
- $\psi_1 = \diamond(\phi^1_1, \dots, \phi^1_n)$ and $\psi_2 = \diamond(\phi^2_1, \dots, \phi^2_n)$, where $\phi^1_j \sim^S \phi^2_j$ for all $1 \le j \le n$. By the induction hypothesis, $\phi^1_j\{\mathbf{t}_1/x\} \sim^S \phi^2_j\{\mathbf{t}_2/x\}$. Hence, $\psi_1\{\mathbf{t}_1/x\} = \diamond(\phi^1_1\{\mathbf{t}_1/x\}, \dots, \phi^1_n\{\mathbf{t}_1/x\}) \sim^S \diamond(\phi^2_1\{\mathbf{t}_2/x\}, \dots, \phi^2_n\{\mathbf{t}_2/x\}) = \psi_2\{\mathbf{t}_2/x\}$.
- $\psi_1 = \mathcal{Q}y\phi^1$ and $\psi_2 = \mathcal{Q}z\phi^2$. Then $\phi^1\{w/y\} \sim^S \phi^2\{w/z\}$ for any fresh variable w. Pick such a fresh variable $w \neq x$. By the induction hypothesis, $\phi^1\{w/y\}\{\mathbf{t}_1/x\} \sim^S \phi^2\{w/z\}\{\mathbf{t}_2/x\}$. By Lemma 5.2.10-1, one of the following cases holds:
  - $x \notin Fv(\psi_1) \cup Fv(\psi_2)$. Then $\psi_1\{\mathbf{t}_1/x\} = \psi_1 \sim^S \psi_2 = \psi_2\{\mathbf{t}_2/x\}$.
  - $x \in Fv(\psi_1) \cap Fv(\psi_2)$. Then $x \neq z$ and $x \neq y$, and it holds that $\psi_1\{\mathbf{t}_1/x\} = \mathcal{Q}y(\phi^1\{\mathbf{t}_1/x\})$ and $\psi_2\{\mathbf{t}_2/x\} = \mathcal{Q}z(\phi^2\{\mathbf{t}_2/x\})$. Since $\mathbf{t}_1, \mathbf{t}_2$ are closed terms, $\phi^1\{w/y\}\{\mathbf{t}_1/x\} = \phi^1\{\mathbf{t}_1/x\}\{w/y\}$ and $\phi^2\{w/z\}\{\mathbf{t}_2/x\} = \phi^2\{\mathbf{t}_2/x\}\{w/z\}$. Hence, $\phi^1\{w/y\}\{\mathbf{t}_1/x\} \sim^S \phi^1\{\mathbf{t}_1/x\}\{w/y\}$, and so $\psi_1\{\mathbf{t}_1/x\} = \mathcal{Q}y(\phi^1\{\mathbf{t}_1/x\}) \sim^S \mathcal{Q}z(\phi^2\{\mathbf{t}_2/x\}) = \psi_2\{\mathbf{t}_2/x\}$.

□

Using the above congruence relation, we can now modify Definition 5.2.3 as follows:

Definition 5.2.12. Let S be an L-structure and $\mathcal{M}$ an Nmatrix for L. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S. A partial S-valuation $v : W \to \mathcal{V}$ is $\sim^S$*-legal in* $\mathcal{M}$ if it is semi-legal in $\mathcal{M}$ and for every $\psi, \varphi \in W$: $\psi \sim^S \varphi$ implies $v(\psi) = v(\varphi)$.

Now we come to the definition of consequence relations induced by Nmatrices, analogous to Definition 5.1.11:

Definition 5.2.13.
- For sets of L-formulas Γ, Δ, we say that $\Gamma \vdash^t_{\mathcal{M}} \Delta$ if for every L-structure S, every S-valuation v which is $\sim^S$-legal in $\mathcal{M}$, and every closed $L(D)$-instance $\Gamma' \cup \Delta'$ of $\Gamma \cup \Delta$: $v \models \Gamma'$ implies $v \models \psi$ for some $\psi \in \Delta'$.
- We say that $\Gamma \vdash^v_{\mathcal{M}} \Delta$ if for every L-structure S and S-valuation v which is $\sim^S$-legal in $\mathcal{M}$: $v \models \Gamma$ implies $v \models \psi$ for some $\psi \in \Delta$.

The following is an extension of Proposition 5.1.12 to the context of Nmatrices:

Proposition 5.2.14. *Let $\mathcal{M}$ be an Nmatrix for L.*

1. $\Gamma \vdash^t_{\mathcal{M}} \psi$ *implies* $\Gamma \vdash^v_{\mathcal{M}} \psi$.
2. *If* $\Gamma \subseteq Frm^{\mathsf{cl}}_L$ *(i.e,* Γ *consists of sentences), then* $\Gamma \vdash^t_{\mathcal{M}} \psi$ *iff* $\Gamma \vdash^v_{\mathcal{M}} \psi$.

Proof. Let us show the proof for the first part. Assume that $\Gamma \vdash^t_{\mathcal{M}} \psi$. Let $S = \langle D, I \rangle$ be an L-structure and v a valuation which is $\sim^S$-legal in $\mathcal{M}$, such that $v \models \Gamma$. Let $\Gamma' \cup \{\psi'\}$ be some closed $L(D)$-instance of $\Gamma \cup \{\psi\}$. Then $v \models \Gamma'$ and by our assumption, $v \models \psi'$. Thus $v \models \psi$ and so $\Gamma \vdash^v_{\mathcal{M}} \psi$.
The proof for the second part is similar to the proof for ordinary matrices.

□

In analogy to the propositional case (see Definition 2.2.6), consequence relations induced by an Nmatrix can be defined not only between sets of formulas, but also between sets of sequents and sequents:

Definition 5.2.15. Let $\mathcal{M}$ be an Nmatrix. Let $S = \langle D, I \rangle$ be an L-structure for $\mathcal{M}$.

1. Let v be an $\mathcal{M}$-legal S-valuation. v is *a model* of a closed sequent $\Gamma \Rightarrow \Delta$, denoted by $v \models \Gamma \Rightarrow \Delta$ if whenever $S, v \models \psi$ for every $\psi \in \Gamma$, there is some $\varphi \in \Delta$, such that $v \models \varphi$. A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$*-valid in* $\langle S, v \rangle$ if for every closed $L(D)$-instance $\Gamma' \Rightarrow \Delta'$ of $\Gamma \Rightarrow \Delta$: $v \models \Gamma' \Rightarrow \Delta'$.
2. For a set of sequents Θ, $\Theta \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ if for every L-structure S and every $\mathcal{M}$-legal S-valuation v: whenever the sequents of Θ are $\mathcal{M}$-valid in $\langle S, v \rangle$, $\Gamma \Rightarrow \Delta$ is also $\mathcal{M}$-valid in $\langle S, v \rangle$.

3. We say that a calculus G is *strongly sound* for an Nmatrix $\mathcal{M}$ if whenever $\Theta \vdash_G \Gamma \Rightarrow \Delta$, also $\Theta \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$. G is *strongly complete* for $\mathcal{M}$ if whenever $\Theta \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, also $\Theta \vdash_G \Gamma \Rightarrow \Delta$. $\mathcal{M}$ is *strongly characteristic* for G if G is both strongly sound and strongly complete for $\mathcal{M}$.

As for analycity, we now prove the following analogue of Proposition 5.2.6:

Proposition 5.2.16. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for L and $S = \langle D, I \rangle$ an L-structure. Then any partial S-valuation which is $\sim^S$-legal in $\mathcal{M}$ can be extended to a full S-valuation which is $\sim^S$-legal in $\mathcal{M}$.*

Proof. Let $S = \langle D, I \rangle$ be an L-structure. Let v_p be some partial S-valuation which is $\sim^S$-legal in $\mathcal{M}$. Suppose that v_p is defined on some set of sentences $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ closed under subsentences with respect to S. We construct an extension of v_p to a full S-valuation v which is $\sim^S$-legal in $\mathcal{M}$.
For every n-ary connective $\diamond$ of L and every $a_1, ..., a_n \in \mathcal{V}$, choose an arbitrary truth-value $\mathbf{b}^{\diamond}_{a_1,...,a_n} \in \tilde{\diamond}(a_1, ..., a_n)$. Similarly, for every quantifier $\mathcal{Q}$ of L and every $B \subseteq P^+(\mathcal{V})$, choose an arbitrary truth-value $\mathbf{b}^{\mathcal{Q}}_B \in \tilde{\mathcal{Q}}(B)$.
Denote by $H_{\sim^S}$ the set of all equivalence classes of $Frm^{\mathsf{cl}}_{L(D)}$ under $\sim^S$. Denote by $[\![\psi]\!]$ the equivalence class of ψ. Define the function $\chi : H_{\sim^S} \to \mathcal{V}$ as follows:

$$\chi([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!]) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$$

$$\chi([\![\diamond(\psi_1, ..., \psi_n)]\!]) = \begin{cases} v_p(\varphi) & \varphi \in ([\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W) \\ \mathbf{b}^{\diamond}_{\chi([\![\psi_1]\!]),...,\chi([\![\psi_n]\!])} & \text{there is no } \varphi \in ([\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W) \end{cases}$$

$$\chi([\![\mathcal{Q}x\psi]\!]) = \begin{cases} v_p(\varphi) & \varphi \in ([\![\mathcal{Q}x\psi]\!] \cap W) \\ \mathbf{b}^{\mathcal{Q}}_{\{\chi([\![\psi\{\overline{a}/x\}]\!]) \mid a \in D\}} & \text{there is no } \varphi \in ([\![\mathcal{Q}x\psi]\!] \cap W) \end{cases}$$

Let us show that χ is well-defined. First of all, note that the above definition does not depend on the choice of $\varphi \in W$ if such φ exists, as for every two $\varphi_1, \varphi_2 \in [\![\psi]\!] \cap W$ for any ψ: $\varphi_1 \sim^S \varphi_2$, and since v_p is $\sim^S$-legal, $v_p(\varphi_1) = v_p(\varphi_2)$. Secondly, we show that the definition does not depend on the representatives of the equivalence class of ψ. We prove that if $\varphi_1, \varphi_2 \in [\![\psi]\!]$ then $\chi([\![\varphi_1]\!]) = \chi([\![\varphi_2]\!])$ by induction on ψ:

- $\psi = p(\mathbf{t}_1, \ldots, \mathbf{t}_n)$. Then since $\varphi_1 \sim^S \varphi_2$, by Lemma 5.2.9, $\varphi_1 = p(\mathbf{s}_1, \ldots, \mathbf{s}_n)$, $\varphi_2 = p(\mathbf{s}'_1, \ldots, \mathbf{s}'_n)$ and $\mathbf{s}_i \sim^S \mathbf{s}'_i$ for all $1 \leq i \leq n$. By Lemma 5.2.10-2: $I(\mathbf{s}_i) = I(\mathbf{s}'_i)$. Hence it holds that $\chi([\![\varphi_1]\!]) = I(p)(I(\mathbf{s}_1), \ldots, I(\mathbf{s}_n)) = I(p)(I(\mathbf{s}'_1), \ldots, I(\mathbf{s}'_n)) = \chi([\![\varphi_2]\!])$.

- $\psi = \diamond(\psi_1, \ldots, \psi_n)$. Then since $\varphi_1 \sim^S \varphi_2$, by Lemma 5.2.9: $\varphi_1 = \diamond(\phi_1, \ldots, \phi_n)$, $\varphi_2 = \diamond(\phi'_1, \ldots, \phi'_n)$ and $\phi_i \sim^S \phi'_i$ for all $1 \leq i \leq n$. If there is some $\varphi \in [\![\psi]\!] \cap W$, then $\chi([\![\varphi_1]\!]) = \chi([\![\varphi_2]\!]) = v_p(\varphi)$. Otherwise, $\chi([\![\varphi_1]\!]) = \mathbf{b}^{\diamond}_{\chi([\![\phi_1]\!]),\ldots,\chi([\![\phi_n]\!])}$ and $\chi([\![\varphi_2]\!]) = \mathbf{b}^{\diamond}_{\chi([\![\phi'_1]\!]),\ldots,\chi([\![\phi'_n]\!])}$. By the induction hypothesis, $\chi([\![\phi_i]\!]) = \chi([\![\phi'_i]\!])$ and so $\chi([\![\varphi_1]\!]) = \chi([\![\varphi_2]\!])$.

- $\psi = \mathcal{Q}x\varphi$. The proof is similar to the previous case.

Next define v as follows for every $\psi \in Frm^{\mathsf{cl}}_L$:

$$v(\psi) = \chi([\![\psi]\!])$$

Obviously, v respects the $\sim^S$ relation. It remains to show that v is legal in $\mathcal{M}$:

- Let $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$. Then $v(\psi) = \chi([\![p(\mathbf{t}_1, \ldots, \mathbf{t}_n)]\!]) = I(p)(I(\mathbf{t}_1), \ldots, I(\mathbf{t}_n))$.

- Let $\psi = \diamond(\psi_1, ..., \psi_n)$. Suppose that there is some $\varphi \in ([\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W$. By Lemma 5.2.9, φ is of the form $\diamond(\varphi_1, ..., \varphi_n)$, where $\varphi_i \sim^S \psi_i$ for all $1 \leq i \leq n$. Since W is closed under subsentences, $\varphi_1, ..., \varphi_n \in W$. By definition of v, $v(\psi) = v_p(\varphi) \in \tilde{\diamond}(v(\varphi_1), ..., v(\varphi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$ (since v_p is legal in $\mathcal{M}$, $v(\psi_i) = v(\varphi_i)$ and by the induction hypothesis, $[\![\varphi_i]\!] = [\![\psi_i]\!]$). Otherwise $v(\psi) = \mathbf{b}^{\diamond}_{\chi([\![\psi_1]\!]),\ldots,\chi([\![\psi_n]\!])} \in \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$.

- Let $\psi = \mathcal{Q}x\phi$. Suppose that there is some $\varphi \in ([\![\mathcal{Q}x\phi]\!] \cap W$. By Lemma 5.2.9, φ is of the form $\mathcal{Q}y\varphi'$, where $\varphi'\{z/x\} \sim^S \phi\{z/y\}$ for a fresh variable z. Since W is closed under subsentences, for every $a \in D$: $\varphi\{\overline{a}/y\} \in W$. Then $v(\mathcal{Q}x\phi) = v_p(\mathcal{Q}y\varphi') \in \tilde{\mathcal{Q}}(\{v(\varphi'\{\overline{a}/y\}) \mid a \in D\})$. By Lemma 5.2.11, $\varphi'\{\overline{a}/y\} = \varphi'\{z/y\}\{\overline{a}/z\} \sim^S \phi\{z/x\}\{\overline{a}/z\} = \phi\{\overline{a}/x\}$. Thus by the induction hypothesis we have $v(\mathcal{Q}x\phi) \in \tilde{\mathcal{Q}}(\{v(\phi\{\overline{a}/x\}) \mid a \in D\})$.
  Otherwise, $v(\mathcal{Q}x\phi) = \mathbf{b}^{\mathcal{Q}}_{\{\chi([\![\phi\{\overline{a}/x\}]\!]) \mid a \in D\}} \in \tilde{\mathcal{Q}}(\{v(\phi\{\overline{a}/x\}) \mid a \in D\})$.

□

We end this section by generalizing the notions of reduction and refinement from Definition 2.2.18 to languages with quantifiers:

Definition 5.2.17. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be two Nmatrices for L.

1. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$, such that:

    - For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$.

- $F(y) \in \tilde{\diamond}_{\mathcal{M}_2}(F(x_1), ..., F(x_n))$ for every n-ary connective $\diamond$ of L and every $x_1, ..., x_n, y \in \mathcal{V}_1$, such that $y \in \tilde{\diamond}_{\mathcal{M}_1}(x_1, ..., x_n)$.
- $F(y) \in \tilde{\mathcal{Q}}_{\mathcal{M}_2}(\{F(z) \mid z \in H\})$ for every quantifier $\mathcal{Q}$ of L, every $y \in \mathcal{V}_1$ and $H \in 2^{\mathcal{V}_1} \setminus \{\emptyset\}$, such that $y \in \tilde{\mathcal{Q}}_{\mathcal{M}_1}(H)$.

2. $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.

Theorem 5.2.18. *If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ then $\vdash^t_{\mathcal{M}_2} \subseteq \vdash^t_{\mathcal{M}_1}$ and $\vdash^v_{\mathcal{M}_2} \subseteq \vdash^v_{\mathcal{M}_1}$.*

Proof. Let $\mathcal{M}_1$ be a refinement of $\mathcal{M}_2$ and suppose that $\Gamma \vdash^t_{\mathcal{M}_2} \psi$. Then there exists a reduction $F : \mathcal{V}_1 \to \mathcal{V}_2$ of $\mathcal{M}_1$ to $\mathcal{M}_2$. Assume for contradiction that $\Gamma \not\vdash^t_{\mathcal{M}_1} \psi$. Then there is some L-structure $S = \langle D, I \rangle$, an S-valuation v which is $\sim^S$-legal in $\mathcal{M}_1$ and a closed $L(D)$-instance $\Gamma' \cup \{\psi'\}$ of $\Gamma \cup \{\psi\}$, such that $v \models \Gamma'$ but $v \not\models \psi'$.
Define the L-structure $S' = \langle D, I' \rangle$, where:

- $I'(c) = I(c)$ and $I(f) = I'(f)$.
- For every $a_1, \ldots, a_n \in D$: $I'(p)(a_1, \ldots, a_n) = F(I(p)(a_1, \ldots, a_n))$.

It is easy to see that for every closed $L(D)$-term $\mathbf{t}$: $I(\mathbf{t}) = I'(\mathbf{t})$.
Next define the S-valuation v' in $\mathcal{M}_2$ as follows:

$$v'(\psi) = F(v(\psi))$$

Let us show that v' is $\sim^S$-legal in $\mathcal{M}_2$. Clearly, v' respects the $\sim^S$-relation (since v is $\sim^S$-legal). It remains to show that v' respects the interpretations of the connectives and quantifiers in $\mathcal{M}_2$:

- $\phi = p(\mathbf{t}_1, \ldots, \mathbf{t}_n)$. Then $v'(\phi) = F(v(p(\mathbf{t}_1, \ldots, \mathbf{t}_n))) = F(I(p)(I(\mathbf{t}_1), \ldots, I(\mathbf{t}_n)))$ $= I'(p)(I'(\mathbf{t}_1), \ldots, I'(\mathbf{t}_n))$.
- $\phi = \diamond(\varphi_1, \ldots, \varphi_n)$. Since $v(\diamond(\varphi_1, \ldots, \varphi_n)) \in \tilde{\diamond}_{\mathcal{M}_1}(v(\varphi_1), \ldots, v(\varphi_n))$, by definition of a refinement, $F(v(\diamond(\varphi_1, \ldots, \varphi_n))) \in \tilde{\diamond}_{\mathcal{M}_2}(F(v(\varphi_1)), \ldots, F(v(\varphi_n)))$, and so it holds that $v'(\diamond(\varphi_1, \ldots, \varphi_n))$ is in $\tilde{\diamond}_{\mathcal{M}_2}(v'(\varphi_1), \ldots, v'(\varphi_n))$.
- $\phi = \mathcal{Q}x\varphi$. Since it holds that $v(\mathcal{Q}x\varphi) \in \tilde{\mathcal{Q}}_{\mathcal{M}_1}(\{v(\varphi\{\overline{a}/x\}) \mid a \in D\})$, $F(v(\mathcal{Q}x\varphi))$ is in the set $\tilde{\mathcal{Q}}_{\mathcal{M}_2}(\{F(v(\varphi\{\overline{a}/x\})) \mid a \in D\})$. Thus $v'(\mathcal{Q}x\varphi) \in \tilde{\mathcal{Q}}_{\mathcal{M}_2}(\{v'(\varphi\{\overline{a}/x\}) \mid a \in D\})$.

We have shown that v' is $\sim^S$-legal in $\mathcal{M}_2$. Since $v \models \Gamma'$ and $v \not\models \psi'$ it must be the case that $v' \models \Gamma'$ and $v' \not\models \psi'$ (recall that by the properties of reduction, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$). Thus $\Gamma \not\vdash^t_{\mathcal{M}_2} \psi$, in contradiction to our assumption.

Now suppose that $\Gamma \vdash^v_{\mathcal{M}_2} \psi$ and assume for contradiction that $\Gamma \not\vdash^v_{\mathcal{M}_1} \psi$. Then there is some L-structure $S = \langle D, I\rangle$ and an S-valuation v which is $\sim^S$-legal in $\mathcal{M}_1$, such that $v \models \Gamma$, but $v \not\models \psi$. Define the S-valuation v', which is $\sim^S$-legal in $\mathcal{M}_2$ like in the proof of the first part above. Then for every closed $L(D)$-formula φ: $v' \models \varphi$ iff $v \models \varphi$. Hence $v' \models \Gamma$, but there is some closed $L(D)$-instance ψ' of ψ, such that $v' \not\models \psi'$. Thus $\Gamma \not\vdash^v_{\mathcal{M}_2} \psi$, in contradiction to our assumption.

$\square$

Remark 5.2.19. Again an important case in which $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1\rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2\rangle$ is when $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, $\tilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every n-ary connective $\diamond$ of L and every $\vec{x} \in \mathcal{V}_1^n$, and $\tilde{\mathcal{Q}}_{\mathcal{M}_1}(H) \subseteq \tilde{\mathcal{Q}}_{\mathcal{M}_2}(H)$ for every quantifier $\mathcal{Q}$ of L and every $H \in 2^{\mathcal{V}_1} \setminus \{\emptyset\}$. It is easy to see that the identity function on $\mathcal{V}_1$ is in this case a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. We will refer to this kind of refinement as *simple*.

5.2.3 The Principle of Void Quantification

In addition to the principles treated in the last subsection, we consider also another principle, which is closely related to the assumption of the non-emptiness of our domain. Namely, it is natural to assume that if a formula ψ' can be obtained from ψ by a deletion (or addition) of void quantifiers (that is, quantifiers that do not bind any variables), then ψ and ψ' should be equivalent, and hence should be assigned the same truth-value in any reasonable semantic framework. This principle seems particularly natural for the first-order quantifiers $\forall$ and $\exists$: for instance, one would definitely expect $\neg\forall x p(c)$ and $\neg p(c)$ to be equivalent. This, however, is not always the case under our current definition of a $\sim^S$-legal valuation (Definition 5.2.12). For an example, consider again the Nmatrix $\mathcal{M}_a = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O}\rangle$ discussed in the previous section, where $\neg$ is interpreted like in Example 2.2.8, and the quantifier $\forall$ is interpreted classically. Clearly, there exists an L-structure S and an S-valuation v legal in $\mathcal{M}_a$, such that $v(\neg\forall x p(c)) = \mathbf{t}$, but $v(\neg p(c)) = \mathbf{f}$.

Our solution is to extend the congruence relation $\sim^S$ to capture the principle of void quantification.

Definition 5.2.20. Let $S = \langle D, I\rangle$ be an L-structure. The relation $\sim^S_{vo}$ on $L(D)$-formulas is the minimal congruence relation on formulas of $L(D)$, which satisfies: (i) $\sim^S \subseteq \sim^S_{vo}$, and (ii) if $\psi \sim^S_{vo} \psi'$ and x does not occur free in ψ, then $\mathcal{Q}x\psi \sim^S_{vo} \psi'$.

Lemma 5.2.21. *Let $S = \langle D, I\rangle$ be an L-structure.*

1. *If $\psi_1 \sim^S_{vo} \psi'_1, \ldots, \psi_n \sim^S_{vo} \psi'_n$, then $\diamond(\psi_1, \ldots, \psi_n) \sim^S_{vo} \diamond(\psi'_1, \ldots, \psi'_n)$.*

2. *If $\psi_1\{w/x\} \sim^S_{vo} \psi_2\{w/y\}$ for a new variable w, then $\mathcal{Q}x\psi_1 \sim^S_{vo} \mathcal{Q}y\psi_2$.*

Proof. The first part follows from the fact that $\sim^S_{vo}$ is a congruence relation. For the second part, assume that $\psi_1\{w/x\} \sim^S_{vo} \psi_2\{w/y\}$ for a new variable w. Again, since $\sim^S_{vo}$ is a congruence relation, $\mathcal{Q}w\psi_1\{w/x\} \sim^S_{vo} \mathcal{Q}w\psi_2\{w/y\}$. But $\mathcal{Q}w\psi_1\{w/x\} \sim^S \mathcal{Q}x\psi_1$ (since these formulas are α-equivalent). Similarly, $\mathcal{Q}w\psi_2\{w/y\} \sim^S \mathcal{Q}y\psi_2$. By transitivity of $\sim^S_{vo}$ and the fact that $\sim^S \subseteq \sim^S_{vo}$, $\mathcal{Q}x\psi_1 \sim^S_{vo} \mathcal{Q}y\psi_2$.

□

The following lemmas can be proved by a tedious induction on $\sim^S_{vo}$:

Lemma 5.2.22. *Let $S = \langle D, I\rangle$ be an L-structure. For every two formulas ψ, φ of $L(D)$, if $\psi \sim^S_{vo} \varphi$ then $Fv(\psi) = Fv(\varphi)$ and one of the following holds:*

- *$\psi = p(\boldsymbol{t}_1, \dots, \boldsymbol{t}_n)$ and $\varphi = p(\boldsymbol{s}_1, \dots, \boldsymbol{s}_n)$, where $\boldsymbol{t}_i \sim^S \boldsymbol{s}_i$ for all $1 \leq i \leq n$.*
- *$\psi = \diamond(\psi_1, \dots, \psi_n)$, $\varphi = \diamond(\varphi_1, \dots, \varphi_n)$ and for all $1 \leq i \leq n$: $\psi_i \sim^S \varphi_i$.*
- *$\psi = \mathcal{Q}x\psi_0$ and $\varphi = \mathcal{Q}y\varphi_0$ and for any fresh variable z: $\psi_0\{z/x\} \sim^S \varphi_0\{z/x\}$.*
- *$\psi = \mathcal{Q}x\psi_0$, $x \notin Fv(\psi_0)$ and $\psi_0 \sim^S_{vo} \varphi$.*
- *$\varphi = \mathcal{Q}x\varphi_0$, $x \notin Fv(\varphi_0)$ and $\varphi_0 \sim^S_{vo} \psi$.*

Lemma 5.2.23. *If ψ' is obtained from ψ by deletion of void quantifiers, then $\psi \sim^S_{vo} \psi'$.*

The following is an analogue of Lemma 5.2.11 for $\sim^S_{vo}$:

Lemma 5.2.24. *Let S be an L-structure, and let $\boldsymbol{t}_1, \boldsymbol{t}_2$ be closed terms of $L(D)$ such that $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$. Let ψ_1, ψ_2 be two $L(D)$-formulas such that $\psi_1 \sim^S_{vo} \psi_2$. Then $\psi_1\{\boldsymbol{t}_1/x\} \sim^S_{vo} \psi_2\{\boldsymbol{t}_2/x\}$.*

Proof. Recall that in the proof of Lemma 5.2.11 we have shown that $(*)$ for every two $L(D)$-terms $\mathbf{s}_1, \mathbf{s}_2$, such that $\mathbf{s}_1 \sim^S \mathbf{s}_2$ it holds that $\mathbf{s}_1\{\mathbf{t}_1/x\} \sim^S \mathbf{s}_2\{\mathbf{t}_2/x\}$. Suppose that $\psi_1 \sim^S_{vo} \psi_2$. Then by Lemma 5.2.22, $(**) Fv(\psi_1) = Fv(\psi_2)$. Denote by $\mathsf{c}(\psi)$ the complexity of a formula ψ. Let x be some variable and $\mathbf{t}_1, \mathbf{t}_2$ two closed terms of $L(D)$, such that $\mathbf{t}_1 \sim^S \mathbf{t}_2$. We now prove that $\psi_1\{\mathbf{t}_1/x\} \sim^S_{vo} \psi_2\{\mathbf{t}_2/x\}$ by induction on $\max\{\mathsf{c}(\psi_1), \mathsf{c}(\psi_2)\}$. The base case: $\psi_1 = p(\mathbf{s}^1_1, \dots, \mathbf{s}^1_n)$ and $\psi_2 = p(\mathbf{s}^2_1, \dots, \mathbf{s}^2_n)$, where $\mathbf{s}^1_j \sim^S \mathbf{s}^2_j$ for all $1 \leq i \leq n$. The claim follows by $(*)$ above. Now assume that the claim holds for every two formulas φ_1, φ_2, such that $\max\{\mathsf{c}(\varphi_1), \mathsf{c}(\varphi_2)\} < l$. Now let ψ_1, ψ_2 be two formulas, such that $\max\{\mathsf{c}(\psi_1), \mathsf{c}(\psi_2)\} = l$. By Lemma 5.2.22, one of the following holds:

- $\psi_1 = \diamond(\phi_1^1, \ldots, \phi_n^1)$ and $\psi_2 = \diamond(\phi_1^2, \ldots, \phi_n^2)$, where $\phi_j^1 \sim_{vo}^S \phi_j^2$ for all $1 \leq j \leq n$. Then it holds that $\mathsf{max}\{\mathsf{c}(\phi_1^1), .., \mathsf{c}(\phi_n^1), \mathsf{c}(\phi_1^2), ..., \mathsf{c}(\phi_n^2)\} < l$. By the induction hypothesis, $\phi_j^1\{\mathbf{t}_1/x\} \sim_{vo}^S \phi_j^2\{\mathbf{t}_2/x\}$. Thus by Lemma 5.2.21, $\psi_1\{\mathbf{t}_1/x\} = \diamond(\phi_1^1\{\mathbf{t}_1/x\}, \ldots, \phi_n^1\{\mathbf{t}_1/x\}) \sim_{vo}^S \diamond(\phi_1^2\{\mathbf{t}_2/x\}, \ldots, \phi_n^2\{\mathbf{t}_2/x\}) = \psi_2\{\mathbf{t}_2/x\}$.

- $\psi_1 = \mathcal{Q}y\phi^1$ and $\psi_2 = \mathcal{Q}z\phi^2$ and $\phi^1\{w/y\} \sim_{vo}^S \phi^2\{w/z\}$ for any fresh variable w. Pick a fresh variable $w \neq x$. Since $\mathsf{max}\{\mathsf{c}(\phi^1\{w/y\}), \mathsf{c}(\phi^2\{w/z\})\} < l$, by the induction hypothesis, $\phi^1\{w/y\}\{\mathbf{t}_1/x\} \sim_{vo}^S \phi^2\{w/z\}\{\mathbf{t}_2/x\}$. By $(**)$, one of the following cases holds:
  - $x \notin Fv(\psi_1) \cup Fv(\psi_2)$. Then $\psi_1\{\mathbf{t}_1/x\} = \psi_1 \sim_{vo}^S \psi_2 = \psi_2\{\mathbf{t}_2/x\}$.
  - $x \in Fv(\psi_1) \cap Fv(\psi_2)$. Then $x \neq z$ and $x \neq y$, and it holds that $\psi_1\{\mathbf{t}_1/x\} = \mathcal{Q}y(\phi^1\{\mathbf{t}_1/x\})$ and $\psi_2\{\mathbf{t}_2/x\} = \mathcal{Q}z(\phi^2\{\mathbf{t}_2/x\})$. Since $\mathbf{t}_1, \mathbf{t}_2$ are closed terms, $\phi^1\{w/y\}\{\mathbf{t}_1/x\} = \phi^1\{\mathbf{t}_1/x\}\{w/y\}$ and $\phi^2\{w/z\}\{\mathbf{t}_2/x\} = \phi^2\{\mathbf{t}_2/x\}\{w/z\}$. Hence, $\phi^1\{\mathbf{t}_1/x\}\{w/y\} \sim_{vo}^S \phi^2\{\mathbf{t}_2/x\}\{w/z\}$, and by Lemma 5.2.21, $\psi_1\{\mathbf{t}_1/x\} = \mathcal{Q}y(\phi^1\{\mathbf{t}_1/x\}) \sim_{vo}^S \mathcal{Q}z(\phi^2\{\mathbf{t}_2/x\}) = \psi_2\{\mathbf{t}_2/x\}$.

- $\psi_1 = \mathcal{Q}y\phi$, $\phi \sim_{vo}^S \psi_2$ and $y \notin Fv(\phi)$. Thus $\mathsf{c}(\psi_1) > \mathsf{c}(\psi_2)$ and $\mathsf{max}\{\mathsf{c}(\psi_2), \mathsf{c}(\phi)\} < l$. By the induction hypothesis, $\phi\{\mathbf{t}_1/x\} \sim_{vo}^S \psi_2\{\mathbf{t}_2/x\}$. Since $x \notin Fv(\phi)$, $\phi\{\mathbf{t}_1/x\} = \phi$, and $\psi_1\{\mathbf{t}_1/x\} = \psi_1 = \mathcal{Q}y\phi \sim_{vo}^S \phi$. By transitivity of $\sim_{vo}^S$, $\psi_1\{\mathbf{t}_1/x\} \sim_{vo}^S \psi_2\{\mathbf{t}_2/x\}$.

- $\psi_2 = \mathcal{Q}y\phi$, $\phi \sim_{vo}^S \psi_1$ and $y \notin Fv(\phi)$. The proof is similar to the previous case.

$\square$

Definition 5.2.25. Let S be an L-structure and $\mathcal{M}$ an Nmatrix for L. Let $W \subseteq Frm^{\mathsf{cl}}_{L(D)}$ be some set of sentences closed under subsentences with respect to S. A partial S-valuation $v : W \to \mathcal{V}$ is $\sim_{vo}^S$*-legal in* $\mathcal{M}$ if it is semi-legal in $\mathcal{M}$ and for every $\psi, \varphi \in W$: $\psi \sim_{vo}^S \varphi$ implies $v(\psi) = v(\varphi)$.

Using the above definition, we can now modify the notions of truth- and validity-based consequence relations from Definition 5.2.13:

Definition 5.2.26. The consequence relations $\vdash^t_{\mathcal{M},vo}$ and $\vdash^v_{\mathcal{M},vo}$ are defined like $\vdash^t_{\mathcal{M}}$ and $\vdash^v_{\mathcal{M}}$ (respectively), but using $\sim_{vo}^S$ rather than $\sim^S$.

The following propositions are the analogues of Proposition 5.2.14 and Theorem 5.2.18 respectively for $\sim_{vo}^S$:

Proposition 5.2.27. *Let $\mathcal{M}$ be an Nmatrix for L.*

1. $\Gamma \vdash^t_{\mathcal{M},vo} \psi$ *implies* $\Gamma \vdash^v_{\mathcal{M},vo} \psi$.
2. *If* $\Gamma \subseteq Frm^{\mathsf{cl}}_L$ *(i.e,* Γ *contains only closed formulas), then* $\Gamma \vdash^t_{\mathcal{M},vo} \psi$ *iff* $\Gamma \vdash^v_{\mathcal{M},vo} \psi$.

Proposition 5.2.28. *If* $\mathcal{M}_1$ *is a refinement of* $\mathcal{M}_2$ *then* $\vdash^t_{\mathcal{M}_2,vo} \subseteq \vdash^t_{\mathcal{M}_1,vo}$ *and* $\vdash^v_{\mathcal{M}_2,vo} \subseteq \vdash^v_{\mathcal{M}_1,vo}$.

It is important to note that analycity for $\sim^S_{vo}$ is *not* always guaranteed. Consider, for instance, an Nmatrix $\mathcal{M}_v = \langle \{\mathbf{t},\mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O}\rangle$ for some first-order language L, with the following interpretation of $\forall$: $\tilde{\forall}[\{H\}] = \{\mathbf{t}\}$ for every $H \subseteq P^+(\{\mathbf{t},\mathbf{f}\})$. Let $S = \langle \{a\}, I\rangle$ be an L-structure, such that $I(c) = a$ and $I(p)(a) = \mathbf{f}$. Let $W = \{p(c)\}$. Then no partial valuation on W can be extended to a full $\mathcal{M}$-legal valuation v which respects $\sim^S_{vo}$.

Next we characterize those Nmatrices in which this problem does not occur. For an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, we define the following condition for an interpretation of a quantifier $\mathcal{Q}$ in $\mathcal{M}$:

$$(\mathbf{V})\ a \in \tilde{\mathcal{Q}}_{\mathcal{M}}(\{a\}) \text{ for every } a \in \mathcal{V}$$

Definition 5.2.29. An Nmatrix $\mathcal{M}$ for L is *V-analytic* if every L-structure S has the property that every partial S-valuation which is $\sim^S_{vo}$-legal in $\mathcal{M}$ can be extended to a full S-valuation which is $\sim^S_{vo}$-legal in $\mathcal{M}$.

Theorem 5.2.30. *Let* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ *be an Nmatrix.* $\mathcal{M}$ *is V-analytic iff the interpretations of all the quantifiers in* $\mathcal{M}$ *satisfy the condition* $(\mathbf{V})$.

Proof. For one direction, suppose that there is some $a \in \mathcal{V}$, such that $a \notin \tilde{\mathcal{Q}}(\{a\})$. Let $p(\mathbf{t}_1, ..., \mathbf{t}_n)$ be some atomic L-sentence. Construct an L-structure S, such that $I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n)) = a$. Let v_p be the partial valuation on $\{p(\mathbf{t}_1, ..., \mathbf{t}_n)\}$ (which is trivially closed under subsentences), such that $v_p(p(\mathbf{t}_1, \dots, \mathbf{t}_n) = a$. For any $\mathcal{M}$-legal full valuation v extending v_p, $v(\forall x p(\mathbf{t}_1, ..., \mathbf{t}_n)) \notin \tilde{\mathcal{Q}}(\{a\})$. Thus v_p has no extension to a full S-valuation that is $\sim^S_{vo}$-legal in $\mathcal{M}$. Hence $\mathcal{M}$ is not V-analytic.

For the converse, suppose that for every $a \in \mathcal{V}$: $a \in \tilde{\mathcal{Q}}(\{a\})$. Let $S = \langle D, I\rangle$ be an L-structure and let v_p be some partial S-valuation which is $\sim^S_{vo}$-legal in $\mathcal{M}$. To construct an extension of v_p to a full S-valuation v which is $\sim^S_{vo}$-legal in $\mathcal{M}$, for every n-ary connective $\diamond$ of L and every $a_1, ..., a_n \in \mathcal{V}$ choose an arbitrary truth-value $\mathbf{b}^{\diamond}_{a_1,...,a_n} \in \tilde{\diamond}(a_1, ..., a_n)$. Similarly, for every $\mathcal{Q}$ in L and every $B \subseteq P^+(\mathcal{V})$, choose a truth-value $\mathbf{b}^{\mathcal{Q}}_B \in \tilde{\mathcal{Q}}(B)$, such that for every $a \in \mathcal{V}$: $\mathbf{b}^{\mathcal{Q}}_{\{a\}} = a$ (such choice is possible, since for every $a \in \mathcal{V}$: $a \in \tilde{\mathcal{Q}}(\{a\})$).

Denote by $H_{\sim^S_{vo}}$ the set of all equivalence classes of $Frm^{\mathsf{cl}}_{L(D)}$ under $\sim^S_{vo}$. Denote by $[\![\psi]\!]$ the equivalence class of ψ. Define the function $\chi : H_{\sim^S_{vo}} \to \mathcal{V}$ as follows:

$$\chi([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!]) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$$

$$\chi(\llbracket \diamond(\psi_1, ..., \psi_n)\rrbracket) = \begin{cases} v_p(\varphi) & \varphi \in (\llbracket \diamond(\psi_1, ..., \psi_n)\rrbracket \cap W) \\ \mathbf{b}^{\diamond}_{\chi(\llbracket\psi_1\rrbracket),...,\chi(\llbracket\psi_n\rrbracket)} & \text{there is no } \varphi \in (\llbracket \diamond(\psi_1, ..., \psi_n)\rrbracket \cap W) \end{cases}$$

$$\chi(\llbracket \mathcal{Q}x\psi\rrbracket) = \begin{cases} v_p(\varphi) & \varphi \in (\llbracket \mathcal{Q}x\psi\rrbracket \cap W) \\ \mathbf{b}^{\mathcal{Q}}_{\{\chi(\llbracket\psi\{\overline{a}/x\}\rrbracket) \mid a\in D\}} & \text{there is no } \varphi \in (\llbracket \mathcal{Q}x\psi\rrbracket \cap W) \end{cases}$$

Let us show that χ is well-defined. First of all, note that the above definition does not depend on the choice of $\varphi \in W$ if such φ exists, as for every two $\varphi_1, \varphi_2 \in \llbracket\psi\rrbracket \cap W$ for any ψ: $\varphi_1 \sim^S_{vo} \varphi_2$, and since v_p is $\sim^S_{vo}$-legal, $v_p(\varphi_1) = v_p(\varphi_2)$. Secondly, we show that the definition does not depend on the representatives of the equivalence class of ψ: we prove that if $\varphi_1 \sim^S_{vo} \varphi_2$ then $\chi(\llbracket\varphi_1\rrbracket) = \chi(\llbracket\varphi_2\rrbracket)$ by induction on $\mathsf{max}\{\mathsf{c}(\varphi_1), \mathsf{max}\{\mathsf{c}(\varphi_2)\}$ (where $\mathsf{c}(\varphi_i)$ is the complexity of φ_i). For the base case, $\varphi_1 = p(\mathbf{s}_1, \dots, \mathbf{s}_n)$, $\varphi_2 = p(\mathbf{s}'_1, \dots, \mathbf{s}'_n)$ and $\mathbf{s}_i \sim^S \mathbf{s}'_i$ for all $1 \leq i \leq n$. By Lemma 5.2.10-2: $I(\mathbf{s}_i) = I(\mathbf{s}'_i)$, so $\chi(\llbracket\varphi_1\rrbracket) = I(p)(I(\mathbf{s}_1), \dots, I(\mathbf{s}_n)) = I(p)(I(\mathbf{s}'_1), \dots, I(\mathbf{s}'_n)) = \chi(\llbracket\varphi_2\rrbracket)$. Suppose that the claim holds for every two sentences with maximal complexity $< l$. Let $\varphi_1 \sim^S_{vo} \varphi_2$, where $\mathsf{max}\{\mathsf{c}(\varphi_1), \mathsf{c}(\varphi_2)\} = l$. If there is some $\varphi \in \llbracket\varphi_1\rrbracket \cap W = \llbracket\varphi_2\rrbracket \cap W$, then $\chi(\llbracket\varphi_1\rrbracket) = \chi(\llbracket\varphi_2\rrbracket) = v_p(\varphi)$ and we are done. Otherwise, by Lemma 5.2.22, one of the following holds:

- $\varphi_1 = \diamond(\phi_1, \dots, \phi_n)$, $\varphi_2 = \diamond(\phi'_1, \dots, \phi'_n)$ and $\phi_i \sim^S \phi'_i$ for all $1 \leq i \leq n$. $\chi(\llbracket\varphi_1\rrbracket) = \mathbf{b}^{\diamond}_{\chi(\llbracket\phi_1\rrbracket),...,\chi(\llbracket\phi_n\rrbracket)}$ and $\chi(\llbracket\varphi_2\rrbracket) = \mathbf{b}^{\diamond}_{\chi(\llbracket\phi'_1\rrbracket),...,\chi(\llbracket\phi'_n\rrbracket)}$. Now we note that it holds that $\mathsf{max}\{\mathsf{c}(\phi_1), ..., \mathsf{c}(\phi_n), \mathsf{c}(\phi'_1), ..., \mathsf{c}(\phi'_n)\} < l$, by the induction hypothesis, $\chi(\llbracket\phi_i\rrbracket) = \chi(\llbracket\phi'_i\rrbracket)$ and so $\chi(\llbracket\varphi_1\rrbracket) = \chi(\llbracket\varphi_2\rrbracket)$.

- $\varphi_1 = \mathcal{Q}x\phi_1$, $\varphi_2 = \mathcal{Q}y\phi_2$ and $\phi_1\{w/x\} \sim^S_{vo} \phi_2\{w/y\}$ for a fresh variable w. By Lemma 5.2.24, $(*)$ for every $a \in D$: $\phi_1\{w/x\}\{\overline{a}/w\} = \phi_1\{\overline{a}/x\} \sim^S_{vo} \phi_2\{\overline{a}/y\} = \phi_2\{w/y\}\{\overline{a}/w\}$. $\chi(\llbracket\varphi_1\rrbracket) = \mathbf{b}^{\mathcal{Q}}_{\{\chi(\llbracket\phi_1\{\overline{a}/x\}\rrbracket) \mid a\in D\}}$ and $\chi(\llbracket\varphi_2\rrbracket) = \mathbf{b}^{\mathcal{Q}}_{\{\chi(\llbracket\phi_2\{\overline{a}/x\}\rrbracket) \mid a\in D\}}$. By $(*)$, $\chi(\llbracket\varphi_1\rrbracket) = \chi(\llbracket\varphi_2\rrbracket)$.

- $\varphi_1 = \mathcal{Q}x\phi$, $x \notin Fv(\phi)$ and $\phi \sim^S_{vo} \varphi_2$. Then $\mathsf{max}\{\mathsf{c}(\phi), \mathsf{c}(\varphi_2)\} < l$, and by the induction hypothesis, $\chi(\llbracket\phi\rrbracket) = \chi(\llbracket\varphi_2\rrbracket)$. By definition of χ, $\chi(\llbracket\mathcal{Q}x\phi\rrbracket) = \mathbf{b}^{\mathcal{Q}}_{\{\chi(\llbracket\phi\{\overline{a}/x\}\rrbracket) \mid a\in D\}} = \mathbf{b}^{\mathcal{Q}}_{\{\chi(\llbracket\phi\rrbracket)\}} = \chi(\llbracket\phi\rrbracket) = \chi(\llbracket\varphi_2\rrbracket)$.

- $\varphi_2 = \mathcal{Q}x\phi$, $x \notin Fv(\phi)$ and $\phi \sim^S_{vo} \varphi_1$. The proof is similar to the previous case.

Next define v as follows for every $\psi \in Frm^{\mathsf{cl}}_L$:

$$v(\psi) = \chi(\llbracket\psi\rrbracket)$$

Obviously, v respects the $\sim^S_{vo}$ relation. We show that $v(\psi)$ is legal in $\mathcal{M}$ by induction on the structure of ψ.

- Let $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$. Then $v(\psi) = \chi([\![p(\mathbf{t}_1, \ldots, \mathbf{t}_n)]\!]) = I(p)(I(\mathbf{t}_1), \ldots, I(\mathbf{t}_n))$.

- Let $\psi = \diamond(\psi_1, ..., \psi_n)$. Suppose that there is some $\varphi \in ([\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W$. Pick one such φ which does not contain any void quantifiers (indeed, if $\varphi \in [\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W$, then since W is closed under subsentences and by Lemma 5.2.23, the sentence φ' obtained from φ by deleting void quantifiers is also in $[\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W$). By Lemma 5.2.9 it must be the case that $\varphi = \diamond(\varphi_1, ..., \varphi_n)$, where $\varphi_i \sim^S_{vo} \psi_i$ for all $1 \leq i \leq n$. Since W is closed under subsentences, $\varphi_1, ..., \varphi_n \in W$. By definition of v, $v(\psi) = v_p(\varphi) \in \tilde{\diamond}(v(\varphi_1), ..., v(\varphi_n)) = \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$ (since v_p is legal in $\mathcal{M}$ and $[\![\varphi_i]\!] = [\![\psi_i]\!]$). Otherwise $v(\psi) = \mathbf{b}^{\diamond}_{\chi([\![\psi_1]\!]), ..., \chi([\![\psi_n]\!])} \in \tilde{\diamond}(v(\psi_1), ..., v(\psi_n))$.

- Let $\psi = \mathcal{Q}x\phi$. Suppose that there is some $\varphi \in ([\![\mathcal{Q}x\phi]\!] \cap W$. Again, pick one such φ which does not contain any void quantifiers. By Lemma 5.2.9, φ must be of the form $\mathcal{Q}y\varphi'$, where $\varphi'\{z/y\} \sim^S_{vo} \phi\{z/x\}$ for a fresh variable z. Since W is closed under subsentences, for every $a \in D$: $\varphi\{\overline{a}/y\} \in W$. Then $v(\mathcal{Q}x\phi) = v_p(\mathcal{Q}y\varphi') \in \tilde{\mathcal{Q}}(\{v(\varphi'\{\overline{a}/y\}) \mid a \in D\})$. By Lemma 5.2.24, $\varphi'\{\overline{a}/y\} = \varphi'\{z/y\}\{\overline{a}/z\} \sim^S_{vo} \phi\{z/x\}\{\overline{a}/z\} = \phi\{\overline{a}/x\}$. Thus we have $v(\mathcal{Q}x\phi) \in \tilde{\mathcal{Q}}(\{v(\phi\{\overline{a}/x\}) \mid a \in D\})$. Otherwise, $v(\mathcal{Q}x\phi) = \mathbf{b}^{\mathcal{Q}}_{\{\chi([\![\phi\{\overline{a}/x\}]\!]) \mid a \in D\}} \in \tilde{\mathcal{Q}}(\{v(\phi\{\overline{a}/x\}) \mid a \in D\})$.

$\square$

An Example: First-order Quantifiers

Let L be a first-order language, which contains the quantifiers $\forall$ and $\exists$. The classical interpretation of these quantifiers (see Example 5.1.3) satisfies the condition (**V**) above. This interpretation can be generalized to an arbitrary number of truth-values as follows:

Definition 5.2.31. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for L. We say that a quantifier $\mathcal{Q}$ is* universally interpreted *in $\mathcal{M}$ if $\tilde{\mathcal{Q}}_{\mathcal{M}}$ for all $H \in 2^{\mathcal{V}} \setminus \{\emptyset\}$:*

$$\widetilde{\mathcal{Q}}(H) \subseteq \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

A quantifier $\mathcal{Q}$ is existentially interpreted *in $\mathcal{M}$ if $\tilde{\mathcal{Q}}_{\mathcal{M}}$ satisfies (**V**) and for all $H \in 2^{\mathcal{V}} \setminus \{\emptyset\}$:*

$$\widetilde{\mathcal{Q}}(H) \subseteq \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Corollary 5.2.32. *Let L be a first-order language and $\mathcal{M}$ an Nmatrix for L with a universal and an existential interpretation of $\forall$ and $\exists$ respectively. Then $\mathcal{M}$ is V-analytic.*

Proof. It is easy to see that any such Nmatrix satisfies the condition **(V)**. The claim follows directly from Theorem 5.2.30. □

Let us now turn to the problem of extending a propositional formal system having a non-deterministic semantics to the first-order level. Let $\mathbf{H}LK^+_\neg$ be a standard Hilbert-type system which corresponds to the logic $LK^+_\neg$ (the characteristic Nmatrix of which is given in Example 2.2.10).

Definition 5.2.33. $\mathbf{QHL}^+_\neg$ *is obtained by adding to* $\mathbf{H}LK^+_\neg$ *the following standard axioms and inference rules for* $\forall$ *and* $\exists$*:*

$$\forall x\psi \supset \psi\{\boldsymbol{t}/x\} \qquad \psi\{\boldsymbol{t}/x\} \supset \exists x\psi$$

$$\frac{(\varphi \supset \theta)}{(\varphi \supset \forall x\theta)} \qquad \frac{(\theta \supset \varphi)}{(\exists x\theta \supset \varphi)}$$

where $\boldsymbol{t}$ *is any term free for* x *in* ψ*, and* x *does not occur free in* φ*.*

Unfortunately, $\mathbf{QHL}^+_\neg$ is not very useful. Due to the absence of axioms for negation, neither the α-equivalence principle, nor the void quantification principle, are derivable in it. For instance, $\not\vdash_{\mathbf{QHL}^+_\neg} \neg\forall xp(x) \leftrightarrow \neg\forall yp(y)$, and $\not\vdash_{\mathbf{QHL}^+_\neg} (\neg\forall xp(c)) \leftrightarrow \neg p(c)$. To handle this, we follow da Costa's approach from [70] and add to $\mathbf{QHL}^+_\neg$ explicit axioms which capture these principles:

Definition 5.2.34. $\sim^{dc}$ is the minimal congruence relation between formulas, which satisfies for $\mathcal{Q} \in \{\forall, \exists\}$:

- If $\psi\{z/x\} \sim^{dc} \psi'\{z/y\}$, where z is fresh, then $\mathcal{Q}x\psi \sim^{dc} \mathcal{Q}y\psi'$.
- If $\psi \sim^{dc} \psi'$ and x does not occur free in ψ, then $\mathcal{Q}x\psi \sim^{dc} \psi'$.

Although the relation $\sim^{dc}$ seems very similar to the relation $\sim^S_{vo}$ from Definition 5.2.20, there are a few differences between the two:

1. $\sim^{dc}$ is a relation between formulas of L, while $\sim^S_{vo}$ is a relation between formulas of $L(D)$.
2. $\sim^S_{vo}$ is defined with respect to some structure S, while $\sim^{dc}$ is purely syntactic.
3. Unlike $\sim^{dc}$, $\sim^S_{vo}$ identifies two sentences ψ, ψ' such that ψ' is obtained from ψ by substituting any number of closed terms for closed terms with the same denotation in S. For instance, let S be an L-structure, such that $I(d) = I(c)$ for two constants $d \neq c$. Then $p(c) \not\sim^{dc} p(d)$, but $p(c) \sim^S_{vo} p(d)$. The motivation for this is related to extending the language with the set of individual constants $\{\overline{a} \mid a \in D\}$. Suppose

we have a closed term $\mathbf{t}$, such that $I(\mathbf{t}) = a \in D$. But a also has an individual constant $\overline{a}$ referring to it. We would like to be able to substitute $\mathbf{t}$ for $\overline{a}$ in every context, as will be shown in the sequel.

The following easy lemma (proved by induction on formulas) summarizes the relation between the two congruences:

Lemma 5.2.35. *Let $S = \langle D, I \rangle$ be an L-structure.*

1. *Let A, B be two L-sentences. If $A \sim^{dc} B$, then $A \sim^{S}_{vo} B$.*
2. *Let A, B be two L-sentences, such that for any two $\boldsymbol{t}_1, \boldsymbol{t}_2 \in Trm^{\mathsf{cl}}_{L(D)}$ occurring in A and B respectively: $I(\boldsymbol{t}_1) \neq I(\boldsymbol{t}_2)$. Then $A \sim^{dc} B$ iff $A \sim^{S}_{vo} B$.*

Now we extend the system $\mathbf{QHL}^{+}_{\neg}$ with an axiom using the $\sim^{dc}$-relation:

Definition 5.2.36. Let **QHL** be the system obtained from $\mathbf{QHL}^{+}_{\neg}$ by adding the axiom **(DC)** $\psi \supset \psi'$ whenever $\psi \sim^{dc} \psi'$.

Definition 5.2.37. Let the Nmatrix $\mathcal{QM}^{B}_{4}$ be the extension of the Nmatrix $\mathcal{M}^{B}_{4}$ (Example 2.2.10) with the following interpretations of $\forall$ and $\exists$:

$$\widetilde{\forall}(H) = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\widetilde{\exists}(H) = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Proposition 5.2.38. $\Gamma \vdash_{\mathbf{QHL}} \psi$ *iff* $\Gamma \vdash^{v}_{\mathcal{QM}^{B}_{4}, vo} \psi$.

The proof is a straightforward modification of the proof of Theorem 6.1.1 below.

5.3 Nmatrices with Multi-ary Quantifers

The notion of a unary quantifier can be further generalized to *multi-ary quantifiers*, which are logical constants that can be applied to more than one formula. If $\mathcal{Q}$ is an n-ary quantifier, x is a variable and $\psi_1, ..., \psi_n$ are formulas, then $\mathcal{Q}x(\psi_1, ..., \psi_n)$ is a formula in which all occurrences of x are bound by $\mathcal{Q}$. In this context the ordinary quantifiers can be thought of as unary quantifiers, while the bounded universal and existential quantifiers $\overline{\forall}$ and $\overline{\exists}$ used in syllogistic reasoning are examples of binary quantifiers (The respective meanings of $\overline{\forall}x(\psi_1, \psi_2)$ and $\overline{\exists}x(\psi_1, \psi_2)$ are $\forall x(\psi_1 \to \psi_2)$ and $\exists x(\psi_1 \wedge \psi_2)$).

Let us first define the notion of a *subformula* in a language with multi-ary quantifiers:

Definition 5.3.1. *Let L be a language with multi-ary quantifiers. For an L-formula ψ, the set $SF_L(\psi)$ of the subformulas of ψ is defined as follows:*

- $SF_L(p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)) = \{p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)\}$
- $SF_L(\diamond(\psi_1, ..., \psi_n)) = SF_L(\psi_1) \cup \ldots \cup SF_L(\psi_1) \cup \{\diamond(\psi_1, ..., \psi_n)\}$
- $SF_L(\mathcal{Q}x(\psi_1, \ldots, \psi_n)) = \bigcup_{1 \le i \le n} \{\psi_i\{\boldsymbol{t}/x\} \mid \boldsymbol{t} \text{ free for } x \text{ in } \psi_i\} \cup$ $\{\mathcal{Q}x(\psi_1, \ldots, \psi_n\}$.

Next we turn to the interpretation of multi-ary quantifiers in Nmatrices. Recall that the interpretation of a unary quantifier $\mathcal{Q}_1$ in an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L is a function $\tilde{\mathcal{Q}}_1 : 2^{\mathcal{V}} \setminus \{\emptyset\} \to \mathcal{V}$. Similarly, an n-ary quantifier will be interpreted by a function $\tilde{\mathcal{Q}}_n : 2^{\mathcal{V}^n} \setminus \{\emptyset\} \to \mathcal{V}$. Thus the following is an extension of Definition 5.2.1 to the level of multi-ary quantifiers:

Definition 5.3.2. An Nmatrix for L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth-values,
- $\mathcal{D}$ is a non-empty proper set of $\mathcal{V}$,
- $\mathcal{O}$ includes a function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective, and a function $\tilde{\mathcal{Q}} : 2^{\mathcal{V}^n} \setminus \{\emptyset\} \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary quantifier.

Example 5.3.3. Consider the Nmatrix $\mathcal{M} = \langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{O} \rangle$ for a language with the standard bounded universal and existential (binary) quantifiers $\overline{\forall}$ and $\overline{\exists}$ described above. In addition, the language contains a binary quantifier $\mathcal{Q}_2$. The interpretations of the quantifiers in $\mathcal{M}$ are given in Figure 5.1.

The congruence relation $\sim^S$ (Definition 5.2.7) is naturally extended to languages with multi-ary quantifiers as follows:

Definition 5.3.4. *Let $S = \langle D, I \rangle$ be an L-structure. The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:*

- *If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, \boldsymbol{t}_2 \sim^S \boldsymbol{t}'_2, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S p(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*
- *If $\psi_i \sim^S \varphi_i$ for all $1 \le i \le n$, then $\diamond(\psi_1, ..., \psi_n) \sim^S \diamond(\varphi_1, ..., \varphi_n)$ for every n-ary connective $\diamond$ of L.*
- *If $\psi_1\{z/x\} \sim^S \varphi_1\{z/y\}, \ldots, \psi_n\{z/x\} \sim^S \varphi_n\{z/y\}$, where x, y are distinct variables and z is a new variable, then $\mathcal{Q}x(\psi_1, \ldots, \psi_n) \sim^S \mathcal{Q}y(\varphi_1, \ldots, \varphi_n)$ for every n-ary quantifier $\mathcal{Q}$ of L.*

H	$\tilde{\tilde{\forall}}(\mathbf{H})$	$\tilde{\tilde{\exists}}(\mathbf{H})$	$\tilde{\mathcal{Q}}_2(\mathbf{H})$
$\{\langle t, t\rangle\}$	$\{t\}$	$\{t\}$	$\{t, f\}$
$\{\langle t, f\rangle\}$	$\{f\}$	$\{f\}$	$\{t\}$
$\{\langle f, f\rangle\}$	$\{t\}$	$\{f\}$	$\{t, f\}$
$\{\langle f, t\rangle\}$	$\{t\}$	$\{t\}$	$\{f\}$
$\{\langle t, t\rangle, \langle t, f\rangle\}$	$\{f\}$	$\{t\}$	$\{t, f\}$
$\{\langle t, t\rangle, \langle f, t\rangle\}$	$\{t\}$	$\{t\}$	$\{t, f\}$
$\{\langle t, t\rangle, \langle f, f\rangle\}$	$\{t\}$	$\{t\}$	$\{t, f\}$
$\{\langle f, t\rangle, \langle t, f\rangle\}$	$\{f\}$	$\{f\}$	$\{t\}$
$\{\langle f, t\rangle, \langle f, f\rangle\}$	$\{t\}$	$\{f\}$	$\{t\}$
$\{\langle t, f\rangle, \langle f, f\rangle\}$	$\{f\}$	$\{f\}$	$\{t\}$
$\{\langle t, t\rangle, \langle t, f\rangle, \langle f, t\rangle\}$	$\{f\}$	$\{t\}$	$\{f\}$
$\{\langle t, t\rangle, \langle f, f\rangle, \langle f, t\rangle\}$	$\{t\}$	$\{t\}$	$\{t, f\}$
$\{\langle f, t\rangle, \langle t, f\rangle, \langle f, f\rangle\}$	$\{f\}$	$\{f\}$	$\{t\}$
$\{\langle f, f\rangle, \langle t, f\rangle, \langle t, t\rangle\}$	$\{f\}$	$\{t\}$	$\{t, f\}$
$\{\langle t, t\rangle, \langle t, f\rangle, \langle f, t\rangle, \langle f, f\rangle\}$	$\{f\}$	$\{t\}$	$\{t\}$

Figure 5.1: The interpretation of the quantifiers in Example 5.3.3

The following is an analogue of Lemma 5.2.9 and is proved similarly:

Lemma 5.3.5. *Let $S = \langle D, I\rangle$ be an L-structure. For every two formulas ψ, φ of $L(D)$, if $\psi \sim^S \varphi$ then one of the following holds:*

- $\psi = p(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ *and* $\varphi = p(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$*, where* $\boldsymbol{t}_i \sim^S \boldsymbol{s}_i$ *for all* $1 \leq i \leq n$.
- $\psi = \diamond(\psi_1, \ldots, \psi_n)$ *and* $\varphi = \diamond(\varphi_1, \ldots, \varphi_n)$ *for some n-ary connective $\diamond$ of L, and for all $1 \leq i \leq n$: $\psi_i \sim^S \varphi_i$.*
- $\psi = \mathcal{Q}x(\psi_1, \ldots, \psi_n)$ *and* $\varphi = \mathcal{Q}y(\varphi_1, \ldots, \varphi_n)$ *for some n-ary quantifier $\mathcal{Q}$ of L, and for any fresh variable z: $\psi_i\{z/x\} \sim^S \varphi_i\{z/x\}$ for all $1 \leq i \leq n$.*

Next all that is needed is to modify the third condition of the second part of Definition 5.2.3 as follows:

$$v(\mathcal{Q}x(\psi_1, ..., \psi_n)) \in \tilde{\mathcal{Q}}_{\mathcal{M}}(\langle\{v(\psi_1\{\overline{a}/x\}), ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\})$$

After this modification, Definitions 5.2.12 and 5.2.13 remain the same. Note, however, that this does not hold for the relation $\sim^S_{vo}$. In fact, the result of any deletion of a void quantifier from a formula $\mathcal{Q}x(\psi_1, \ldots, \psi_n)$ (for $n > 1$) is not a valid wff. Note also that a void n-ary quantifier $\mathcal{Q}$ behaves like an n-ary connective (this is the reason why propositional connectives are not considered in the sequel).

5.4 Generalized Nmatrices with (n, k)-ary Quantifiers

The notion of multi-ary quantifiers can be further generalized to (n, k)-*ary quantifiers*. An (n, k)-ary quantifier ([94, 119]) is a generalized logical connective, which binds k variables and connects n formulas. Any n-ary propositional connective can be thought of as an $(n, 0)$-ary quantifier. For instance, the standard $\wedge$ connective binds no variables and connects two formulas: $\wedge(\psi_1, \psi_2)$. The standard first-order quantifiers $\exists$ and $\forall$ are $(1, 1)$-quantifiers, as they bind one variable and connect one formula: $\forall x\psi, \exists x\psi$. Bounded universal and existential quantifiers used in syllogistic reasoning ($\forall x(p(x) \to q(x))$ and $\exists x(p(x) \wedge q(x))$) can be represented as (2,1)-ary quantifiers $\overline{\forall}$ and $\overline{\exists}$, binding one variable and connecting two formulas: $\overline{\forall}x(p(x), q(x))$ and $\overline{\exists}x(p(x), q(x))$. An example of (n, k)-ary quantifiers for $k > 1$ are Henkin quantifiers ([89, 91]). The simplest Henkin quantifier Q_H binds 4 variables and connects one formula:

$$Q_H\ x_1x_2y_1y_2\ \psi(x_1, x_2, y_1, y_2) := \begin{array}{cc} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{array} \psi(x_1, x_2, y_1, y_2)$$

In this way of recording combinations of quantifiers, dependency relations between variables are expressed as follows: an existentially quantified variable depends on those universally quantified variables which are on the left of it in the same row.

In what follows, L is a language with (n, k)-ary quantifiers. As before, we assume that L has no propositional connectives (as a propositional n-ary connective can be thought of as an $(n, 0)$-ary quantifier). We write $\mathcal{Q}\overrightarrow{x}A$ instead of $\mathcal{Q}x_1...x_kA$, and $\psi\{\overrightarrow{\mathbf{t}}/\overrightarrow{z}\}$ instead of $\psi\{\mathbf{t}_1/z_1, ..., \mathbf{t}_k/z_k\}$.

It is clear that the interpretation of (n, k)-ary quantifiers using distributions like for multi-ary quantifiers, is not sufficient for the case of $k > 1$. Using them, we cannot capture any kind of dependencies between elements of the domain. For instance, there is no way we can express the fact that there exists an element b in the domain, such that for *every* element a, $p(a, b)$ holds. It is clear that a more general interpretation of a quantifier is needed.

We will generalize the interpretation of quantifiers as follows. Given an L-structure $S = \langle D, I\rangle$, an interpretation of an (n,k)-ary quantifier $\mathcal{Q}$ in S is an operation $\tilde{\mathcal{Q}}_S : (D^k \to \mathcal{V}^n) \to P^+(\mathcal{V})$, which for every function (from k-ary vectors of the domain elements to n-ary vectors of truth-values) returns a non-empty set of truth-values.

Definition 5.4.1. A *generalized non-deterministic matrix* (GNmatrix) for L is a tuple $\mathcal{M} =< \mathcal{V}, \mathcal{D}, \mathcal{O} >$, where:

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
- For every (n,k)-ary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$[3] includes a corresponding operation $\tilde{\mathcal{Q}}_S : (D^k \to \mathcal{V}^n) \to P^+(\mathcal{V})$ for every L-structure $S = \langle D, I\rangle$.

A 2GNmatrix is any GNmatrix with $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$.

Below we consider the following examples:

1. Given an L-structure $S = \langle D, I\rangle$, the standard $(1,1)$-ary quantifier $\forall$ is interpreted as follows for any $g \in D \to \{t,f\}$: $\tilde{\forall}_S(g) = \{t\}$ if for every $a \in D$, $g(a) = t$, and $\tilde{\forall}_S(g) = \{f\}$ otherwise. The standard $(1,1)$-ary quantifier $\exists$ is interpreted as follows for any $g \in D \to \{t,f\}$: $\tilde{\exists}_S(g) = \{t\}$ if there exists some $a \in D$, such that $g(a) = t$, and $\tilde{\exists}_S(g) = \{f\}$ otherwise.

2. Given an L-structure $S = \langle D, I\rangle$, the $(1,2)$-ary bounded universal quantifier $\overline{\forall}$ is interpreted as follows: for any $g \in D \to \{t,f\}^2$, $\tilde{\overline{\forall}}_S(g) = \{t\}$ if for every $a \in D$, $g(a) \neq \langle t, f\rangle$, and $\tilde{\overline{\forall}}_S(g) = \{f\}$ otherwise. The $(1,2)$-ary bounded existential quantifier $\overline{\exists}$ is interpreted as follows: for any $g \in D \to \{t,f\}^2$, $\tilde{\overline{\forall}}_S(g) = \{t\}$ if there exists some $a \in D$, such that $g(a) = \langle t, t\rangle$, and $\tilde{\overline{\forall}}_S(g) = \{f\}$ otherwise.

3. Consider the $(2,2)$-ary quantifier $\mathcal{Q}$, with the intended meaning of $\mathcal{Q}xy(\psi_1, \psi_2)$ as $\exists y \forall x(\psi_1(x,y) \wedge \neg\psi_2(x,y))$. Its interpretation for every L-structure $S = \langle D, I\rangle$, every $g \in D^2 \to \{t,f\}^2$ is as follows: $\tilde{Q}_S(g) = t$ iff there exists some $a \in D$, such that for every $b \in D$: $g(a,b) = \langle t, f\rangle$.

4. Consider the $(4,1)$-ary Henkin quantifier $\mathcal{Q}_H$ discussed above. Its interpretation for every L-structure $S = \langle D, I\rangle$ and every $g \in D^4 \to \{t,f\}$ is as follows: $\tilde{\mathcal{Q}}^H_S(g) = \{t\}$ if for every $a \in D$ there exists some $b \in D$ and for every $c \in D$ there exists some $d \in D$, such that $g(a,b,c,d) = t$. $\tilde{\mathcal{Q}}^H_S(g) = \{f\}$ otherwise.

[3]In the current definition, $\mathcal{O}$ is not a class and the tuple $\langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is not well-defined. We can overcome this technical problem by assuming that the domains of all the structures are prefixes of the set of natural numbers. A more general solution to this problem is a question for further research.

The congruence relation $\sim^S$ (Definition 5.2.7) is naturally extended to languages with multi-ary quantifiers as follows:

Definition 5.4.2. The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:

- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, \mathbf{t}_2 \sim^S \mathbf{t}'_2, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $p(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S p(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.
- If $\psi_1\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_1\{\overrightarrow{z}/\overrightarrow{y}\}, ..., \psi_n\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_n\{\overrightarrow{z}/\overrightarrow{y}\}$, where $\overrightarrow{x} = x_1...x_k$ and $\overrightarrow{y} = y_1...y_k$ are distinct variables and $\overrightarrow{z} = z_1...z_k$ are new distinct variables, then for any (n, k)-ary quantifier $\mathcal{Q}$ of L also $\mathcal{Q}\overrightarrow{x}(\psi_1, ..., \psi_n) \sim^S \mathcal{Q}\overrightarrow{y}(\varphi_1, ..., \varphi_n)$.

The following is a generalization of Lemma 5.2.11:

Lemma 5.4.3. *Let S be an L-structure for a GNmatrix $\mathcal{M}$. Let ψ, ψ' be formulas of $L(D)$. Let $\boldsymbol{t}_1, ..., \boldsymbol{t}_n, \boldsymbol{t}'_1, ..., \boldsymbol{t}'_n$ be closed terms of $L(D)$, such that $\boldsymbol{t}_i \sim^S \boldsymbol{t}'_i$ for every $1 \leq i \leq n$. Then whenever $\psi \sim^S \psi'$, also $\psi\{\overrightarrow{\boldsymbol{t}}/\overrightarrow{x}\} \sim^S \psi'\{\overrightarrow{\boldsymbol{t}'}/\overrightarrow{x}\}$.*

The notion of an $\sim^S$-legal valuation (Definition 5.2.12) is extended as follows:

Definition 5.4.4. Let $S = \langle D, I\rangle$ be an L-structure for a GNmatrix $\mathcal{M}$. An S-valuation v is *$\sim^S$-legal in $\mathcal{M}$* if it satisfies the following conditions:

- $v(\psi) = v(\psi')$ for every two sentences ψ, ψ' of $L(D)$, such that $\psi \sim^S \psi'$.
- $v(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$.
- For every (n, k)-ary quantifier $\mathcal{Q}$ of L, $v(\mathcal{Q}x_1, ..., x_k(\psi_1, ..., \psi_n)$ is in the set $\tilde{\mathcal{Q}}_S(\lambda a_1, ..., a_k \in D.\langle v(\psi_1\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}), ..., v(\psi_n\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\})\rangle)$.

After these modifications, Definition 5.2.13 remains the same, using the notion of a GNmatrix instead of an Nmatrix.

Chapter 6

Application: Nmatrices for First-order LFIs

In this chapter we apply the framework of Nmatrices with first-order quantifiers introduced in Section 5.2.3 to provide modular semantics for first-order paraconsistent logics. The propositional fragments of these logics were already discussed in section 2.3. Below we extend these results to the full first-order level. The work presented below is mainly based on [34, 32].
For simplicity of presentation, the logics will be formulated in terms of Hilbert-style systems, rather than in terms of abstract consequence relations.
In this chapter we fix the first-order language $L_C = \{\vee, \wedge, \supset, \neg, \circ, \forall, \exists\}$.

6.1 LFIs with Finite Nmatrices

Our starting point will be the basic paraconsistent system **QHB**, obtained from **QHL** (Definition 5.2.36) by the addition of the following schemata:

$$(\mathbf{n})\ \varphi \vee \neg\varphi \qquad (\mathbf{b})\ (\circ\varphi \wedge \neg\varphi \wedge \varphi) \supset \psi$$

QHB is the obvious first-order extension of the Hilbert-style axiomatization of the logic **B** from Example 2.3.6 (which is the basic logic of formal inconsistency from [59, 62], where it is called mbC). Accordingly, in this section we shall refer to **QHB** simply as **B**.

Let us start by providing a characteristic Nmatrix for **B**. It is a straightforward extension of the Nmatrix $\mathcal{M}_5^B$ from Example 2.3.6:

Theorem 6.1.1. *Let $\mathcal{QM}_5^B$ be the extension of $\mathcal{M}_5^B$ with the following interpretations* *of quantifiers:*

$$\widetilde{\forall}(H) = \begin{cases} \mathcal{D} & \textit{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \textit{otherwise} \end{cases} \quad \widetilde{\exists}(H) = \begin{cases} \mathcal{D} & \textit{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \textit{otherwise} \end{cases}$$

Then $\Gamma \vdash^{v}_{\mathcal{QM}^B_5, vo} \psi_0$ *iff* $\Gamma \vdash_{\mathbf{B}} \psi_0$.

Proof. The proof of soundness is a straightforward modification of the proof for the propositional case (Theorem 2.3.4). It is easy to verify that the quantifier schemata and inference rules are sound with respect to $\mathcal{QM}^B_5$.

For completeness, assume that $\Gamma \not\vdash_{\mathbf{B}} \psi_0$. First note that by definition of the interpretation of $\forall$ in $\mathcal{QM}^B_5$, $\forall x\varphi \vdash_{\mathcal{QM}^B_5, vo} \varphi$ and $\varphi \vdash_{\mathcal{QM}^B_5, vo} \forall x\varphi$ for every formula φ and every variable x. Obviously the same relations hold between φ and $\forall x\varphi$ also in $\mathbf{B}$. It follows that we may assume that all formulas in $\Gamma \cup \{\psi_0\}$ are sentences. It is also easy to see that we may restrict ourselves to sentences in L_r, the language consisting of all the constants, function, and predicate symbols occurring in $\Gamma \cup \{\psi_0\}$. Now suppose that $\Gamma \not\vdash_{\mathbf{B}} \psi_0$. We will construct an L_r-structure S and an S-valuation v which is $\sim^S_{vo}$-legal in $\mathcal{QM}^B_5$, such that $v \models \Gamma$, but $v \not\models \psi_0$. Let L' be the language obtained from L_r by adding a countably infinite set of new constants. It is a standard matter to show (using a usual Henkin-type construction) that Γ can be extended to a maximal set Γ^* of sentences in L', such that: (i) $\Gamma^* \not\vdash_{\mathbf{B}} \psi_0$, (ii) $\Gamma \subseteq \Gamma^*$, (iii) For every L'-sentence $\exists x\psi \in \Gamma^*$ there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \in \Gamma^*$, and (iv) For every L'-sentence $\forall x\psi \notin \Gamma^*$, there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \notin \Gamma^*$. (The last property follows from property (iii), the deduction theorem for $\mathbf{B}$, and the fact that for any $x \notin Fv(\varphi)$, $(\forall x\psi \supset \varphi) \supset \exists x(\psi \supset \varphi)$ is provable in $\mathbf{B}$.) It follows that Γ^* has the following properties:

1. If $\psi \notin \Gamma^*$, then $\psi \supset \psi_0 \in \Gamma^*$,
2. $\psi \vee \varphi \in \Gamma^*$ iff either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$,
3. $\psi \wedge \varphi \in \Gamma^*$ iff both $\varphi \in \Gamma^*$ and $\psi \in \Gamma^*$,
4. $\varphi \supset \psi \in \Gamma^*$ iff either $\varphi \notin \Gamma^*$ or $\psi \in \Gamma^*$,
5. Either $\psi \in \Gamma^*$ or $\neg\psi \in \Gamma^*$,
6. If ψ and $\neg\psi$ are both in Γ^*, then $\circ\psi \notin \Gamma^*$,
7. If $\psi \in \Gamma^*$, then for every L'-sentence ψ' such that $\psi' \sim^{dc} \psi$: $\psi' \in \Gamma^*$,
8. If $\forall x\theta \in \Gamma^*$, then for every closed L'-term $\mathbf{t}$: $\theta\{\mathbf{t}/x\} \in \Gamma^*$. If $\forall x\theta \notin \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L', such that $\theta\{\mathbf{t}_\theta/x\} \notin \Gamma^*$,

9. If $\exists x\theta \in \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L', such that $\theta\{\mathbf{t}_\theta/x\} \in \Gamma^*$. If $\exists x\theta \notin \Gamma^*$, then for every closed term $\mathbf{t}$ of L': $\theta\{\mathbf{t}/x\} \notin \Gamma^*$.

The L'-structure $S = \langle D, I\rangle$ is defined as follows:

- D is the set of all the closed terms of L'.
- For every constant c of L': $I(c) = c$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I(f)(\mathbf{t}_1, ..., \mathbf{t}_n) = f(\mathbf{t}_1, ..., \mathbf{t}_n)$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I(p)(\mathbf{t}_1, ..., \mathbf{t}_n) = \langle x, y, z\rangle$, where $x, y, z \in \{0, 1\}$ and (i) $x = 1$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$, (ii) $y = 1$ iff $\neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$, (iii) $z = 1$ iff $\circ p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$.

The following lemma can be proved by induction on $\mathbf{t}$:

Lemma 6.1.2. *$I(\boldsymbol{t}) = \boldsymbol{t}$ for every $\boldsymbol{t} \in D$.*

Note that in the extended language $L'(D)$ we now have an individual constant $\overline{\mathbf{t}}$ for every term $\mathbf{t} \in D$. For an $L'(D)$-formula ψ and an $L'(D)$-term $\mathbf{t}$, the L'-formula $\widehat{\psi}$ and the L'-term $\widehat{\mathbf{t}}$ are defined as follows:

- $\widehat{x} = x$ for any variable x of L.
- $\widehat{c} = c$ for any constant c of L.
- $\widehat{\overline{\mathbf{t}}} = \mathbf{t}$ for any $\mathbf{t} \in D$.
- $\widehat{f(\mathbf{t}_1, ..., \mathbf{t}_n)} = f(\widehat{\mathbf{t}}_1, ..., \widehat{\mathbf{t}}_n)$.
- $\widehat{p(\mathbf{t}_1, ..., \mathbf{t}_n)} = p(\widehat{\mathbf{t}}_1, ..., \widehat{\mathbf{t}}_n)$.
- $\widehat{\diamond(\psi_1, ..., \psi_n)} = \diamond(\widehat{\psi}_1, ..., \widehat{\psi}_n)$.
- $\widehat{\mathcal{Q}x\psi} = \mathcal{Q}x\widehat{\psi}$.

In other words, $\widehat{\psi}$ ($\widehat{\mathbf{t}}$) is obtained by replacing all individual constants of the form $\overline{\mathbf{s}}$ (where $\mathbf{s} \in D$) occurring in ψ ($\mathbf{t}$) by the respective (closed) term $\mathbf{s}$.

Lemma 6.1.3. *1. For any $\psi \in Frm^{\mathsf{cl}}_{L'(D)}$: $\psi \sim^S \widehat{\psi}$ (and so also $\psi \sim^S_{vo} \widehat{\psi}$).*

2. For any $\psi, \varphi \in Frm^{\mathsf{cl}}_{L'(D)}$: if $\psi \sim^S_{vo} \varphi$, then $\widehat{\psi} \sim^{dc} \widehat{\varphi}$.

3. For every $\psi \in Frm^{\mathsf{cl}}_{L'(D)}$ and every $\boldsymbol{t} \in D$: $\widehat{\psi\{\boldsymbol{t}/x\}} = \widehat{\psi}\{\boldsymbol{t}/x\}$.

Proof. First let us prove that for every $\mathbf{t} \in Frm^{\mathsf{cl}}_{L'(D)}$: $I(\mathbf{t}) = I(\widehat{\mathbf{t}})$. The proof is by induction on the structure of $\mathbf{t}$. For the case when $\mathbf{t}$ is a constant of L', the claim is trivial as $\mathbf{t} = \widehat{\mathbf{t}}$. For the case when $\mathbf{t}$ is an individual constant $\overline{\mathbf{s}}$ of $L'(D)$ for some $\mathbf{s} \in D$, by Lemma 6.1.2 it follows that $I(\mathbf{t}) = \mathbf{s} = I(\mathbf{s}) = I(\widehat{\mathbf{t}})$. Now let $\mathbf{t} = f(\mathbf{s}_1, \dots, \mathbf{s}_n)$. Then $\widehat{\mathbf{t}} = f(\widehat{\mathbf{s}}_1, \dots, \widehat{\mathbf{s}}_n)$ and by the induction hypothesis $I(\mathbf{s}_i) = I(\widehat{\mathbf{s}}_i)$ for all $1 \leq i \leq n$. Hence, $I(\mathbf{t}) = I(f)(I(\mathbf{s}_1), \dots, I(\mathbf{s}_n)) = I(f)(I(\widehat{\mathbf{s}}_1), \dots, I(\widehat{\mathbf{s}}_n)) = I(\widehat{\mathbf{t}})$. Now the proof of the first item follows by an easy induction on the structure of ψ. For the second item, suppose that $\psi \sim^S_{vo} \varphi$. Then (since by the first part, $\psi \sim^S \widehat{\psi}$ and $\varphi \sim^S \widehat{\varphi}$), by the transitivity of $\sim^S_{vo}$ it follows that $\widehat{\psi} \sim^S_{vo} \widehat{\varphi}$. Since $\widehat{\psi}, \widehat{\varphi}$ are L'-sentences, the claim follows by Lemma 5.2.35-2 (since by Lemma 6.1.2, for every $\mathbf{t}_1, \mathbf{t}_2 \in D$, $\mathbf{t}_1 \neq \mathbf{t}_2$ implies $I(\mathbf{t}_1) \neq I(\mathbf{t}_2)$). The third item is again proved by a tedious induction on the structure of ψ.

$\square$

The refuting S-valuation $v : Frm^{\mathsf{cl}}_{L'(D)} \to \mathcal{V}$ is defined as follows:

$$v(\psi) = \langle x_\psi, y_\psi, z_\psi \rangle$$

where $x_\psi, y_\psi, z_\psi \in \{0, 1\}$ and: (i) $x_\psi = 1$ iff $\widehat{\psi} \in \Gamma^*$, (ii) $y_\psi = 1$ iff $\widehat{\neg\psi} \in \Gamma^*$, (iii) $z_\psi = 1$ iff $\widehat{\circ\psi} \in \Gamma^*$.

Next we prove that v is $\sim^S_{vo}$-legal in $\mathcal{QM}^B_5$. Let ψ, ψ' be two $L'(D)$-sentences, such that $\psi \sim^S_{vo} \psi'$. Then by Lemma 6.1.3, $\widehat{\psi} \sim^{dc} \widehat{\psi'}$, and by property 7 of Γ^*, $\widehat{\psi} \in \Gamma^*$ iff $\widehat{\psi'} \in \Gamma^*$. Similarly, since $\neg\psi \sim^S_{vo} \neg\psi'$ and $\circ\psi \sim^S_{vo} \circ\psi'$, $\neg\widehat{\psi} = \widehat{\neg\psi} \sim^{dc} \widehat{\neg\psi'} = \neg\widehat{\psi'}$ and $\widehat{\circ\psi} \sim^{dc} \widehat{\circ\psi'}$. Thus $\widehat{\neg\psi} \in \Gamma^*$ iff $\widehat{\neg\psi'} \in \Gamma^*$ and $\widehat{\circ\psi} \in \Gamma^*$ iff $\widehat{\circ\psi'} \in \Gamma^*$. Hence $v(\psi) = v(\psi')$ and so v respects the $\sim^S_{vo}$ relation.
It remains to check that v respects the interpretations of the connectives and quantifiers in $\mathcal{QM}_5$. This is guaranteed by the properties of Γ^*. We prove this for the case of $\forall$:

- Let $\forall x\psi$ be an $L'(D)$-sentence, such that $\{v(\psi\{\overline{a}/x\}) \mid a \in D\} \subseteq \mathcal{D}$. Then for every $\mathbf{t} \in D$, $v(\psi\{\overline{\mathbf{t}}/x\}) \in \mathcal{D}$. By Lemma 5.2.24 (recall that by Lemma 5.2.10-2, $\mathbf{t} \sim^S \overline{\mathbf{t}}$ for any $\mathbf{t} \in D$) it holds that $\psi\{\mathbf{t}/x\} \sim^S_{vo} \psi\{\overline{\mathbf{t}}/x\}$, and since v respects the $\sim^S_{vo}$ relation, $v(\psi\{\mathbf{t}/x\}) \in \mathcal{D}$ for every $\mathbf{t} \in D$. Since $\psi \sim^S_{vo} \widehat{\psi}$, by Lemma 5.2.24 again also $\psi\{\mathbf{t}/x\} \sim^S_{vo} \widehat{\psi}\{\mathbf{t}/x\}$. Thus $v(\widehat{\psi}\{\mathbf{t}/x\}) \in \mathcal{D}$ for every $\mathbf{t} \in D$. By property 8 of Γ^*, $\widehat{\forall x\psi} = \forall x\widehat{\psi} \in \Gamma^*$, hence $v(\forall x\psi) \notin \mathcal{D}$.

- Let $\forall x\psi$ be an $L'(D)$-sentence, such that $\{v(\psi\{\overline{a}/x\}) \mid a \in D\} \cap \mathcal{F} \neq \emptyset$. The proof that $v(\forall x\psi) \notin \mathcal{F}$ is similar to the previous case.

Clearly, for every L'-sentence ψ: $v(\psi) \in \mathcal{D}$ iff $\psi \in \Gamma^*$. So $v \models \Gamma$ (recall that $\Gamma \subseteq \Gamma^*$), but $v \not\models \psi_0$. $\square$

$$(\mathbf{a}_\forall)\ \forall x \circ \varphi \supset (\circ(\forall x \varphi))$$

$$(\mathbf{a}_\exists)\ \forall x \circ \varphi \supset (\circ(\exists x \varphi))$$

$$(\mathbf{o}_\forall)\ \exists x \circ \varphi \supset (\circ(\forall x \varphi))$$

$$(\mathbf{o}_\exists)\ \exists x \circ \varphi \supset (\circ(\exists x \varphi))$$

$$(\mathbf{v}_\forall)\ \ \circ(\forall x \varphi)$$

$$(\mathbf{v}_\exists)\ \ \circ(\exists x \varphi)$$

Figure 6.1: Quantifier-related Axioms

Now that we have provided semantics for the basic system **B**, we turn to the family of extensions of **B** with various combinations of axioms from $HLFIR$ (Definition 2.3.1), to which we add the quantifier-related axioms (considered e.g. in [57]) which are listed in Figure 6.1. These axioms capture the different ways of propagation of consistency in quantified formulas, and are generalizations of the corresponding propositional schemata from Figure 2.2.

Definition 6.1.4. Let $\mathcal{QR} = HLFIR \cup \{(\mathbf{a}_\forall), (\mathbf{a}_\exists), (\mathbf{o}_\forall), (\mathbf{o}_\exists), (\mathbf{v}_\forall), (\mathbf{v}_\exists)\}$. For a set $S \subseteq \mathcal{QR}$, $\mathbf{B}[S]$ is the system obtained by adding the axioms in S to **B**.

Notation 6.1.5. We denote by $\{(\mathbf{a})\}$ the set $\{(\mathbf{a})_\diamond \mid \diamond \in \{\wedge, \vee, \supset\} \cup \{(\mathbf{a})_\mathcal{Q} \mid \mathcal{Q} \in \{\forall, \exists\}\}$. Similarly for $(\mathbf{o})$ and $(\mathbf{v})$.

Like in the propositional case, the systems obtained by adding some set of axioms from $\mathcal{QR}$ to **B** can be characterized by the simple refinement of the basic Nmatrix $\mathcal{QM}_5^B$ (Theorem 6.1.1) induced by the conditions corresponding to the axioms from $\mathcal{QR}$. Below we define these semantic conditions:

Definition 6.1.6. Let $\mathsf{Con} = \{\langle x, y, 1\rangle \mid x, y \in \{0, 1\}\}$.

- For $r \in HLFIR$, $\mathrm{C}(r)$ is defined like in Definition 2.3.3.
- $\mathrm{C}(\mathbf{a}_\mathcal{Q})$: If $H \subseteq \mathsf{Con}$, then $\tilde{\mathcal{Q}}(H) \subseteq \mathsf{Con}$
- $\mathrm{C}(\mathbf{o}_\mathcal{Q})$: If $H \cap \mathsf{Con} \neq \emptyset$, then $\tilde{\mathcal{Q}}(H) \subseteq \mathsf{Con}$
- $\mathrm{C}(\mathbf{v}_\mathcal{Q})$: $\tilde{\mathcal{Q}}(H) \subseteq \mathsf{Con}$ for every non-empty $H \subseteq \mathcal{V}_5$

For $S \subseteq \mathcal{QR}$, $\mathrm{C}(S) = \{\mathrm{C}(r) \mid r \in S\}$, and $\mathcal{QM}_5^B[S]$ is the weakest simple refinement of $\mathcal{QM}_5^B$ in which all the conditions in $\mathrm{C}(S)$ are satisfied.

Let us explain, for instance, how C($\mathbf{a}_{\mathcal{Q}}$) is obtained. To guarantee the validity of $\forall x \circ \varphi \supset \circ(\mathcal{Q}x\varphi)$, the following must hold for every L_C-structure S and every S-valuation v: whenever $v(\forall x \circ \varphi) \in \mathcal{D}$, also $v(\circ(\mathcal{Q}x\varphi)) \in \mathcal{D}$. Suppose that $v(\forall x \circ \varphi) \in \mathcal{D}$. Then $\tilde{\forall}(\{v(\circ\varphi\{\overline{a}/x\}) \mid a \in D\}) \subseteq \mathcal{D}$ and for every $a \in D$: $v(\varphi\{\overline{a}/x\}) \in \{t, f\}$. If C($\mathbf{a}_{\mathcal{Q}}$) holds (that is for every $H \subseteq \mathsf{Con}$, $\tilde{\mathcal{Q}}(H) \subseteq \mathsf{Con}$), then whenever $v(\forall x \circ \varphi) \in \mathcal{D}$, it also holds that $v(\mathcal{Q}x\varphi) \in \mathsf{Con}$ and so $v(\circ\mathcal{Q}x\varphi) \in \mathcal{D}$, leading to the validity of **($\mathbf{a}_{\mathcal{Q}}$)**. The explanations for the conditions for **($\mathbf{o}_{\mathcal{Q}}$)** and **($\mathbf{v}_{\mathcal{Q}}$)** are quite similar.

Example 6.1.7. Let $S_i = \{(\mathbf{i})\}$, $S_o = S_i \cup \{(\mathbf{o})\}$ and $S_a = S_i \cup \{(\mathbf{a})\}$. The interpretations of $\forall$ and $\exists$ are defined in $\mathcal{QM}_5^B[S_i]$, $\mathcal{QM}_5^B[S_o]$ and $\mathcal{QM}_5^B[S_a]$ (respectively) as follows[1] (note that $\mathcal{QM}_5^B[S_o]$ and $\mathcal{QM}_5^B[S_a]$ are two different simple refinements of $\mathcal{QM}_5^B[S_i]$):

$\mathcal{QM}_5^B[S_i]$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t, I\}$	$\{t, I\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t, I\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t, I\}$

$\mathcal{QM}_5^B[S_o]$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t\}$	$\{t\}$
$\{f, I\}$	$\{f\}$	$\{t\}$
$\{t, f, I\}$	$\{f\}$	$\{t\}$

$\mathcal{QM}_5^B[S_a]$:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t, I\}$	$\{t, I\}$
$\{t, f\}$	$\{f\}$	$\{t\}$
$\{t, I\}$	$\{t, I\}$	$\{t, I\}$
$\{f, I\}$	$\{f\}$	$\{t, I\}$
$\{t, f, I\}$	$\{f\}$	$\{t, I\}$

Theorem 6.1.8. *For $S \subseteq \mathcal{QR}$, $\Gamma \vdash^v_{\mathcal{QM}_5^B[S],vo} \psi$ iff $\Gamma \vdash_{\mathbf{B}[S]} \psi$.*

[1] Recall that by C($\mathbf{i_1}$) and C($\mathbf{i_2}$) the truth-values t_I and f_I are deleted and we are left with only three truth-values: t, f and I.

Proof. The proof of completeness is a straightforward modification of the proof of Theorem 6.1.1. Γ is again extended to a maximal set Γ^*, which satisfies the properties 1-9. It is easy to see that in this case Γ^* also satisfies additional properties:

10. If $(\mathbf{a})_{\mathcal{Q}} \in S$, then whenever $\circ \mathcal{Q}x\psi \notin \Gamma^*$, also $\forall x \circ \psi \notin \Gamma^*$,

11. If $(\mathbf{o})_{\mathcal{Q}} \in S$, then whenever $\circ \mathcal{Q}x\psi \notin \Gamma^*$, also $\exists x \circ \psi \notin \Gamma^*$,

12. If $(\mathbf{v})_{\mathcal{Q}} \in S$, then $\circ \mathcal{Q}x\psi \in \Gamma^*$ for every L'-sentence $\mathcal{Q}x\psi$.

Now the L'-structure S and the refuting S-valuation v are defined exactly like in the proof of Theorem 6.1.1. The proof that v respects the $\sim^S_{vo}$ relation and the interpretations of $\mathcal{QM}^B_5$ is also similar. It remains to check that the additional conditions imposed on $\mathcal{QM}^B_5[S]$ by the schemata in S are respected by the valuation v. We show the proof for the case when $(\mathbf{o}_{\mathcal{Q}}) \in S$. Let $\mathcal{Q}x\psi$ be an $L'(D)$-sentence, such that $H_\psi = \{v(\psi\{\overline{a}/x\}) \mid a \in D\}$ satisfies $H_\psi \cap \mathsf{Con} \neq \emptyset$. Then there is some $\overline{\mathbf{t}} \in D$, such that $v(\psi\{\overline{\mathbf{t}}/x\}) \in \mathsf{Con}$. By Lemma 5.2.24, $\psi\{\mathbf{t}/x\} \sim^S_{vo} \psi\{\overline{\mathbf{t}}/x\}$. Since v respects the $\sim^S_{vo}$ relation, $v(\psi\{\mathbf{t}/x\}) \in \mathsf{Con}$. By definition of v, $\circ(\widehat{\psi\{\mathbf{t}/x\}}) \in \Gamma^*$. By Lemma 6.1.3, $(\circ(\widehat{\psi\{\mathbf{t}/x\}})) = (\widehat{\circ\psi})\{\mathbf{t}/x\}$. Hence by property 9 of Γ^*, $\exists x \circ (\widehat{\psi}) = \exists x(\widehat{\circ\psi}) \in \Gamma^*$. By property 11 of Γ^*, $(\widehat{\circ\mathcal{Q}x\psi}) = \circ\mathcal{Q}x(\widehat{\psi}) \in \Gamma^*$. By definition of v, $v(\mathcal{Q}x\psi) \in \mathsf{Con}$.
The proof for the rest of the cases is similar. ∎

Corollary 6.1.9. *For every $S \subseteq \mathcal{QR}$, $\mathcal{QM}^B_5[S]$ is V-analytic.*

Proof. It is easy to verify that the interpretations of $\forall$ and $\exists$ in $\mathcal{QM}^B_5[S]$ are universal and existential respectively. The claim follows by Corollary 5.2.32. ∎

6.2 LFIs with Infinite Nmatrices

We now turn to first-order systems which include the problematic axiom $(\mathbf{l})$ (see Figure 2.2). By Theorem 2.3.8 it follows that such systems can have no finite characteristic Nmatrices already on the propositional level. This theorem is extended in [20] also to systems which include the following alternatives[2] of $(\mathbf{l})$:

Definition 6.2.1. *The set Ax' consists of the following schemata:*

[2] In his original formulation of the hierarchy of C-systems ([70], da Costa chose the formula $\neg(\varphi \wedge \neg\varphi)$ to represent the consistency of φ. It turns out that choosing the formula $\neg(\neg\varphi \wedge \varphi)$ instead leads to a different hierarchy of systems, using the axiom $(\mathbf{d})$ instead of $(\mathbf{l})$. $(\mathbf{h})$ is a combination of these two axioms.

(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

(d) $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$

(h) $(\neg(\varphi \wedge \neg\varphi) \vee \neg(\neg\varphi \wedge \varphi)) \supset \circ\varphi$

For $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{h}\}$ and $S \subseteq \mathcal{QR}$, $\mathbf{B}y[S]$ is the system obtained from $\mathbf{B}[S]$ by adding the schema y.

Notation 6.2.2. We shall denote $\mathbf{B}y[S]$ by $\mathbf{B}ys$, where s is a string consisting of the names of the axioms in S. For instance, we write **Blce** instead of $\mathbf{Bl}[\{(\mathbf{c}), (\mathbf{e})\}]$. If both $(\mathbf{x_1})$ and $(\mathbf{x_2})$ are in S for $\mathbf{x} \in \{\mathbf{i}, \mathbf{k}\}$, we abbreviate it by $\mathbf{x}$. Also, if $\mathbf{x_y}$ is in S for every $\mathbf{y} \in \{\supset, \wedge, \vee\}$ and some $\mathbf{x} \in \{\mathbf{a}, \mathbf{o}, \mathbf{v}\}$, we shall write $\mathbf{x_p}$. Similarly, if $\mathbf{x_y}$ is in S for every $\mathbf{y} \in \{\forall, \exists\}$ and some $\mathbf{x} \in \{\mathbf{a}, \mathbf{o}, \mathbf{v}\}$, we shall write $\mathbf{x_Q}$. For both $\mathbf{x_p}$ and $\mathbf{x_Q}$ we shall write $\mathbf{x}$.

Example 6.2.3. da Costa's original first-order logic C_1^* is the $\circ$-free fragment of **Bcia** (note that the axioms $(\mathbf{a}_\forall)$ and $(\mathbf{a}_\exists)$ are also included).

We start by providing semantics for the systems $\mathbf{B}y$, where y is any axiom from Ax'. It is easy to see that any of the schemata from Ax' entails in $\mathbf{B}$ both $(\mathbf{k}_1)$ and $(\mathbf{k}_2)$. Recall that the semantic effect of these two axioms is to delete t_I and f_I from the basic Nmatrix $\mathcal{QM}_5^B$. Thus the infinite Nmatrices provided in this section are all refinements (although not simple, see Definition 2.2.18) of the *three-valued* Nmatrix $\mathcal{QM}_5^B[\{(\mathbf{k_1}), (\mathbf{k_2})\}]$. To provide some informal intuition about the infinite semantics, note that what all the schemata $(\mathbf{l}), (\mathbf{b}), (\mathbf{h})$ have in common is a conjunction of a formula with its negation. Consider for instance the schema **(l)** $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$. Its validity is guaranteed only if $v(\neg(\varphi \wedge \neg\varphi)) \notin \mathcal{D}$ whenever $v(\circ\varphi) \notin \mathcal{D}$. Informally, to ensure this, we need to be able to isolate a conjunction of an "inconsistent" formula ψ with its own negation from conjunctions of ψ with other formulas. This can be done by enforcing an intimate connection between the truth-value of an "inconsistent" formula and the truth-value of its negation. This, in turn, requires a supply of infinitely many truth-values.

The following definition is an extension of Definition 6.2.4:

Definition 6.2.4. Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$. $\mathcal{QM}_3^B\mathbf{l}$ is the Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where:

1. $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$

2. $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$

3. $\mathcal{O}$ is defined by:

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$\tilde{\forall}(H) = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\tilde{\exists}(H) = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

The Nmatrix $\mathcal{QM}_3^B\mathbf{d}$ is defined like $\mathcal{QM}_3^B\mathbf{l}$, except that $\tilde{\wedge}$ is defined as follows:

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

The Nmatrix $\mathcal{QM}_3^B\mathbf{h}$ is defined like $\mathcal{QM}_3^B\mathbf{l}$, except that $\tilde{\wedge}$ is defined as follows:

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & (\text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\}) \text{ or } (b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\}) \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Theorem 6.2.5. *For $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{h}\}$, $\Gamma \vdash^v_{\mathcal{QM}_3^B, vo} \psi$ iff $\Gamma \vdash_{\mathbf{B}y} \psi$.*

Proof. We do the proof for the case of **QBl**. The proofs in the other two cases are similar. *Soundness:* Define the function $F : \mathcal{T} \cup \mathcal{I} \cup \mathcal{F} \to \{t, I, f\}$ as follows:

$$F(x) = \begin{cases} f & x \in \mathcal{F} \\ t & x \in \mathcal{T} \\ I & x \in \mathcal{I} \end{cases}$$

It is easy to see that F is a reduction (see Definition 5.2.17) of $\mathcal{QM}_3^B\mathbf{l}$ to $\mathcal{QM}_5^B\mathbf{k}$, and so $\mathcal{QM}_3^B\mathbf{l}$ is a refinement of $\mathcal{QM}_5^B\mathbf{k}$. By Theorem 5.2.28, $\vdash_{\mathcal{QM}_5^B\mathbf{k},vo} \subseteq \vdash_{\mathcal{QM}_3^B\mathbf{l},vo}$. To prove soundness, it remains to show that **(l)** is $\mathcal{QM}_3^B\mathbf{l}$-valid. Let S be an L_C-structure and v an S-valuation which is $\sim_{vo}^S$-legal in $\mathcal{QM}_3\mathbf{l}$ and for which $v(\circ\psi) \in \mathcal{F}$. Then $v(\psi) = I_j^i$ for some i and j. Hence $v(\neg\psi) \in \{I_j^{i+1}, t_j^{i+1}\}$ and so $v(\psi \wedge \neg\psi) \in \mathcal{T}$ and $v(\neg(\psi \wedge \neg\psi)) \in \mathcal{F}$. Hence **(l)** is valid in $\mathcal{QM}_3^B\mathbf{l}$.

Completeness: Assume that $\mathbf{\Gamma} \nvdash_{\mathbf{QBl}} \psi_0$. Like in the proof of Theorem 6.1.1, we may assume that all the elements of $\Gamma \cup \psi_0$ are sentences. We proceed again with a Henkin construction to get a maximal theory Γ^*, such that $\Gamma^* \nvdash_{\mathbf{QBl}} \psi_0$ over the extended language L', and Γ^* satisfies the properties 1-9 from the proof of Theorem 6.1.1. In addition, using the (**l**) axiom, it is easy to show that Γ^* also satisfies the property (10) If $\circ\psi \notin \Gamma^*$, then $\neg(\psi \wedge \neg\psi) \notin \Gamma^*$.

Let D be the set of all the closed terms of L'. We define the L'-structure $S = \langle D, I\rangle$ as follows. For every constant c of L': $I(c) = c$, and for every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I(f)(\mathbf{t}_1, ..., \mathbf{t}_n) = f(\mathbf{t}_1, ..., \mathbf{t}_n)$. For the definition of the interpretation of predicate symbols of L', let Cl be the set of all the equivalence classes of $L'(D)$-sentences under $\sim_{vo}^S$ (note that $\sim_{vo}^S$ is already determined by the interpretations of closed terms in S). For every $\mathcal{E} \in Cl$, call a sentence ψ a *minimal representative* of $\mathcal{E}$ if ψ the least number of quantifiers of all the sentences in $\mathcal{E}$. (For instance, the sentences $\forall x p(c)$ and $p(c)$ are in the same equivalence class, but $\forall x p(c)$ is not a minimal representative, since $p(c)$ has less quantifiers). It is easy to see that all the minimal representatives of an equivalence class are α-equivalent. Let $\lambda i.\alpha_i$ be an enumeration of all the equivalence classes of $L_C(D)$-sentences under $\sim_{vo}^S$, the minimal representatives of which do not begin with $\neg$ (for instance, the minimal representative of $[\![\forall x \neg p(c)]\!]$ begins with $\neg$). It is easy to see that for any equivalence class $[\![\psi]\!]$, there are unique $n_{[\![\psi]\!]}, k_{[\![\psi]\!]}$ such that for every $A \in [\![\psi]\!]$, $A = \overline{\neg}_{k_{[\![\psi]\!]}}\varphi$ for some $\varphi \in \alpha_{n_{[\![\psi]\!]}}$, where $\overline{\neg}_k\theta$ is a sentence obtained from θ by adding k preceding negation symbols and any number of preceding void quantifiers (note that for any atomic sentence $p(\mathbf{t}_1, ..., \mathbf{t}_n)$, $k([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!]) = 0$). For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$, define:

$$I(p)(\mathbf{t}_1, ..., \mathbf{t}_n) = \begin{cases} f & p(\mathbf{t}_1, ..., \mathbf{t}_n) \notin \Gamma^* \\ t^0_{n([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!])} & \neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \notin \Gamma^* \\ I^0_{n([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!])} & p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*, \neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^* \end{cases}$$

For an $L'(D)$-term $\mathbf{t}$ and an $L'(D)$-formula ψ the L'-term $\widehat{\mathbf{t}}$ and the L'-formula $\widehat{\psi}$ are defined like in the proof of Theorem 6.1.1. Note that the Lemmas 6.1.2 and 6.1.3 also hold here. The valuation v is now defined as follows:

$$v(\psi) = \begin{cases} f & \widehat{\psi} \notin \Gamma^* \\ t_{n(\llbracket\psi\rrbracket)}^{k(\llbracket\psi\rrbracket)} & \widehat{(\neg\psi)} \notin \Gamma^* \\ I_{n(\llbracket\psi\rrbracket)}^{k(\llbracket\psi\rrbracket)} & \widehat{\psi} \in \Gamma^*, \widehat{(\neg\psi)} \in \Gamma^* \end{cases}$$

It remains to show that v is $\sim^S_{vo}$-legal in $\mathcal{QM}^B_3\mathbf{l}$. Let A, B be $L'(D)$-formulas such that $A \sim^S_{vo} B$. Then $n_{\llbracket A\rrbracket} = n_{\llbracket B\rrbracket}$, and $k_{\llbracket A\rrbracket} = k_{\llbracket B\rrbracket}$. Also, $\neg A \sim^S_{vo} \neg B$, and by Lemma 6.1.3-2 $\widehat{A} \sim^{dc} \widehat{B}$ and $\widehat{\neg A} \sim^{dc} \widehat{\neg B}$. By property 7 of Γ^*, $\widehat{A} \in \Gamma^*$ iff $\widehat{B} \in \Gamma^*$ and $\widehat{\neg A} \in \Gamma^*$ iff $\widehat{\neg B} \in \Gamma^*$. Thus by definition of v, $v(A) = v(B)$ and so v respects the $\sim^S_{vo}$ relation.
The proof that v respects the operations corresponding to $\vee$, $\supset$, $\forall$ and $\exists$ is like in the proof of Theorem 6.1.1. We consider next the cases of $\circ$, $\neg$ and $\wedge$:

$\circ$**:** That $v(\circ\psi) = f$ in case $v(\psi) \in \mathcal{I}$ is shown as in the proof of Theorem 6.1.1. Assume next that $v(\psi) \in \mathcal{T} \cup \mathcal{F}$. Then either $\widehat{\psi} \notin \Gamma^*$, or $\widehat{\neg\psi} \notin \Gamma^*$. By property 3 of Γ^*, it follows that $\widehat{\psi \wedge \neg\psi} \notin \Gamma^*$, and so by property 5 of Γ^*, $\neg(\widehat{\psi \wedge \neg\psi}) \in \Gamma^*$. Hence $\circ\widehat{\psi} \in \Gamma^*$ by property 10, and so $v(\circ\psi) \in \mathcal{D}$.

$\neg$**:** The proofs that $v(\psi) = f$ implies $v(\neg\psi) \in \mathcal{D}$ and that $v(\psi) \in \mathcal{T}$ implies $v(\neg\psi) = f$ are like in the proof of Theorem 6.1.1. Assume next that $v(\psi) = I^k_n$. Then both $\widehat{\psi}$ and $\widehat{\neg\psi}$ are in Γ^*, and $\psi = \neg_k\varphi$ where $\varphi \in \alpha_n$. Thus $\neg\psi = \neg_{k+1}\varphi$ for $\varphi \in \alpha_n$, and so $n_{\llbracket\neg\psi\rrbracket} = n$, $k_{\llbracket\neg\psi\rrbracket} = k + 1$. It follows by definition of v that $v(\neg\psi)$ is either I^{k+1}_n or t^{k+1}_n (depending on whether $\neg\neg\psi$ is in Γ^* or not).

$\wedge$**:** The proofs that if $v(\psi_1) = f$ or $v(\psi_2) = f$ then $v(\psi_1 \wedge \psi_2) = f$, and that $v(\psi_1 \wedge \psi_2) \in \mathcal{D}$ otherwise, are like in the proof of Theorem 6.1.1. Assume next that $v(\psi_1) = I^k_n$ and $v(\psi_2) \in \{I^{k+1}_n, t^{k+1}_n\}$. Then both $\widehat{\psi}_1$ and $\widehat{\psi}_2$ are in Γ^*, and so $\widehat{\psi_1 \wedge \psi_2} \in \Gamma^*$. Also, $\psi_1 = \neg_k\varphi_1$, $\psi_2 = \neg_{k+1}\varphi_2$ for $\varphi_1, \varphi_2 \in \alpha_n$. It follows that $\psi_2 \sim^S \neg\psi_1$ and $\psi_1 \wedge \psi_2 \sim^S_{vo} \psi_1 \wedge \neg\psi_1$. By Lemma 6.1.3, $\widehat{\psi_1 \wedge \psi_2} \sim^{dc} \widehat{\psi_1 \wedge \neg\psi_1}$. By property 7 of Γ^*, $\widehat{\psi_1 \wedge \neg\psi_1} \in \Gamma^*$, and so $\widehat{\psi}_1, \widehat{\neg\psi}_1 \in \Gamma^*$. This entails that $\widehat{\circ\psi}_1 \notin \Gamma^*$. Hence property 10 implies that $\neg(\widehat{\psi_1 \wedge \neg\psi_1}) \notin \Gamma^*$. Hence $v(\psi_1 \wedge \psi_2) \in \mathcal{T}$.

Obviously, $v(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma^*$, and so also for every $\psi \in Gamma$, while $v(\psi_0) = f$. Hence $\Gamma \not\vdash_{\mathcal{QM}^B_3\mathbf{l}} \psi_0$.

$\square$

The semantic effects of adding the schemata from $\mathcal{QR}$ to $\mathbf{B}y$ are defined similarly to the finite case (Definition 6.1.6). For $S \subseteq \mathcal{QR}$, the Nmatrices $\mathcal{QM}^B_3\mathbf{l}[S]$, $\mathcal{QM}^B_3\mathbf{d}[S]$ and $\mathcal{QM}^B_3\mathbf{h}[S]$ we denote by $\mathcal{QM}^B_3\mathbf{l}$, $\mathcal{QM}^B_3\mathbf{d}$ and $\mathcal{QM}^B_3\mathbf{h}$ respectively, the weakest simple refinements of in which all the conditions corresponding to the schemata in S hold. Like in the finite case, it is easy to check that for any $S \subseteq \mathcal{QR}$ and $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{h}\}$, the set of conditions in S is coherent, the interpretations of the connectives and the quantifiers of $\mathcal{QM}_3^B y[S]$ never return empty sets and so $\mathcal{QM}_3^B y[S]$ is well-defined.

Theorem 6.2.6. *For $S \subseteq \mathcal{QR}$ and $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{h}\}$, $\Gamma \vdash^v_{\mathcal{QM}_3^B[S],vo} \psi$ iff $\Gamma \vdash_{\mathbf{B}y[S]} \psi$.*

Proof. Since $\mathcal{QM}_3^B y[S]$ is a refinement of $\mathcal{QM}_3^B[S]$. Hence by Theorem 5.2.28 it holds that $\vdash_{\mathcal{QM}_3^B[S],vo} \subseteq \vdash_{\mathcal{QM}_3^B y[S],vo}$. It is also easy to check that for any schema in S, the relevant condition guarantees its validity in $\mathcal{QM}_3^B y[S]$, and so soundness follows. The proof of completeness is a straightforward extension of the proof of Theorem 6.2.5.

□

Corollary 6.2.7. *Let $\Gamma \cup \psi$ be a set of L_C-formulas, in which $\circ$ does not occur. Then $\Gamma \vdash_{\mathbf{Blca}} \psi$ iff $\Gamma \vdash_{\mathbf{Blcia}} \psi$.*

Proof. It can be easily checked that the only difference between the Nmatrices $\mathcal{QM}_3$**lcia** and $\mathcal{QM}_3$**lca** is in their interpretation of $\circ$. □

Corollary 6.2.8. *Let the Nmatrix $\mathcal{QM}_3^B C_1^*$ for $\mathcal{L}_{\text{cl}}$ be obtained from the Nmatrix $\mathcal{QM}_3$**lcia** for L_C (or $\mathcal{QM}_3^B$**lca**) by discarding the interpretation of $\circ$. Then $\mathcal{QM}_3^B C_1^*$[3] is a characteristic Nmatrix for C_1^*.*

Proof. By an extension of the (purely syntactic) proof of theorem 107 of [62], it is possible to show that **QBlcia** is a conservative extension of C_1^*, hence the claim follows. □

Remark 6.2.9. da Costa's C_1 is usually considered to be the $\circ$-free analogue of the propositional fragment of **Blcia** (called **Cila** in [57, 62]). However, from the above corollaries it follows that it is equally justified to identify it with **Cla**, the propositional fragment of **Blca**. A similar observation applies to C_1^*.

It is important to note that, like the finite Nmatrices from the previous section, all of the Nmatrices provided above are V-alalytic:

Corollary 6.2.10. *For every $S \subseteq \mathcal{QR}$ and $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{h}\}$, $\mathcal{QM}_3^B y[S]$ is V-analytic.*

Proof. It is easy to verify that the interpretations of $\forall$ and $\exists$ in $\mathcal{QM}_3^B y[S]$ are universal and existential respectively. The claim follows by Corollary 5.2.32. □

We end this section by applying the V-analycity of the framework to prove an important proof-theoretical property of the first-order LFIs studied here.

[3]This Nmatrix is an extension to the first-order case of the propositional Nmatrix $\mathcal{M}_{C_1}$ from Example 2.3.11.

Definition 6.2.11. *Let* $\mathbf{S}$ *be a system which includes positive classical logic. Two sentences* A *and* B *are* logically indistinguishable *in* $\mathbf{S}$ *if* $\varphi(A) \vdash_{\mathbf{S}} \varphi(B)$ *and* $\varphi(B) \vdash_{\mathbf{S}} \varphi(A)$ *for every sentence* $\varphi(\psi)$ *in the language of* $\mathbf{S}$*.*

The following is an extension of a similar theorem from [20], where it is proved for propositional systems weaker than the propositional fragments of $\mathbf{QBbcia_pe}$ and $\mathbf{QBbio_pe}$.

Theorem 6.2.12. *Let* $\mathbf{S}$ *be a system over a first-order language* L *which includes* $\{\neg, \supset\}$*, and assume that* $A \vdash_{\mathbf{S}} B$ *whenever* $A \sim^{dc} B$*. If one of the following holds, then two sentences* A, B *are logically indistinguishable in* $\mathbf{S}$ *iff* $A \sim^{dc} B$*:*

1. $\mathbf{QBbcia_pwv_Q}$ *is an extension of* $\mathbf{S}$*.*
2. $\mathbf{QBbcia_pev_Q}$ *is an extension of* $\mathbf{S}$*.*
3. $\mathbf{QBbive}$ *is an extension of* $\mathbf{S}$

Proof. The proof is an extension of the proof from [20] to the first-order level. □

Remark 6.2.13. Extensions of **QBcio** do not have the property described above. In fact, it can be shown that $\circ(A \supset A)$ and $\circ(B \supset B)$ are logically indistinguishable in **QBcio** for any two sentences A and B (it is shown in [62] for the propositional case). Extensions of **QBiew** also do not have the above property. In fact, it can be shown that **QBiew** collapses into classical logic, where any two logically equivalent formulas are logically indistinguishable.

Chapter 7

Application: Canonical Calculi with Quantifiers

In this chapter we extend the theory of propositional canonical Gentzen-type calculi (see Chapter 3) to languages with quantifiers. For simplicity of presentation, we assume that language L does not include any propositional connectives (as the latter can be thought of as multi-ary quantifiers which bind no variables).

7.1 Multi-ary Quantifiers

We start our investigation with multi-ary quantifiers. In what follows, L is a language with such quantifiers.
The results in this section are mainly based on [131, 133, 35].

7.1.1 Extending the Notion of Canonical Calculi

We start by proposing a precise characterization of "canonical Gentzen-type rules and systems" with multi-ary quantifiers. Let us first explain the intuition behind our approach. Using an introduction rule for an n-ary quantifier $\mathcal{Q}$, one should be able to derive a sequent of the form $\Gamma \Rightarrow \mathcal{Q}x(\psi_1, ..., \psi_n), \Delta$ or of the form $\Gamma, \mathcal{Q}x(\psi_1, ..., \psi_n) \Rightarrow \Delta$, based on some information about the subformulas of $\mathcal{Q}x(\psi_1, ..., \psi_n)$ contained in the premises of the rule. For instance, consider the following standard rules for the unary quantifier $\forall$:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where $\mathbf{t}, z$ are free for w in A and z does not occur free in the conclusion. Our key observation is that the internal structure of A, as well as the exact term $\mathbf{t}$ or variable

w used, are immaterial for the meaning of $\forall$. What is important here is the sequent on which A appears, as well as whether a term $\mathbf{t}$ or a variable z is used.
It follows that the internal structure of the formulas of L used in the description of a rule can be abstracted by using a simplified language, i.e., the formulas of L in an introduction rule of a n-ary quantifier, can be represented by *atomic* formulas with unary predicate symbols. The case when the substituted term is any L-term, will be signified by a constant, and the case when it is a variable satisfying the above conditions - by a variable. In other words, constants serve as term variables, while variables are eigenvariables. Thus in addition to our original language L with multi-ary quantifiers we use also other simplified languages for schematic representation of the rules.

Definition 7.1.1. For $n \geq 1$ and a set of constants Con, $L_n(Con)$ is the language with n unary predicate symbols $p_1, ..., p_n$ and the set of constants Con (and no quantifiers).

We assume that for every n-ary quantifier $\mathcal{Q}$ of L, $L_n(Con)$ is a subset of L. This assumption is not necessary, but it makes the presentation easier, as will be explained in the sequel. Henceforth, whenever the set Con is clear from context, we will write L_n instead of $L_n(Con)$.

Next we formalize the notion of a canonical rule and its application.

Definition 7.1.2. A *canonical rule of arity* n is an expression of the form $[\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C]$, where $m \geq 0$, C is either $\Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x))$ or $\mathcal{Q}x(p_1(x), ..., p_n(x)) \Rightarrow$ for some n-ary quantifier $\mathcal{Q}$ of L and for every $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ is a clause[1] over L_n.

For an actual application of a canonical rule, we need to instantiate it within some context. For this we need some notion of a *mapping* from the terms and formulas of L_n to the terms and formulas of L, which handles with care the choice of terms and variables of L.

Definition 7.1.3. Let $R = [\Theta/C]$ be a canonical rule, where $C = (\mathcal{Q}x(p_1(x), ..., p_n(x)) \Rightarrow\ )$ or $C = (\Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x)))$. Let Γ be a set of L-formulas and z some variable of L. An $\langle R, \Gamma, z\rangle$*-mapping* is any function χ from the predicate symbols, terms and formulas of L_n to formulas and terms of L, satisfying the following conditions:

- For every $1 \leq i \leq n$, $\chi(p_i)$ is an L-formula.
- $\chi(y)$ is a variable of L.
- $\chi(x) \neq \chi(y)$ for every two variables $x \neq y$.

[1]By a clause we mean as usual a sequent consisting only of atomic formulas.

- $\chi(c)$ is an L-term, such that $\chi(x)$ does not occur in $\chi(c)$ for any variable x occurring in Θ.
- For every $1 \leq i \leq n$, whenever $p_i(\mathbf{t})$ occurs in Θ, $\chi(\mathbf{t})$ is a term free for z in $\chi(p_i)$, and if $\mathbf{t}$ is a variable, then $\chi(\mathbf{t})$ does not occur free in $\Gamma \cup \{\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))\}$.
- $\chi(p_i(\mathbf{t})) = \chi(p_i)\{\chi(\mathbf{t})/z\}$.

We extend χ to sets of L_n-formulas as follows:

$$\chi(\Delta) = \{\chi(\psi) \mid \psi \in \Delta\}$$

Given a schematic representation of a rule and an instantiation mapping, we can define an application of a rule as follows.

Definition 7.1.4. An *application* of a rule $R = [\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\mathcal{Q}x(p_1(x), ..., p_n(x)) \Rightarrow]$ is any inference step of the form:

$$\frac{\{\Gamma, \chi(\Pi_i) \Rightarrow \Delta, \chi(\Sigma_i)\}_{1\leq i\leq m}}{\Gamma, \mathcal{Q}z\ (\chi(p_1), ..., \chi(p_n)) \Rightarrow \Delta}$$

where z is some variable, Γ, Δ are any sets of L-formulas and χ is some $\langle R, \Gamma \cup \Delta, z\rangle$-mapping.
An application of a canonical rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x))$ is defined similarly.

Below we demonstrate the above definitions by a number of examples.

Example 7.1.5. 1. The standard introduction rules for the unary quantifiers $\forall$ and $\exists$ can be formulated as follows:

$$[\{p_1(c) \Rightarrow\}/\forall x\, p_1(x) \Rightarrow] \quad [\{\Rightarrow p_1(x)\}/ \Rightarrow \forall x\, p_1(x)]$$

$$[\{\Rightarrow p_1(d)\}/ \Rightarrow \exists x p_1(x)] \quad [\{p_1(x) \Rightarrow\}/\exists x\, p_1(x) \Rightarrow]$$

Applications of these rules have the forms:

$$\frac{\Gamma, \psi\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, \psi \Rightarrow \Delta}\ (\forall \Rightarrow) \quad \frac{\Gamma \Rightarrow \psi\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, \psi, \Delta}\ (\Rightarrow \forall)$$

$$\frac{\Gamma \Rightarrow \psi\{\mathbf{t}/w\}, \Delta}{\Gamma \Rightarrow \exists w\, A, \Delta}\ (\Rightarrow \exists) \quad \frac{\Gamma, \psi\{z/w\} \Rightarrow \Delta}{\Gamma, \exists w\, \psi \Rightarrow \Delta}\ (\exists \Rightarrow)$$

where z is free for w in ψ, z is not free in $\Gamma \cup \Delta \cup \{\forall w \psi\}$, and $\mathbf{t}$ is any term free for w in ψ.

2. Consider the bounded existential and universal $(2,1)$-ary quantifiers $\overline{\forall}$ and $\overline{\exists}$ (corresponding to $\forall x.p_1(x) \to p_2(x)$ and $\exists x.p_1(x) \wedge p_2(x)$ used in syllogistic reasoning). Their corresponding rules can be formulated as follows:

$$[\{p_2(c) \Rightarrow\ ,\ \Rightarrow p_1(c)\}/\overline{\forall} x\ (p_1(x), p_2(x)) \Rightarrow]$$

$$[\{p_1(x) \Rightarrow p_2(x)\}/ \Rightarrow \overline{\forall} x\ (p_1(x), p_2(x))]$$

$$[\{p_1(x), p_2(x) \Rightarrow\}/\overline{\exists}\ x(p_1(x), p_2(x)) \Rightarrow]$$

$$[\{\Rightarrow p_1(c)\ ,\ \Rightarrow p_2(c)\}/ \Rightarrow \overline{\exists} x(p_1(x), p_2(x))]$$

Applications of these rules are of the form:

$$\frac{\Gamma, \psi_2\{\mathbf{t}/z\} \Rightarrow \Delta \quad \Gamma \Rightarrow \psi_1\{\mathbf{t}/z\}, \Delta}{\Gamma, \overline{\forall} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma, \psi_1\{y/z\} \Rightarrow \psi_2\{y/z\}, \Delta}{\Gamma \Rightarrow \overline{\forall} z\ (\psi_1, \psi_2), \Delta}$$

$$\frac{\Gamma, \psi_1\{y/z\}, \psi_2\{y/z\} \Rightarrow \Delta}{\Gamma, \overline{\exists} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi_1\{\mathbf{t}/x\}, \Delta \quad \Gamma \Rightarrow \psi_2\{\mathbf{t}/x\}, \Delta}{\Gamma \Rightarrow \overline{\exists} z\ (\psi_1, \psi_2), \Delta}$$

where $\mathbf{t}$ and y are free for z in ψ_1 and ψ_2, y does not occur free in $\Gamma \cup \Delta \cup \{\overline{\exists} z(\psi_1, \psi_2)\}$.

When extending the notion of "canonical calculi" to languages with quantifiers, there are two important additions that were not present on the propositional level. The first is generalizing the logical axioms to capture the α-equivalence principle. Consider, for instance, the classical Gentzen-type rules for $\forall$. The sequent $\psi \Rightarrow \psi'$ is derivable using these rules for any two α-equivalent formulas ψ, ψ'. However, if we discard one of the rules, this sequent is no longer derivable in the resulting calculus. Hence, the derivability of the α-axiom is not guaranteed in a canonical calculus, and so we add this axiom explicitly. The second addition is that of the *substitution rule*, the importance of which will become clear in the sequel.

Definition 7.1.6. For any language L, an L-formula ψ' is a *L-instance of* ψ if ψ' is of the form $\psi\{\mathbf{t}_1/x_1, ..., \mathbf{t}_n/x_n\}$, where $\mathbf{t}_1, ..., \mathbf{t}_n$ are L-terms free in ψ for $x_1, ..., x_n$ respectively. An L-instance Ω' (Θ') of a sequent Ω (a set of sequents Θ) is defined similarly.

The following definition extends Definition 3.3.12 to languages with multi-ary quantifiers:

Definition 7.1.7. 1. An *alpha-axiom* is a sequent of the form $\psi \Rightarrow \psi'$, where $\psi \equiv_\alpha \psi'$.

2. The *substitution rule* is defined as follows:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}\ \textbf{Sub}$$

   where $\Gamma' \Rightarrow \Delta'$ is any L-instance of $\Gamma \Rightarrow \Delta$.

3. A calculus with multi-ary quantifiers is *canonical* if it consists of: (i) All alpha-axioms, (ii) The rules of cut, weakening and substitution, and (iii) A finite number of canonical rules.

The notion of coherence for calculi with quantifiers is defined similarly to the propositional case (see Definition 3.1.4). The only addition is the use of renaming, the purpose of which is to avoid clashing names of constants and variables in different canonical rules:

Definition 7.1.8. For two sets of clauses Θ_1, Θ_2 over L_n, $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is a set $\Theta_1 \cup \Theta_2'$, where Θ_2' is obtained from Θ_2 by a fresh renaming of constants and variables which occur in Θ_1.

Definition 7.1.9. *A canonical calculus G is* coherent *if for every pair of canonical rules of the form $\Theta_1/\ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, the set of clauses $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically inconsistent (i.e., the empty sequent can be derived from it using cuts and substitutions).*

Note that the principle of renaming of clashing constants and variables is similar to the one used in first-order resolution. The importance of this principle for the definition of coherence will be demonstrated in the sequel (Remark 7.1.34 below).

Proposition 7.1.10. *The coherence of a canonical calculus G is decidable.*

Proof. The question of classical consistency of a finite set of clauses without function symbols (over L_n) can be shown to be equivalent to satisfiability of a finite set of universal formulas with no function symbols. This is decidable (by an obvious application of Herbrand's theorem). □

7.1.2 2Nmatrices, Strong Cut-elimination and Coherence

Recall that for propositional Gentzen-type calculi there is a correspondence between coherence, 2Nmatrices and cut elimination (Theorem 3.1.6). Below we establish a similar correspondence for canonical calculi with multi-ary quantifiers, however this time using *strong* cut-elimination instead of the standard one. Interestingly enough, for languages with multi-ary quantifiers, standard cut-elimination no longer implies coherence, as we shall see below.

The notion of strong cut-elimination for propositional calculi (Definition 3.3.15) is extended to languages with quantifiers as follows:

Notation 7.1.11. We say that a sequent Ω (a set of sequents Θ) satisfies the *free-variable condition* if the set of variables occurring bound in Ω (Θ) is disjoint from the set of variables occurring free in it.

Definition 7.1.12.

1. Let Θ be a set of sequents and Ω a sequent over L. A proof of Ω from Θ is Θ-cut-free if all cuts in it are on substitution instances of formulas from Θ.
2. A calculus G with quantifiers admits *strong cut-elimination* if for every set of sequents Θ and every sequent Ω, such that $\Theta \cup \{\Omega\}$ satisfy the free-variable condition, whenever $\Theta \vdash_G \Omega$, Ω has a Θ-cut-free proof from Θ in G.

Now recall that in addition to the language L, for each n-ary canonical rule we have a simplified language L_n used for formulating the rules. For the semantics of these languages, we shall use two-valued L_n-structures (defined similarly to standard two-valued L-structures). To make the distinction clearer, we shall use the metavariable S for L-structures and $\mathcal{W}$ for L_n-structures. Since the formulas of L_n are always atomic, the truth-value of an L_n-sentence depends only on the given structure (and not on valuations). Hence we have the following natural definition of the semantics for L_n:

Definition 7.1.13. An L_n-structure $\mathcal{W} = \langle D, I\rangle$ *satisfies* an atomic L_n-sentence $p(\mathbf{t})$ if $I(p)(I(\mathbf{t})) = t$. $\mathcal{W}$ satisfies an L_n-clause $\Gamma \Rightarrow \Delta$ consisting of sentences if either $\mathcal{W}$ satisfies some sentence in Δ, or it does not satisfy some sentence in Γ. An L_n-clause is *$\mathcal{W}$-valid* if $\mathcal{W}$ satisfies each of its closed $L_n(D)$-instances. A set of L_n-clauses is *$\mathcal{W}$-valid* if each of its clauses is *$\mathcal{W}$-valid*. A set of L_n-clauses is *satisfiable* if there is some L_n-structure $\mathcal{W}$ in which it is valid.

Remark 7.1.14. By the well-known completeness of first-order resolution, a set of L_n-clauses Θ is classically consistent iff Θ is satisfiable. Hence, checking whether a canonical calculus is coherent can be reduced to checking satisfiability of sets of clauses.

Definition 7.1.15. Let $\mathcal{E} \in P^+(\{t, f\}^n)$. A set of L_n-clauses Θ is $\mathcal{E}$-characteristic if it is $\mathcal{W}$-valid for some L_n-structure $\mathcal{W} = \langle D, I\rangle$ in which $\{\langle I(p_1)(a), ..., I(p_n)(a)\rangle \mid a \in D\} = \mathcal{E}$.

In this context it is convenient to define a special kind of L_n-structures which we call *canonical* structures, which are sufficient to reflect the behavior of all possible L_n-structures.

Definition 7.1.16. *Let $\mathcal{E} \in P^+(\{t, f\}^n)$. An L_n-structure $\mathcal{W} = \langle D, I\rangle$ is $\mathcal{E}$-canonical if $D = \mathcal{E}$ and for every $b = \langle s_1, ..., s_n\rangle \in D$: $I(p_i)(b) = a_i$ for every $1 \leq i \leq n$.*

Lemma 7.1.17. *If Θ is $\mathcal{E}$-characteristic, then Θ is $\mathcal{W}$-valid for some $\mathcal{E}$-canonical L_n-structure $\mathcal{W}$.*

Proof. Suppose that Θ is $\mathcal{E}$-characteristic. Then Θ valid in an L_n-structure $\mathcal{W} = \langle D, I\rangle$, where $\{\langle I(p_1)(a), ..., I(p_n)(a)\rangle \mid a \in D\} = \mathcal{E}$. Define the L_n-structure $\mathcal{W}' = \langle I', D'\rangle$ as follows: $D' = \mathcal{E}$, $I'(c) = \langle I(p_1)(I(c)), ..., I(p_n)(I(c))\rangle$ for every constant c occurring in Θ, and for every $1 \leq i \leq n$: $I'(p_i)(\langle s_1, ..., s_n\rangle) = t$ iff $s_i = t$. Clearly, $\mathcal{W}'$ is $\mathcal{E}$-canonical. It is also easy to verify that Θ is also valid in $\mathcal{W}'$. □

Corollary 7.1.18. *For any $\mathcal{E} \in P^+(\{t, f\}^n)$ and any finite set of L_n-clauses Θ, the question whether Θ is $\mathcal{E}$-characteristic is decidable.*

Proof. Follows directly from Lemma 7.1.17 and the fact that there are finitely many $\mathcal{E}$-canonical structures.

□

Lemma 7.1.19. *Let $\mathcal{E} \in P^+(\{t, f\}^n)$. Let Θ_1 and Θ_2 be two $\mathcal{E}$-characteristic sets of L_n-clauses with disjoint sets of constants. Then so the set $\Theta_1 \cup \Theta_2$ is also $\mathcal{E}$-characteristic.*

Proof. If Θ_1 and Θ_2 are $\mathcal{E}$-characteristic, then by Lemma 7.1.17 there are $\mathcal{E}$-canonical structures $\mathcal{W}_1$ and $\mathcal{W}_2$, in which Θ_1 and Θ_2 are valid respectively. The only difference between different $\mathcal{E}$-canonical structures is in the interpretation of constants, and since the sets of constants occurring in Θ_1 and Θ_2' are disjoint, an extended $\mathcal{E}$-canonical structure for the language containing the constants of both Θ_1 and Θ_2 can be easily constructed, in which $\Theta_1 \cup \Theta_2$ is valid.

□

Definition 7.1.20. Let G be a canonical calculus. A 2Nmatrix $\mathcal{M}$ is *suitable* for G if for every canonical introduction rule $R = [\Theta/C]$ for an n-ary quantifier $\mathcal{Q}$, it holds that $\tilde{\mathcal{Q}}_{\mathcal{M}}(\mathcal{E}) = \{v_C\}$ whenever Θ is $\mathcal{E}$-characteristic, where $v_C = t$ if R is a right introduction rule, and $v_C = f$ if R is a left one.

The following theorem establishes the strong soundness of a calculus G for any 2Nmatrix suitable for G:

Theorem 7.1.21. *Let G be a canonical calculus and $\mathcal{M}$ a 2Nmatrix suitable for G. Then for every set of sequents Θ and every sequent Ω: whenever $\Theta \vdash_G \Omega$, also $\Theta \vdash_{\mathcal{M}} \Omega$.*

Proof. Suppose that $\mathcal{M}$ is suitable for G and $\Theta \vdash_G \Omega$. Let $S = \langle D, I\rangle$ be some L-structure and v - an $\mathcal{M}$-legal S-valuation, such that Θ is $\mathcal{M}$-valid in $\langle S, v\rangle$. We show that Ω is $\mathcal{M}$-valid in $\langle S, v\rangle$. It is easy to see that the axioms, structural rules, substitution and cut are sound with respect to $\mathcal{M}$. It remains to show that for every application of a canonical rule R of G: if the premises of R are $\mathcal{M}$-valid in $\langle S, v\rangle$, then its conclusion is $\mathcal{M}$-valid in $\langle S, v\rangle$. Suppose w.l.o.g. that R has the form $[\Theta_R / \Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x))]$ where $\Theta_R = \{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$. Then any application of R is of the form:

$$\frac{\{\Gamma, \chi(\Sigma_j) \Rightarrow \chi(\Pi_j), \Delta\}_{1\leq j\leq m}}{\Gamma, \Rightarrow \Delta, \mathcal{Q}z(\chi(p_1), ..., \chi(p_n))}$$

where χ is some $\langle R, \Gamma\cup\Delta, z\rangle$-mapping. Suppose that $\{\Gamma, \chi(\Sigma_j) \Rightarrow \chi(\Pi_j), \Delta\}_{1\leq j\leq m}$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. We will now show that $\Gamma \Rightarrow \Delta, \mathcal{Q}z(\chi(p_1), ..., \chi(p_n))$ is also $\mathcal{M}$-valid in $\langle S, v\rangle$.

Notation 7.1.22. By a substitution we shall mean below a mapping from variables of L to closed $L(D)$-terms. For a substitution σ, we say that $\Gamma' \Rightarrow \Delta'$ is the σ-*instance* of $\Gamma \Rightarrow \Delta$ if it is obtained from $\Gamma \Rightarrow \Delta$ by replacing each variable x occurring free in $\Gamma \cup \Delta$ by the closed term $\sigma(x)$. Denote by $\sigma(\psi)$ ($\sigma(\mathbf{t})$) the sentence (closed term) obtained from ψ ($\mathbf{t}$) by replacing each variable x occurring free in ψ ($\mathbf{t}$) by the closed term $\sigma(x)$. Denote $\sigma(\Gamma) = \{\sigma(\psi) \mid \psi \in \Gamma\}$.

Let $\sigma(\Gamma) \Rightarrow \sigma(\Delta), \sigma(\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))$ be the σ-instance of $\Gamma \Rightarrow \Delta, \mathcal{Q}z(\chi(p_1), ..., \chi(p_n)$ for some substitution σ and suppose that $v \not\models \sigma(\Gamma) \Rightarrow \sigma(\Delta)$ (otherwise $v \models \sigma(\Gamma) \Rightarrow \sigma(\Delta), \sigma(\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))$ and we are done).
For $\psi \in \{\chi(p_1), \ldots, \chi(p_n)\}$ denote by $\widetilde{\psi}$ the $L(D)$-formula obtained from ψ by substituting $\sigma(w)$ for every free occurrence of $w \in Fv(\psi) - \{z\}$. Suppose that $\mathcal{E}$ is the set $\{\langle v(\widetilde{\chi(p_1)}\{\overline{a}/z\}), ..., v(\widetilde{\chi(p_n)}\{\overline{a}/z\})\rangle \mid a \in D\}$. We now show that Θ_R is $\mathcal{E}$-characteristic and so it must be the case that $\tilde{\mathcal{Q}}(\mathcal{E}) = \{t\}$. Since v is $\mathcal{M}$-valid, it follows that $v \models \mathcal{Q}z(\widetilde{\chi(p_1)}, \ldots, \widetilde{\chi(p_n)}) = \sigma(\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))$ and $v \models \sigma(\Gamma) \Rightarrow \sigma(\Delta), \sigma(\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))$.

Construct the L_n-structure $\mathcal{W} = \langle D', I'\rangle$ as follows: (i) $D' = D$, (ii) for every $a \in D$: $I'(p_i)(a) = v(\widetilde{\chi(p_i)}\{\overline{a}/z\})$, and (iii) for every constant c, $I'(c) = I(\sigma(\chi(c)))$. Let us show that $\Theta_R = \{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$ is valid in $\mathcal{W}$. Let $1 \leq j \leq m$ and let $\eta(\Sigma_i) \Rightarrow \eta(\Pi_j)$ be the η-instance of $\Sigma_j \Rightarrow \Pi_j$ for some substitution η. Suppose that $\mathcal{W}$ satisfies all the formulas in $\eta(\Sigma_j)$. Now we show that $\mathcal{W}$ satisfies some formula from $\eta(\Pi_j)$. Let σ' be the following substitution:

$$\sigma'(x) = \begin{cases} \overline{I'(\eta(y))} & x = \chi(y) \text{ and } y \text{ occurs free in } \Theta_R \\ \sigma(x) & \text{otherwise} \end{cases}$$

Note that σ' is well-defined, since for every two different variables x, y: $\chi(x) \neq \chi(y)$ (recall Definition 7.1.3). Let $\psi \in \chi(\Sigma_j) \cup \chi(\Pi_j)$. Then there is some $1 \leq i_\psi \leq m$, such that $p_{i_\psi}(\mathbf{t}) \in \Sigma_j \cup \Pi_j$ and $\psi = \chi(p_{i_\psi})\{\chi(\mathbf{t})/z\}$. We show that $v(\sigma'(\psi)) = I'(p_{i_\psi})(I'(\eta(\mathbf{t})))$. One of the following cases holds:

- $\mathbf{t}$ is some constant c. Then $\psi = \chi(p_{i_\psi})\{\chi(c)/z\}$, where $\chi(c)$ is some term free for z in $\chi(p_{i_\psi})$, such that for any variable y occurring in Θ_R, $\chi(y)$ does not occur free in $\chi(c)$. And so we have:

  $$v(\sigma'(\psi)) = v(\sigma'(\chi(p_{i_\psi})\{\chi(c)/z\})) = v(\widetilde{\chi(p_{i_\psi})}\{\sigma'(\chi(c))/z\}) = v(\widetilde{\chi(p_{i_\psi})}\{\sigma(\chi(c))/z\})$$

  (Recall that for every variable y occurring in Θ_R, $\chi(y)$ does not occur free in $\mathcal{Q}z(\chi(p_1), ..., \chi(p_n))$, and so $\sigma(y) = \sigma'(y)$ for any such variable). By Lemma 5.2.11-2 and the legality of v:

  $$v(\widetilde{\chi(p_{i_\psi})}\{\sigma(\chi(c))/z\}) = v(\widetilde{\chi(p_{i_\psi})}\{\overline{I(\sigma(\chi(c)))}/z\})$$

  By definition of I', $I'(c) = I(\sigma(\chi(c)))$ and so:

  $$v(\widetilde{\chi(p_{i_\psi})}\{\overline{I(\sigma(\chi(c)))}/z\}) = v(\widetilde{\chi(p_{i_\psi})}\{\overline{I'(c)}/z\}) = I'(p_{i_\psi})(I'(c)) = I'(p_{i_\psi})(I'(\eta(c)))$$

- $\mathbf{t}$ is some variable y. Then $\psi = \chi(p_{i_\psi})\{\chi(y)/z\}$ where $\chi(y)$ does not occur free in $\Gamma \cup \Delta \cup \{\mathcal{Q}z(\psi_1, ..., \psi_n)\}$ and is free for z in $\chi(p_{i_\psi})$. Let $a = I'(\eta(y))$. Then by definition of σ', $\sigma'(\chi(y)) = \overline{a}$ and so:

  $$v(\sigma'(\psi)) = v(\sigma'(\chi(p_{i_\psi})\{\chi(y)/z\}) = v(\widetilde{\chi(p_{i_\psi})}\{\sigma'(\chi(y))/z\}) =$$

  $$= v(\widetilde{\chi(p_{i_\psi})}\{\overline{a}/z\}) = I'(p_{i_\psi})(a) = I'(p_{i_\psi})(I'(\eta(y)))$$

Thus we have shown that $v(\sigma'(\psi)) = I'(p_{i_\psi})(I'(\eta(\mathbf{t})))$ for every $\psi \in \chi(\Sigma_j) \cup \chi(\Pi_j)$. Since we assumed that $\mathcal{W}$ satisfies $\eta(\Sigma_j)$, it follows that (i) $v \models \sigma'(\chi(\Sigma_j))$. Recall also that we have assumed that (ii) $\Gamma, \chi(\Sigma_j) \Rightarrow \chi(\Pi_j), \Delta$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. Now since there is no variable y occurring in Θ_R, such that $\chi(y)$ occurs in $\Gamma \cup \Delta$, it follows that $\sigma(\varphi) = \sigma'(\varphi)$ for any $\varphi \in \Gamma \cup \Delta$. Note that $\sigma(\Gamma), \sigma'(\chi(\Sigma_j)) \Rightarrow \sigma'(\chi(\Pi_j)), \sigma(\Delta)$ is a closed $L(D)$-instance of $\Gamma, \chi(\Sigma_j) \Rightarrow \chi(\Pi_j), \Delta$, so by (i) and (ii) above there is some $\theta \in \chi(\Pi_j)$, such that $v(\sigma'(\theta)) = t$. Hence $I'(p_{i_\theta})(I'(\eta(\mathbf{t}))) = t$ for some $p_{i_\theta}(\mathbf{t}) \in \Pi_j$ and $\mathcal{W}$ satisfies

$\eta(\Sigma_j) \Rightarrow \eta(\Pi_j)$.
Thus we have shown that $\Theta_R = \{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$ is $\mathcal{W}$-valid. Moreover, by definition of $\mathcal{W}$, $\{\langle I'(p_1)(a),\ldots,I'(p_n)(a)\rangle \mid a \in D\} = \{\langle v(\widetilde{\chi(p_1)}\{\overline{a}/z\}),...,v(\widetilde{\chi(p_n)}\{\overline{a}/z\})\rangle \mid a \in D\} = \mathcal{E}$ and so Θ_R is $\mathcal{E}$-characteristic. Since $\mathcal{M}$ is suitable for G, $\tilde{\mathcal{Q}}(\mathcal{E}) = \{t\}$ and by the $\mathcal{M}$-legality of v, it follows that $v \models \mathcal{Q}z(\phi_1,\ldots,\phi_n)$. Thus $\Gamma \Rightarrow \Delta, \mathcal{Q}z(\chi(p_1),...,\chi(p_n))$ is $\mathcal{M}$-valid in $\langle S, v\rangle$ (since each of its closed $L(D)$-instances is satisfied by v in S.)

□

Now we come to the construction of a characteristic 2Nmatrix for every coherent canonical calculus.

Definition 7.1.23. *Let G be a coherent canonical calculus. The Nmatrix $\mathcal{M}_G$ for L is defined as follows for every n-ary quantifier $\mathcal{Q}$ of L and every $\mathcal{E} \in P^+(\{t,f\}^n)$:*

$$\tilde{\mathcal{Q}}_{\mathcal{M}_G}(\mathcal{E}) = \begin{cases} \{t\} & \textit{there is some } [\Theta/\Rightarrow \mathcal{Q}x(p_1(x),\ldots,p_n(x))] \in G, \\ & \textit{where } \Theta \textit{ is } \mathcal{E}\textit{-characteristic} \\ \{f\} & \textit{there is some } [\Theta/\mathcal{Q}x(p_1(x),\ldots,p_n(x)) \Rightarrow] \in G, \\ & \textit{where } \Theta \textit{ is } \mathcal{E}\textit{-characteristic} \\ \{t,f\} & \textit{otherwise} \end{cases}$$

First of all, note that by Corollary 7.1.18, the above definition is constructive. Next, let us show that $\mathcal{M}_G$ is well-defined. Assume by contradiction that there are rules $[\Theta_1/\Rightarrow \mathcal{Q}x(p_1(x),\ldots,p_n(x))]$ and $[\Theta_2/\mathcal{Q}x(p_1(x),\ldots,p_n(x)) \Rightarrow]$, such that both Θ_1 and Θ_2 are $\mathcal{E}$-characteristic, and so is Θ_2' which is obtained from Θ_2 by fresh renamings of constants and variables. By Lemma 7.1.19, $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \Theta_1 \cup \Theta_2'$ is $\mathcal{E}$-characteristic, and so also consistent, in contradiction to the coherence of G.

Let us demonstrate the construction of a characteristic 2Nmatrix by some simple examples.

Example 7.1.24. 1. It is easy to see that for any canonical coherent calculus G including the standard unary rules for $\forall$ and $\exists$ from Example 7.1.5-2:

$$\tilde{\forall}_{\mathcal{M}_G}(\{t,f\}) = \tilde{\forall}_{\mathcal{M}_G}(\{f\}) = \tilde{\exists}_{\mathcal{M}_G}(\{f\}) = \{f\}$$

$$\tilde{\forall}_{\mathcal{M}_G}(\{t\}) = \tilde{\exists}_{\mathcal{M}_G}(\{t,f\}) = \tilde{\exists}_{\mathcal{M}_G}(\{t\}) = \{t\}$$

2. Consider the canonical calculus G' consisting of the following three binary rules from Example 7.1.5:

$$[\{p_1(x) \Rightarrow p_2(x)\}/ \Rightarrow \overline{\forall}x\ (p_1(x), p_2(x))]$$

$$[\{p_2(c) \Rightarrow \ , \ \Rightarrow p_1(c)\}/\overline{\forall} v_1(p_1(x), p_2(x)) \Rightarrow]$$

$$[\{\Rightarrow p_1(c) \ , \ \Rightarrow p_2(c)\}/ \Rightarrow \overline{\exists} x(p_1(x), p_2(x))]$$

G' is obviously coherent. The 2Nmatrix $\mathcal{M}_{G'}$ is defined as follows for every $\mathcal{E} \in P^+(\{t,f\}^2)$:

$$\tilde{\overline{\forall}}(\mathcal{E}) = \begin{cases} \{t\} & \textit{if } \langle t,f\rangle \notin \mathcal{E} \\ \{f\} & \textit{otherwise} \end{cases} \qquad \tilde{\overline{\exists}}(\mathcal{E}) = \begin{cases} \{t\} & \textit{if } \langle t,t\rangle \in \mathcal{E} \\ \{t,f\} & \textit{otherwise} \end{cases}$$

The first rule dictates the condition that $\overline{\forall}(\mathcal{E}) = \{t\}$ for the case of $\langle t,f\rangle \notin \mathcal{E}$. The second rule dictates the condition that $\overline{\forall}(\mathcal{E}) = \{f\}$ for the case that $\langle t,f\rangle \in \mathcal{E}$. Since G' is coherent, these conditions are non-contradictory. The third rule dictates the condition that $\overline{\exists}(\mathcal{E}) = \{t\}$ in the case that $\langle t,t\rangle \in \mathcal{E}$. There is no rule which dictates conditions for the case of $\langle t,t\rangle \notin \mathcal{E}$, and so the interpretation in this case is non-deterministic.

3. Consider the canonical calculus G'' consisting of the following trenary rule:

$$[\{p_2(x), p_3(x) \Rightarrow\}/\mathcal{Q}x(p_1(x), p_2(x), p_3(x)) \Rightarrow]$$

Of course, G'' is coherent. The 2Nmatrix $\mathcal{M}_{G''}$ is defined as follows for every $\mathcal{E} \in P^+(\{t,f\}^3)$:

$$\tilde{\overline{\mathcal{Q}}}(\mathcal{E}) = \begin{cases} \{f\} & \textit{if } \mathcal{E} \subseteq \{\langle t,t,f\rangle, \langle t,f,t\rangle, \langle t,f,f\rangle, \langle f,t,f\rangle, \langle f,f,t\rangle, \langle f,f,f\rangle\} \\ \{t,f\} & \textit{if } \langle f,t,t\rangle \in \mathcal{E} \textit{ or } \langle t,t,t\rangle \in \mathcal{E} \end{cases}$$

Theorem 7.1.25. *Let G be a coherent canonical calculus. Then $\mathcal{M}_G$ is strongly characteristic for G.*

Proof. It is easy to see that $\mathcal{M}_G$ is suitable for G. Strong soundness follows by Proposition 7.1.21. For strong completeness, we shall need the following proposition:

Proposition 7.1.26. *Let G be a coherent calculus. Let $\mathcal{S}$ be a set of sequents and $\Gamma \Rightarrow \Delta$ - a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition (see Notation 7.1.11). If $\Gamma \Rightarrow \Delta$ has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G, then $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$.*

Proof. Let $\mathcal{S}$ be a set of sequents and $\Gamma \Rightarrow \Delta$ a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. Suppose that $\Gamma \Rightarrow \Delta$ has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G. To show that $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$, we construct a structure S and an $\mathcal{M}$-legal valuation v, such that the sequents of $\mathcal{S}$ are $\mathcal{M}_G$-valid in $\langle S, v\rangle$, while $\Gamma \Rightarrow \Delta$ is not.
It is easy to see that we can limit ourselves to the language L^*, which is a subset of L,

consisting of all the constants and predicate and function symbols, occurring in $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$.

Let $\mathbf{T}$ be the set of all the terms in L^* which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$ and $\mathcal{S}$. It is a standard matter to show that Γ, Δ can be extended to two (possibly infinite) sets Γ', Δ' (where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), satisfying the following properties:

1. For every finite $\Gamma_1 \subseteq \Gamma'$ and $\Delta_1 \subseteq \Delta'$, $\Gamma_1 \Rightarrow \Delta_1$ has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G.

2. There are no $\psi \in \Gamma'$ and $\varphi \in \Delta'$, such that $\psi \equiv_\alpha \varphi$.

3. If $[\Theta / \Rightarrow (p_1(x), \ldots, p_n(x))] \in G$ ($[\Theta/(p_1(x), \ldots, p_n(x)) \Rightarrow] \in G$) for $\Theta = \{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m}$, then there is some $1 \leq j \leq m$, such that:

    - For every constant c: if $p_i(c) \in \Sigma_j$ ($p_i(c) \in \Pi_j$) for some $1 \leq i \leq n$, then $\psi_i\{\mathbf{t}/z\} \in \Gamma'$ ($\psi_i\{\mathbf{t}/z\} \in \Delta'$) for every term $\mathbf{t} \in \mathbf{T}$.
    - For every variable y, there exists some $\mathbf{t}_y \in \mathbf{T}$, such that whenever $p_i(y) \in \Sigma_j$ ($p_i(y) \in \Pi_j$) for some $1 \leq i \leq n$, then $\psi_i\{\mathbf{t}_y/z\} \in \Gamma'$ ($\psi_i\{\mathbf{t}_y/z\} \in \Delta'$).

    (Note that every $\mathbf{t} \in \mathbf{T}$ is free for z in ψ_i above for every $1 \leq i \leq n$.)

4. For every formula ψ occurring in $\mathcal{S}$, every closed L^*-instance ψ' of ψ is in $\Gamma' \cup \Delta'$.

(Note that the last condition can be satisfied because cuts on formulas from $\mathcal{S}$ are allowed in a $\mathcal{S}$-cut-free proof.)

Let $S = \langle D, I \rangle$ be the L^*-structure defined as follows:

- $D = \mathbf{T}$.
- $I(c) = c$ for every constant c of L^*.
- $I(f)(\mathbf{t}_1, ..., \mathbf{t}_n) = f(\mathbf{t}_1, ..., \mathbf{t}_n)$ for every n-ary function symbol f.
- $I(p)(\mathbf{t}_1, ..., \mathbf{t}_n) = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$ for every n-ary predicate symbol p of L^*.

For an L^*-formula ψ (an L^*-term $\mathbf{t}$), denote by $\sigma^*(\psi)$ ($\sigma^*(\mathbf{t})$) the closed $L^*(D)$-formula ($L^*(D)$-term) obtained from ψ ($\mathbf{t}$) by replacing every variable x occurring free in ψ ($\mathbf{t}$) for $\overline{x}$. (Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has an individual constant referring to it in $L^*(D)$.)

For an $L(D)$-formula ψ (an $L(D)$-term $\mathbf{t}$), the formula $\widehat{\psi}$ ($\widehat{\mathbf{t}}$) are defined like in the proof of Theorem 6.1.1 (in other words, $\widehat{\psi}$ and $\widehat{\mathbf{t}}$ are obtained from ψ ($\mathbf{t}$) by replacing every individual constant of the form $\overline{\mathbf{s}}$ for some $\mathbf{s} \in \mathbf{T}$ by the term $\mathbf{s}$). The following lemma is proved by a tedious induction on the structure of $\mathbf{t}$ and ψ:

Lemma 7.1.27. *Let $\boldsymbol{t}$ be an $L^*(D)$-term and ψ - an $L^*(D)$-formula.*

1. *For any z, x: $\widehat{\boldsymbol{t}}\{z/x\} = \widehat{\boldsymbol{t}\{z/x\}}$ and $\widehat{\psi}\{z/x\} = \widehat{\psi\{z/x\}}$.*
2. *$\psi \sim^S \sigma^*(\widehat{\psi})$.*
3. *For every $\psi \in \Gamma' \cup \Delta'$: $\widehat{\sigma^*(\psi)} = \psi$.*

Next define the S-valuation v as follows:

- $v(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n))$.
- If there is some $\varphi \in \Gamma' \cup \Delta'$, such that $\varphi \equiv_\alpha \widehat{\mathcal{Q}x(\psi_1, ..., \psi_n)}$, then $v(\mathcal{Q}x(\psi_1, ..., \psi_n)) = t$ iff $\varphi \in \Gamma'$. Otherwise $v(\mathcal{Q}x(\psi_1, ..., \psi_n)) = t$ iff $\tilde{\mathcal{Q}}(\{\langle v(\psi_1\{\overline{a}/x\}), ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\}) = \{t\}$.

Lemma 7.1.28. 1. *$I^*(\sigma^*(\boldsymbol{t})) = \boldsymbol{t}$ for every $\boldsymbol{t} \in \mathbf{T}$.*

2. *For every two $L^*(D)$-formulas ψ, ψ': if $\psi \equiv_\alpha \psi'$, then $\sigma^*(\psi) \equiv_\alpha \sigma^*(\psi')$.*
3. *For every two $L^*(D)$-sentences ψ, ψ': if $\psi \sim^S \psi'$, then $\widehat{\psi} \equiv_\alpha \widehat{\psi'}$.*

Proof. The claims are proven by induction on $\mathbf{t}$ in the first case, and on ψ and ψ' in the second and third cases.

Lemma 7.1.29. *For every $\psi \in \Gamma' \cup \Delta'$: $v(\sigma^*(\psi)) = t$ iff $\psi \in \Gamma'$.*

Proof. If ψ is an atomic formula of the form $p(\mathbf{t}_1, ..., \mathbf{t}_n)$, then it holds that $v(\sigma^*(\psi)) = I(p)(I(\sigma^*(\mathbf{t}_1)), ..., I(\sigma^*(\mathbf{t}_n)))$. Note[2] that for every $1 \leq i \leq n$, $\mathbf{t}_i \in \mathbf{T}$. By Lemma 7.1.28-1, $I(\sigma^*(\mathbf{t}_i)) = \mathbf{t}_i$, and by the definition of I, $v(\sigma^*(\psi)) = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$.
Otherwise $\psi = \mathcal{Q}(\psi_1, ..., \psi_n)$. If $\psi \in \Gamma'$, then by Lemma 7.1.27-3 $\widehat{\sigma^*(\psi)} = \psi \in \Gamma'$ and so $v(\sigma^*(\psi)) = t$. If $\psi \in \Delta'$ then by property 2 of $\Gamma' \cup \Delta'$ it cannot be the case that there is some $\varphi \in \Gamma'$, such that $\varphi \equiv_\alpha \widehat{\sigma^*(\psi)} = \psi$ and so $v(\sigma^*(\psi)) = f$.

□

Lemma 7.1.30. *v is legal in $\mathcal{M}_G$.*

[2]This is obvious if $\mathbf{t}_i$ does not occur in the set $\{\Gamma \Rightarrow \Delta\} \cup \mathcal{S}$. If it occurs in this set, then by the free-variable condition $\mathbf{t}_i$ does not contain variables bound in this set and so $\mathbf{t}_i \in \mathbf{T}$ by definition of $\mathbf{T}$.

Proof. First we need to show that v respects the $\sim^S$-relation. First it is easy to show by induction that for every two closed $L^*(D)$-terms $\mathbf{t}, \mathbf{s}$: $\mathbf{t} \sim^S \mathbf{s}$ implies $I(\mathbf{t}) = I(\mathbf{s})$. Next suppose that $\psi \sim^S \psi'$. By Lemma 5.3.5, one of the following cases holds:

- $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$, $\psi' = p(\mathbf{s}_1, ..., \mathbf{s}_n)$ and $\mathbf{t}_i \sim^S \mathbf{s}_i$ for every $1 \leq i \leq n$. Then by the property above $I(\mathbf{t}_i) = I(\mathbf{s}_i)$ and by definition of v: $v(p(\mathbf{t}_1, ..., \mathbf{t}_n)) = I(p)(I(\mathbf{t}_1), ..., I(\mathbf{t}_n)) = I(p)(I(\mathbf{s}_1), ..., I(\mathbf{s}_n))$
$= v(p(\mathbf{s}_1, ..., \mathbf{s}_n))$.

- $\psi = \mathcal{Q}x(\psi_1, ..., \psi_n)$, $\psi' = \mathcal{Q}y(\psi'_1, ..., \psi'_n)$ and for every $1 \leq i \leq n$: $\psi_i\{z/x\} \sim^S \psi'_i\{z/y\}$ for a fresh variable z. By Lemma 5.2.11-2, for every $a \in D$: $\psi_i\{z/x\}\{\overline{a}/z\} = \psi_i\{\overline{a}/x\} \sim^S \psi'_i\{\overline{a}/y\} = \psi_i\{z/y\}\{\overline{a}/z\}$. By the induction hypothesis, it holds that $\{\langle v(\psi_1\{\overline{a}/x\}), ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\} = \{\langle v(\psi'_1\{\overline{a}/x\}), ..., v(\psi'_n\{\overline{a}/x\})\rangle \mid a \in D\}$. One of the following cases holds:

  - There is no $\varphi \in \Gamma' \cup \Delta'$, such that $\varphi \equiv_\alpha \widehat{\psi}$ or $\varphi \equiv_\alpha \widehat{\psi'}$. Then $v(\mathcal{Q}x(\psi_1, ..., \psi_n)) = t$ iff $\{\langle v(\psi_1\{\overline{a}/x\}), ..., v(\psi_n\{\overline{a}/x\})\rangle \mid a \in D\} = t$ iff
  $\{\langle v(\psi'_1\{\overline{a}/x\}), ..., v(\psi'_n\{\overline{a}/x\})\rangle \mid a \in D\} = t$ iff $v(\mathcal{Q}y(\psi'_1, ..., \psi'_n)) = t$.
  - There is some $\varphi \in \Gamma' \cup \Delta'$, such that $\varphi \equiv_\alpha \widehat{\psi}$. By Lemma 7.1.28-3, $\widehat{\psi} \equiv_\alpha \widehat{\psi'}$, and so $v(\psi) = v(\psi') = t$ iff $\varphi \in \Gamma'$.
  - There is some $\varphi \in \Gamma' \cup \Delta'$, such that $\varphi \equiv_\alpha \widehat{\psi'}$. Similarly to the previous case, $v(\psi) = v(\psi') = t$ iff $\varphi \in \Gamma$.

It remains to show that v respects the interpretations of the quantifiers in $\mathcal{M}_G$. Suppose by contradiction that there is some $L^*(D)$-sentence $\varphi = \mathcal{Q}z(\psi_1, ..., \psi_n)$, such that $v(\varphi) \notin \tilde{\mathcal{Q}}(H_\varphi)$, where $H_\varphi = \{\langle v(\psi_1\{\overline{a}/z\}), ..., v(\psi_n\{\overline{a}/z\})\rangle \mid a \in D\}$. From the definition of v, it must be the case that:

(a) there is some L-formula $\theta \in \Gamma' \cup \Delta'$, such that $\theta \equiv_\alpha \widehat{\varphi}$, and $v(\varphi) = t$ iff $\theta \in \Gamma'$.

(Indeed, if there is no L-formula $\theta \in \Gamma' \cup \Delta'$, such that $\theta \equiv_\alpha \widehat{\varphi}$, then by definition of v, $v(\varphi)$ is always in $\tilde{\mathcal{Q}}(H_\varphi)$, so this case is not possible.)
Suppose w.l.o.g. that $\tilde{\mathcal{Q}}(H_\varphi) = \{t\}$ and $v(\varphi) = f$. By definition of $\mathcal{M}_G$ and the fact that $\tilde{\mathcal{Q}}(H_\varphi)$ is a singleton, it must be the case that there is some canonical rule $[\{\Sigma_k \Rightarrow \Pi_k\}_{1 \leq k \leq m} / \Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x))]$ in G, such that $\{\Sigma_k \Rightarrow \Pi_k\}_{1 \leq k \leq m}$ is H_φ-characteristic.

(b) Then there is some L_n-structure $\mathcal{W} = \langle D_\mathcal{W}, I_\mathcal{W}\rangle$, such that $\{\Sigma_k \Rightarrow \Pi_k\}_{1 \leq k \leq m}$ is valid in $\mathcal{W}$ and $\{\langle I_\mathcal{W}(p_1)(a), \ldots, I_\mathcal{W}(p_n)(a)\rangle \mid a \in D_\mathcal{W}\} = H_\varphi$.

Now $\varphi = \mathcal{Q}z(\psi_1, ..., \psi_n)$ and $\theta \equiv_\alpha \widehat{\varphi}$, so θ is of the form $\mathcal{Q}w(\varphi_1, ..., \varphi_n)$. By Lemma 7.1.28-2, $\sigma^*(\theta) \equiv_\alpha \sigma^*(\widehat{\varphi})$, and so $\sigma^*(\theta) \sim^S \sigma^*(\widehat{\varphi})$. By Lemma 7.1.27-2, $\sigma^*(\widehat{\varphi}) \sim^S \varphi$, and thus $\sigma^*(\theta) \sim^S \varphi$. Let ϕ_i be the formula obtained from φ_i by replacing every $x \in Fv(\varphi_i) - \{w\}$ by $\sigma^*(x)$. Then $\sigma^*(\theta) = \mathcal{Q}w(\phi_1, \dots, \phi_n)$ and $\varphi = \mathcal{Q}z(\psi_1, \dots, \psi_n)$. By Lemma 5.3.5, $\phi_i\{r/w\} \sim^S \psi_i\{r/z\}$ for a fresh variable r. By Lemma 5.2.11, $\phi_i\{\overline{a}/w\} = \phi_i\{r/w\}\{\overline{a}/r\} \sim^S \psi_i\{r/z\}\{\overline{a}/z\} = \psi_i\{\overline{a}/z\}$ for every $a \in D$ and every $1 \le i \le n$. We have already shown that v respects the $\sim^S$-relation, and so $v(\phi_i\{\overline{a}/w\}) = v(\psi_i\{\overline{a}/z\})$. Thus $H_\varphi = \{\langle v(\phi_1\{\overline{a}/w\}), ..., v(\phi_n\{\overline{a}/w\})\rangle \mid a \in D\}$.
Since $v(\varphi) = f$, it follows from (a) that $\theta = \mathcal{Q}w(\varphi_1, ..., \varphi_n) \in \Delta'$. Then by property 3 of $\Gamma' \cup \Delta'$, there is some $1 \le j \le m$, such that whenever $p_i(y) \in \Sigma_j$ ($p_i(y) \in \Pi_j$), there is some $\mathbf{t}_y \in \mathbf{T}$, such that $\varphi_i\{\mathbf{t}_y/w\} \in \Gamma'$ ($\varphi_i\{\mathbf{t}_y/w\} \in \Delta'$). By Lemma 7.1.29, $v(\sigma^*(\varphi_i\{\mathbf{t}_y/w\})) = v(\phi_i\{\sigma^*(\mathbf{t}_y)/w\}) = t$ ($v(\sigma^*(\varphi_i\{\mathbf{t}_y/w\})) = f$). Since $\mathcal{W}$ is H_φ-characteristic, there is some $a \in D_\mathcal{W}$, such that $I_\mathcal{W}(p_i)(a) = v(\phi_i\{\sigma^*(\mathbf{t}_y)/w\}) = t$ ($I_\mathcal{W}(p_i)(a) = f$). Pick one such a_y for every variable y occurring in $\Sigma_j \cup \Pi_j$.
Let us now show that $\Sigma_j \Rightarrow \Pi_j$ is not valid in $\mathcal{W}$ (in contradiction to (b)). Denote by $\mu(\psi)$ the closed formula obtained from ψ by replacing every variable x occurring free in ψ by $\overline{a}_y$. Let $\Sigma'_j \Rightarrow \Pi'_j$ be the closed $L_n(D_\mathcal{W})$-instance of $\Sigma_j \Rightarrow \Pi_j$, where $\Sigma'_j = \{\mu(\psi) \mid \psi \in \Sigma_j\}$ and $\Pi'_j = \{\mu(\psi) \mid \psi \in \Pi_j\}$. We now show that $\mathcal{W}$ does not satisfy $\Sigma'_j \Rightarrow \Pi'_j$. Let $p(\mathbf{t}) \in \Sigma'_j$. If $\mathbf{t}$ is some variable y, then $I_\mathcal{W}(p_i)(\mu(y)) = I_\mathcal{W}(p_i)(I_\mathcal{W}(\overline{a}_y)) = I_\mathcal{W}(p_i)(a_y) = t$. Otherwise $\mathbf{t}$ is some constant c. By property 3 of $\Gamma' \cup \Delta'$, for every $\mathbf{t} \in \mathbf{T}$: $\varphi_i\{\mathbf{t}/x\} \in \Gamma'$. By Lemma 7.1.29, $v(\sigma^*(\varphi_i\{\mathbf{t}/w\})) = v(\phi_i\{\sigma^*(\mathbf{t})/w\}) = t$. Thus for every $\mathbf{t} \in \mathbf{T}$: $v(\phi_i\{\sigma^*(\mathbf{t})/w\}) = v(\phi_i\{\overline{\mathbf{t}}/w\}) = t$. By (b), $I_\mathcal{W}(p_c)(I_\mathcal{W}(c)) = t$. The proof that whenever $p(\mathbf{t}) \in \Pi'_j$, $I(p)(I(\mathbf{t})) = f$ is similar. Hence $\Sigma_j \Rightarrow \Pi_j$ is not valid in $\mathcal{W}$.
We have shown that v respects the interpretations of the quantifiers in $\mathcal{M}_G$. □

Lemma 7.1.31. *For every sequent $\Sigma \Rightarrow \Pi \in \mathcal{S}$, $\Sigma \Rightarrow \Pi$ is $\mathcal{M}_G$-valid in $\langle S, v\rangle$.*

Proof. Suppose for contradiction that there is some $\Sigma \Rightarrow \Pi \in \mathcal{S}$, which is not $\mathcal{M}_G$-valid in $\langle S, v\rangle$. Then there is some closed $L^*(D)$-instance $\Sigma' \Rightarrow \Pi'$ of $\Sigma \Rightarrow \Pi$, which is not satisfied by v in S. For $\varphi \in \Sigma \cup \Pi$, denote by $\mu(\varphi)$ the corresponding closed $L^*(D)$-instance of φ in $\Sigma' \cup \Pi'$. Then if $\varphi \in \Sigma$, $v \models \mu(\varphi)$, and if $\varphi \in \Pi$: $v \not\models \mu(\varphi)$. Note that for every $\phi \in \Sigma \cup \Pi$, $\widehat{\mu(\phi)}$ is a substitution instance of ϕ. By property 5 of $\Gamma' \cup \Delta'$: $\widehat{\mu(\phi)} \in \Gamma' \cup \Delta'$. By Lemma 7.1.29, if $\widehat{\mu(\phi)} \in \Gamma'$ then $v(\sigma^*(\widehat{\mu(\phi)})) = t$, and if $\widehat{\mu(\phi)} \in \Delta'$ then $v(\sigma^*(\widehat{\mu(\phi)})) = f$. By Lemma 7.1.27-2, $\mu(\phi) \sim^S \sigma^*(\widehat{\mu(\phi)})$. By Lemma 7.1.30, v is $\mathcal{M}_G$-legal, so it respects the $\sim^S$-relation and for every $\phi \in \Sigma \cup \Pi$: $v(\mu(\phi)) = v(\sigma^*(\widehat{\mu(\phi)}))$. Thus $\widehat{\mu(\Sigma)} \subseteq \Gamma'$ and $\widehat{\mu(\Pi)} \subseteq \Delta'$ (where $\widehat{\mu(\Sigma)} = \{\mu(\theta) \mid \hat{\mu(\theta)} \in \Sigma\}$ and similarly for Π). But $\widehat{\mu(\Sigma)} \Rightarrow \widehat{\mu(\Pi)}$ has a $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G by the substitution rule, in contradiction to property 1 of $\Gamma' \cup \Delta'$. □

We have shown that (i) v is legal in $\mathcal{M}_G$, (ii) for every $\psi \in \Gamma' \cup \Delta'$: $v(\sigma^*(\psi)) = t$ iff $\psi \in \Gamma'$, and (iii) the sequents in $\mathcal{S}$ are $\mathcal{M}_G$-valid in $\langle S, v\rangle$. From (ii) it follows that $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}_G$-valid in $\langle S, v\rangle$, which completes the proof. □

Finally, for strong completeness of G for $\mathcal{M}_G$, assume that $\mathcal{S} \not\vdash_G \Gamma \Rightarrow \Delta$. If $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ does not satisfy the free-variable condition, obtain $\mathcal{S}' \cup \{\Gamma' \Rightarrow \Delta'\}$ by renaming the bound variables, so that $\mathcal{S}' \cup \{\Gamma' \Rightarrow \Delta'\}$ satisfies the condition (otherwise, take $\Gamma' \Rightarrow \Delta'$ and $\mathcal{S}'$ to be $\Gamma \Rightarrow \Delta$ and $\mathcal{S}$ respectively). Then $\Gamma' \Rightarrow \Delta'$ has no proof from $\mathcal{S}'$ in G (otherwise we could obtain a proof of $\Gamma \Rightarrow \Delta$ from $\mathcal{S}$ by using cuts on logical axioms), and so it also has no $\mathcal{S}'$-cut-free proof from $\mathcal{S}'$ in G. By proposition 7.1.26, $\mathcal{S}' \not\vdash_{\mathcal{M}_G} \Gamma' \Rightarrow \Delta'$. That is, there is an L-structure S and an $\mathcal{M}_G$-legal valuation v, such that the sequents in $\mathcal{S}'$ are $\mathcal{M}_G$-valid in $\langle S, v\rangle$, while $\Gamma' \Rightarrow \Delta'$ is not. Since v respects the $\equiv_\alpha$-relation, the sequents of $\mathcal{S}$ are also $\mathcal{M}_G$-valid in $\langle S, v\rangle$, while $\Gamma \Rightarrow \Delta$ is not. And so $\mathcal{S} \not\vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$.

□

Corollary 7.1.32. *Any coherent calculus admits strong cut-elimination.*

Proof. Let G be a coherent calculus. Let $\mathcal{S}$ be a set of sequents and $\Gamma \Rightarrow \Delta$ a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. Suppose that $\Theta \vdash_G \Gamma \Rightarrow \Delta$. Then by Theorem 7.1.25, $\Theta \vdash_{\mathcal{M}_G} \Gamma \Rightarrow \Delta$. By Proposition 7.1.26, $\Gamma \Rightarrow \Delta$ has no $\mathcal{S}$-cut-free proof from $\mathcal{S}$ in G.

□

Now we come to the main theorem, establishing a connection between the coherence of a canonical calculus G, the existence of a strongly characteristic 2Nmatrix for G and *strong cut-elimination* in G.

Theorem 7.1.33. *Let G be a canonical calculus. Then the following statements concerning G are equivalent:*

1. *G is coherent.*
2. *G has a strongly characteristic 2Nmatrix.*
3. *G admits strong cut-elimination.*

Proof. First we prove that (2) $\Rightarrow$ (1). Suppose that G has a strongly characteristic 2Nmatrix $\mathcal{M}$ and assume for contradiction that G is not coherent. Then there exist two rules $R_1 = [\Theta_1/\Rightarrow A]$ and $R_2 = [\Theta_2/A \Rightarrow]$ in G, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent, where $A = \mathcal{Q}x(p_1(x), ..., p_n(x))$. Recall that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \Theta_1 \cup \Theta_2'$, where Θ_2' is obtained from Θ_2 by renaming constants and variables that occur also in Θ_1 (see

Definition 7.1.8). For simplicity[3] we assume that the fresh constants used for renaming are all in L. Let $\Theta_1 = \{\Sigma^1_j \Rightarrow \Pi^1_j\}_{1\leq j\leq m}$ and $\Theta'_2 = \{\Sigma^2_j \Rightarrow \Pi^2_j\}_{1\leq j\leq r}$. Since $\Theta_1 \cup \Theta'_2$ is classically consistent, there exists an L_n-structure $\mathcal{W} = \langle D, I\rangle$, in which both Θ_1 and Θ'_2 are valid (Remark 7.1.14). Recall that we also assume that L_n is a subset of L[4] and so the following are applications of R_1 and R_2 respectively:

$$\frac{\{\Sigma^1_j \Rightarrow \Pi^1_j\}_{1\leq j\leq m}}{\Rightarrow \mathcal{Q}x(p_1(x), ..., p_n(x))} \qquad \frac{\{\Sigma^2_j \Rightarrow \Pi^2_j\}_{1\leq j\leq m}}{\mathcal{Q}x(p_1(x), ..., p_n(x)) \Rightarrow}$$

Let S be any extension of $\mathcal{W}$ to L and v - any $\mathcal{M}$-legal S-valuation. It is easy to see that the premises of the applications above are $\mathcal{M}$-valid in $\langle S, v\rangle$ (since the premises contain atomic formulas). But then by Theorem 7.1.21, both $\Rightarrow \mathcal{Q}v_1(p_1(v_1), ..., p_n(v_1))$ and $\mathcal{Q}v_1(p_1(v_1), ..., p_n(v_1)) \Rightarrow$ should also be $\mathcal{M}$-valid in $\langle S, v\rangle$, which is of course impossible.

Next, we prove that (3) $\Rightarrow$ (1). Let G be a canonical calculus which admits strong cut-elimination. Suppose by contradiction that G is not coherent. Then there are two dual rules of G: $\Theta_1/ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent. $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) \cup \{\Rightarrow\}$ satisfy the free-variable condition, since only atomic formulas are involved and no variables are bound there. It is easy to see that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) \vdash_G \Rightarrow A$ and $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) \vdash_G A \Rightarrow$. By using cut, $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) \vdash_G \Rightarrow$. But $\Rightarrow$ has no $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$-cut-free proof in G from $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ (since $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is consistent), in contradiction to the fact that G admits strong cut-elimination.

Finally, both (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3), follow from Theorem 7.1.25 and Corollary 7.1.32. $\square$

Remark 7.1.34. At this point it should be noted that the renaming of clashing constants in the definition of coherence (see Definition 7.1.9) is crucial. Consider, for instance, a canonical calculus G consisting of the introduction rules $[\{p_1(c) \Rightarrow\ ;\ \Rightarrow p_1(c')\}/ \Rightarrow \mathcal{Q}x\ p_1(x)]$ and $[\{p_1(c'') \Rightarrow\ ;\ \Rightarrow p_1(c)\}/\mathcal{Q}x\ p(x) \Rightarrow]$ for a unary quantifier $\mathcal{Q}$. Without renaming of clashing constants, we would conclude that the set $\{p_1(c) \Rightarrow\ ;\ \Rightarrow p_1(c')\ ;\ p_1(c'') \Rightarrow, \Rightarrow p_1(c)\}$ is classically inconsistent. However, G obviously has no strongly characteristic 2Nmatrix, since the rules dictate contradicting requirements for $\tilde{\mathcal{Q}}(\{t, f\})$. But if we perform renaming first, obtaining the set $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \{p_1(c) \Rightarrow\ ,\ \Rightarrow p_1(c')\ ,\ p_1(c'') \Rightarrow, \Rightarrow p_1(c''')\}$, we shall see that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent and so G is not coherent. Hence, by Theorem 7.1.33, G has no strongly characteristic 2Nmatrix.

[3]This assumption is not necessary and is used only for simplification of presentation, since we can instantiate the constants by any L-terms.

[4]This assumption is again not essential for the proof, but it simplifies the presentation.

Finally we turn to the relation between coherence and standard cut-elimination. Clearly, since strong cut-elimination implies the standard one, by Theorem 7.1.25 coherence is a sufficient condition for standard cut-elimination. In the more restricted canonical systems of [131] with unary quantifiers it also is a necessary condition. However, the following example shows that it does not hold even for the case of binary quantifiers.

Example 7.1.35. Consider, for instance, the following canonical calculus G_0 consisting of the following two inference rules: $[\Theta_1/ \Rightarrow \mathcal{Q}x(p_1(x), p_2(x))]$ and $[\Theta_2/\mathcal{Q}x(p_1(x), p_2(x)) \Rightarrow\ ]$, where:

$$\Theta_1 = \Theta_2 = \{p_1(x) \Rightarrow p_2(x)\,; \Rightarrow p_1(c_1)\,; \Rightarrow p_2(c_1)\,; p_1(c_2) \Rightarrow\,; p_2(c_2) \Rightarrow\,; p_1(c_3) \Rightarrow\,; \Rightarrow p_2(c_3)\}$$

Clearly, G_0 is not coherent. We now sketch a proof that the only sequents provable in G_0 are logical axioms. This immediately implies that G_0 admits cut-elimination.
To prove this it suffices to show that for every rule of G_0: if its premises are logical axioms, then its conclusion is a logical axiom. Suppose for contradiction that we can apply one of the rules on logical axioms and obtain a conclusion which is not a logical axiom. Suppose, without loss of generality, that it is the first rule. Then the application would be of the form:

$$\frac{\Gamma, \chi(p_1)\{\chi(x)/w\} \Rightarrow \Delta, \chi(p_2)\{\chi(x)/w\} \quad ... \quad \Gamma \Rightarrow \chi(p_1)\{\chi(c_1)/w\}, \Delta \quad \Gamma \Rightarrow \chi(p_2)\{\chi(c_1)/w\}, \Delta}{\Gamma \Rightarrow \mathcal{Q}w(\chi(p_1), \chi(p_2)), \Delta}$$

Since the proved sequent is not a logical axiom, (*) there are no $\psi \in \Gamma$ and $\varphi \in \Delta$, such that $\psi \equiv_\alpha \varphi$. Moreover, since $\Gamma, \chi(p_1)\{\chi(v_1)/w\} \Rightarrow \Delta, \chi(p_2)\{\chi(y)/w\}$ is a logical axiom, either (i) there is some $\theta \in \Delta$, such that $\theta \equiv_\alpha \chi(p_1)\{\chi(x)/w\}$, (ii) there is some $\theta \in \Gamma$, such that $\theta \equiv_\alpha \chi(p_2)\{\chi(x)/w\}$, or (iii) $\chi(p_1)(\chi(x)/w) \equiv_\alpha \chi(p_2)\{\chi(x)/w\}$. Suppose (i) holds, i.e. there is some some $\theta \in \Delta$, such that $\theta \equiv_\alpha \chi(p_1)\{\chi(x)/w\}$. Then since $\chi(x)$ cannot occur free in Δ, $w \notin Fv(\theta)$, and so $w \notin Fv(\chi(p_1))$. Hence, $\chi(p_1)\{\chi(c_1)/w\} = \chi(p_1)\{\chi(x)/w\} = \chi(p_1)$. Now since $\Gamma \Rightarrow \chi(p_1)\{\chi(c_1)/w\}, \Delta$ is a logical axiom, and due to (*), there is some $\phi \in \Gamma$, such that $\phi \equiv_\alpha \chi(p_1)\{\chi(c_1)/w\}$. But since $\chi(p_1)\{\chi(c_1)/w\} = \chi(p_1)\{\chi(x)/w\}$, $\theta \equiv_\alpha \phi$, $\theta \in \Delta$ and $\phi \in \Gamma$, in contradiction to (*). The case (ii) is treated similarly using the constant c_2. The case (iii) is handled using the constant c_3.
Thus, only logical axioms are provable in G_0 and so it admits standard cut-elimination, although it is not coherent.

Hence coherence is not a necessary condition for cut-elimination in canonical calculi with multi-ary quantifiers.

7.2 (n,k)-ary Quantifiers

Below we extend the results of the previous section to languages with (n, k)-ary quantifiers, using the extended GNmatrices (Definition 5.4.1) instead of ordinary Nmatrices. The results below are mainly based on [33, 36].

The framework of canonical calculi defined in Section 7.1.1 can be naturally extended to (n, k)-ary quantifiers as follows. Instead of a simplified language $L_n(Con)$ (Definition 7.1.1), we shall use a language $L_n^k(Con)$ defined as follows:

Definition 7.2.1. For $k \geq 0$, $n \geq 1$ and a set of constants Con, $L_n^k(Con)$ is the language with n k-ary predicate symbols $p_1, ..., p_n$ and the set of constants Con (and no quantifiers or connectives).

As before, whenever the set Con is clear from context, we will write L_n^k instead of $L_n^k(Con)$. The semantics for these languages will be provided using L_n^k-structures, which are two-valued structures defined similarly to L_n-structures from the previous section.

The following are natural extensions of Definitions 7.1.2, 7.1.3 and 7.1.4 to the (n, k)-ary case:

Definition 7.2.2. A *canonical rule of arity* (n, k) is an expression which has the form $[\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C]$, where C is either $\Rightarrow \mathcal{Q}x_1...x_k(p_1(x_1, ..., x_k), ..., p_n(x_1, ..., x_k))$, or $\mathcal{Q}x_1...x_k(p_1(x_1, ..., x_k), ..., p_n(x_1, ..., x_k)) \Rightarrow$ and $m \geq 0$ for some (n, k)-ary quantifier $\mathcal{Q}$ of L and for every $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ is a clause over L_n^k.

Definition 7.2.3. Let $R = [\Theta/C]$ be an (n, k)-ary canonical rule, where C is of one of the forms $(\mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}), ..., p_n(\overrightarrow{x})) \Rightarrow)$ or $(\Rightarrow \mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}), ..., p_n(\overrightarrow{x})))$. Let Γ be a set of L-formulas and $z_1, ..., z_k$ - distinct variables of L. An $\langle R, \Gamma, z_1, ..., z_k\rangle$*-mapping* is any function χ from the predicate symbols, terms and formulas of L_n^k to formulas and terms of L, satisfying the following conditions:

- For every $1 \leq i \leq n$, $\chi(p_i)$ is an L-formula, $\chi(y)$ is a variable of L, and $\chi(x) \neq \chi(y)$ for every two variables $x \neq y$. $\chi(c)$ is an L-term, such that $\chi(x)$ does not occur in $\chi(c)$ for any variable x occurring in Θ.

- For every $1 \leq i \leq n$, whenever $p_i(\mathbf{t}_1, ..., \mathbf{t}_k)$ occurs in Θ, for every $1 \leq j \leq k$: $\chi(\mathbf{t}_j)$ is a term free for z_j in $\chi(p_i)$, and if $\mathbf{t}_j$ is a variable, then $\chi(\mathbf{t}_j)$ does not occur free in $\Gamma \cup \{\mathcal{Q}z_1...z_k(\chi(p_1), ..., \chi(p_n))\}$.

- $\chi(p_i(\mathbf{t}_1, ..., \mathbf{t}_k)) = \chi(p_i)\{\chi(\mathbf{t}_1)/z_1, ..., \chi(\mathbf{t}_k)/z_k\}$.

χ is extended to sets of L_n^k-formulas as follows: $\chi(\Delta) = \{\chi(\psi) \mid \psi \in \Delta\}$.

Definition 7.2.4. An *application* of $R = [\{\Pi_i \Rightarrow \Sigma_i\}_{1\le i\le m}/\mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}), ..., p_n(\overrightarrow{x})) \Rightarrow]$ is any inference step of the form:

$$\frac{\{\Gamma, \chi(\Pi_i) \Rightarrow \Delta, \chi(\Sigma_i)\}_{1\le i\le m}}{\Gamma, \mathcal{Q}z_1...z_k\ (\chi(p_1), ..., \chi(p_n)) \Rightarrow \Delta}$$

where $z_1, ..., z_k$ are variables, Γ, Δ are any sets of L-formulas and χ is some $\langle R, \Gamma \cup \Delta, z_1, ..., z_k\rangle$-mapping.
An application of a canonical rule of the form $[\{\Pi_i \Rightarrow \Sigma_i\}_{1\le i\le m}/ \Rightarrow \mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}), ..., p_n(\overrightarrow{x}))]$ is defined similarly.

The definitions of the notions of canonical calculi (Definition 7.1.7) and of coherence (Definition 7.1.9) remain the same.

To provide semantics for canonical calculi with (n,k)-ary quantifiers, we will need the following technical notion:

Definition 7.2.5. *Let $\mathcal{W} = \langle D, I\rangle$ be an L_n^k-structure. The* functional distribution *of $\mathcal{W}$ is defined as follows: $FDist_{\mathcal{W}} = \lambda a_1, ..., a_k \in D.\langle I(p_1)(a_1, ..., a_k), ..., I(p_n)(a_1, ..., a_k)\rangle$.*

The characteristic GNmatrix for every coherent canonical calculus with (n,k)-ary quantifiers is defined as follows:

Definition 7.2.6. Let G be a coherent canonical calculus. For every L-structure $S = \langle D, I\rangle$, the GNmatrix $\mathcal{M}_G$ contains the operation $\tilde{\mathcal{Q}}_S$ defined as follows. For every (n,k)-ary quantifier $\mathcal{Q}$ of L and every $g \in D^k \to \{t, f\}^n$:

$$\tilde{\mathcal{Q}}_S(g) = \begin{cases} \{t\} & [\Theta/ \Rightarrow \mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}), \ldots, p_n(\overrightarrow{x}))] \in G \textit{ and there is some } \mathcal{W} = \langle D, I_{\mathcal{W}}\rangle \\ & \textit{such that } FDist_{\mathcal{W}} = g \textit{ and } \Theta \textit{ is valid in } \mathcal{W}. \\ \{f\} & [\Theta/\mathcal{Q}\overrightarrow{x}(p_1(\overrightarrow{x}), \ldots, p_n(\overrightarrow{x})) \Rightarrow] \in G \textit{ and there is some } \mathcal{W} = \langle D, I_{\mathcal{W}}\rangle \\ & \textit{such that } FDist_{\mathcal{W}} = g \textit{ and } \Theta \textit{ is valid in } \mathcal{W}. \\ \{t, f\} & \textit{otherwise} \end{cases}$$

It should be noted that as opposed to the Definition 7.1.23 , the above definition is not constructive. This is because the question whether Θ is valid in some L_n^k-structure with a given functional distribution is not generally decidable. Next, let us show that

$\mathcal{M}_G$ is well-defined. Assume by contradiction that there are two rules $[\Theta_1/\Rightarrow A]$ and $[\Theta_2/A \Rightarrow]$, such that there exist two L_n^k-structures $\mathcal{W}_1 = \langle D, I_1\rangle$ and $\mathcal{W}_2 = \langle D, I_2\rangle$, which satisfy: $FDist_{\mathcal{W}_1} = FDist_{\mathcal{W}_2}$ and Θ_i is valid in $\mathcal{W}_i$ for $i \in \{1, 2\}$. But then $\mathcal{W}_1$ and $\mathcal{W}_2$ only differ in their interpretations of constants from Θ_1 and Θ_2, and we can easily construct an L_n^k-structure $\mathcal{W}_3 = \langle D, I_3\rangle$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is valid in $\mathcal{W}_3$ (the renaming is essential since it may be the case that the same constant occurs both in Θ_1 and Θ_2). And so $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent, in contradiction to the coherence of G.

Let us demonstrate the construction of $\mathcal{M}_G$ for some coherent canonical calculi.

Example 7.2.7. 1. The canonical calculus G_1 consists of (1,1)-ary rule $[\Rightarrow p(x)/\Rightarrow \forall x p(x)]$. G_1 is (trivially) coherent. For every L-structure $S = \langle D, I\rangle$, $\mathcal{M}_{G_1}$ contains the operation $\tilde{\forall}_S$ defined as follows: for every $g \in D \to \mathcal{V}$,

$$\tilde{\forall}_S(g) = \begin{cases} \{t\} & \mathit{if\ for\ all}\ a \in D:\ g(a) = t \\ \{t, f\} & \mathit{otherwise} \end{cases}$$

2. The canonical calculus G_2 consists of the following rules:

$$[\{p_1(x) \Rightarrow p_2(x)\}/\Rightarrow \overline{\forall} x\ (p_1(x), p_2(x))] \quad [\{p_2(c) \Rightarrow\ ,\ \Rightarrow p_1(c)\}/\overline{\forall} x(p_1(x), p_2(x)) \Rightarrow]$$

$$[\{\Rightarrow p_1(c)\ ,\ \Rightarrow p_2(c)\}/\Rightarrow \overline{\exists} x(p_1(x), p_2(x))]$$

G' is obviously coherent. The operations $\tilde{\overline{\forall}}_S$ and $\tilde{\overline{\exists}}_S$ in $\mathcal{M}_{G_2}$ are defined as follows: for every $g \in D \to \{t, f\}^2$,

$$\tilde{\overline{\forall}}_S(g) = \begin{cases} \{t\} & \mathit{if\ there\ are\ no\ such}\ a, b \in D,\ \mathit{that}\ g(a, b) = \langle t, f\rangle \\ \{f\} & \mathit{otherwise} \end{cases}$$

$$\tilde{\overline{\exists}}_S(g) = \begin{cases} \{t\} & \mathit{if\ there\ are}\ a, b \in D, s.t.\ g(a, b) = \langle t, t\rangle \\ \{t, f\} & \mathit{otherwise} \end{cases}$$

The rule (i) dictates the condition that $\overline{\forall}_S(g) = \{t\}$ for the case that there are no $a, b \in D$, s.t. $g(a, b) = \langle t, f\rangle$. The rule (ii) dictates the condition that $\overline{\forall}_S(g) = \{f\}$ for the case that there are such $a, b \in D$. Since G_2 is coherent, the dictated conditions are non-contradictory. The rule (iii) dictates the condition that $\overline{\exists}_S(g) = \{t\}$ in the case that there are $a, b \in D$, s.t. $g(a, b) = \langle t, t\rangle$. There is no rule which dictates conditions for the case of $\langle t, t\rangle \notin H$, and so the interpretation in this case is non-deterministic.

3. Consider the $(2,2)$-ary rule $\{p_1(x,y) \Rightarrow\ ;\ \Rightarrow p_2(c,x)\}/ \Rightarrow \mathcal{Q}v_1v_2(p_1(x,y), p_2(x,y))$. The canonical calculus G_3 consisting of this rule is (trivially) coherent. For a vector $v = \langle a_1, ..., a_n\rangle$, denote by $(v)_i$ the i-th element of v. For every L-structure $S = \langle D, I\rangle$, $\mathcal{M}_{G_3}$ contains the operation $\tilde{\mathcal{Q}}_S$ defined as follows for every $g \in D^2 \to \{t,f\}^2$:

$$\tilde{\mathcal{Q}}_S(g) = \begin{cases} \{t\} & \textit{if there is some } a \in D, \textit{ s.t. for every } b, c \in D \\ & (g(b,c))_1 = f \textit{ and } (g(a,b))_2 = t \\ \{t,f\} & \textit{otherwise} \end{cases}$$

The following theorem extends Theorem 7.1.25:

Theorem 7.2.8. *Let G be a coherent canonical calculus with (n,k)-ary quantifiers. Then $\mathcal{M}_G$ is strongly characteristic for G.*

Proof. The proof of strong soundness is along the lines of the proof of Theorem 7.1.20. Strong completeness follows (similarly to the proof of Theorem 7.1.25) from a generalization of Proposition 7.1.26 for (n,k)-ary quantifiers, which can be proved in a similar way, using GNmatrices instead of Nmatrices. □

Hence the correspondence from Theorem 7.1.33 can extended to the case of (n,k)-ary quantifiers as follows:

Theorem 7.2.9. *Let G be a canonical calculus. Then the following statements concerning G are equivalent:*

1. *G is coherent.*

2. *G has a strongly characteristic 2GNmatrix.*

3. *G admits strong cut-elimination.*

The proof is quite similar to the proof of Theorem 7.1.33.

Chapter 8

Summary and Further Work

In this thesis the framework of Nmatrices was extended in a number of directions. On the propositional level we have studied the general theory of canonical systems. It was shown that in coherent canonical Gentzen-type calculi there is a correspondence between the existence for a calculus of a deterministic matrix, invertibility of its logical rules and axiom-expansion. The theory of canonical systems was then extended to the much more general case of signed calculi. Finally, first steps were made in investigating the usefulness of Nmatrices in distance-based approaches to reasoning under uncertainty.
We have then further generalized the framework of Nmatrices to languages with quantifiers. We considered here three types of generalized quantifiers: unary, multi-ary and (n, k)-ary ones, and proposed ways to interpret such quantifiers in Nmatrices (based on various approaches to deterministic quantifiers in the literature). Problems related to the principles of α-equivalence, identity and void quantification, which were not evident on the propositional level, were resolved using special congruence relations between formulas, and some general analycity results were proved. As one application of the extended framework, we have provided modular non-deterministic semantics for a large family of first-order paraconsistent logics. As another application, we have generalized the theory of canonical systems to languages with quantifiers.

The framework of Nmatrices has already demonstrated its usefulness and great potential in a number of areas. Nevertheless, the field is still at the early stages of its development, with new lines of research, potential applications, promising extensions, and theoretical problems emerging all the time. The main directions for further research are to extensively develop, refine, and enlarge the framework, and to further exploit its potential in the various directions to which the research that was done so far has naturally led. This includes the following lines of research:

- Foundations of Logic:

General theory of Nmatrices: There are by now many examples of how the use of Nmatrices makes it possible to provide in a modular way concrete, useful semantics for families of logics (this is in contrast to the use of the usual deterministic algebraic semantics, in which one can only consider a system as a whole). One such example is the family of paraconsistent logics investigated in Chapter 6. Other examples are shown in [19, 20, 21, 17]. An important research direction is developing a general foundational methodology for doing this (based on Nmatrices). In addition to providing very useful semantic tools, such a methodology will provide a better understanding of logical systems, logical constants, and inference rules.

Canonical systems: The theory of canonical systems needs to be further extended to more complex quantifiers. This may shed a new light into proof-theoretical investigations of Henkin quantifiers. Note that the semantic treatment of (n, k)-ary quantifiers using GNmatrices, proposed in Section 5.4, is already sufficient to handle Henkin quantifiers. However, the language used for describing canonical systems with (n, k)-ary quantifiers in Section 7.2 is not expressive enough to capture such quantifiers. Enriching the language to capture the complex dependencies (possibly along the lines of [92]) is inevitable, at the price of the decidability of coherence. Another important direction is to extend the results on the connection between cut-elimination and Nmatrices to systems less restrictive than the canonical ones. This may lead to a general theory of the cut-elimination phenomena, on which both of the methods of resolution and tableaux are based (see [16]). This, in turn, might open the door to an extensive uniform system for automated reasoning in classical and non-classical logics.

Combining Nmatrices and probability: There is an important aspect of non-deterministic operations in a given Nmatrix $\mathcal{M}$, which is not reflected by its set of legal valuations. Consider for example the 2-valued Nmatrix $\mathcal{M}$ with the following operations:

$\rightarrow$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\sim$	
$\mathbf{t}$	$\{\mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}, \mathbf{f}\}$

$\neg$	
$\mathbf{t}$	$\{\mathbf{t}, \mathbf{f}\}$
$\mathbf{f}$	$\{\mathbf{t}\}$

The "truth tables" corresponding to the formulas $\varphi = p \rightarrow \sim\sim p$ and $\psi = p \rightarrow \neg\neg p$ are easily seen to be identical. Assume now that all choices are done completely at random, and it is known that $v(p) = \mathbf{t}$. Then the probability that $v(\varphi) = \mathbf{t}$ is $1/2$, while the probability that $v(\psi) = \mathbf{t}$ is $3/4$. To account for this difference Nmatrices can be generalized by allowing *weights* to be assigned to each element of $\widetilde{\diamond}(x_1, \ldots, x_n)$ (so that the sum of these weights is 1). Probably the best way to treat weights would be to use signed formulas (Chapter 3), where the meaning of $a : \psi$ is: "The probability that $v(\psi)$ is designated is greater than a". Investigation of relations with and applications to the fields of probabilistic reasoning, fuzzy logic, and verification of non-deterministic programs seem a promising line of research.

Non-deterministic structures: The framework of Nmatrices can be further extended by introducing yet another degree of non-determinism at the first-order level and beyond by allowing non-deterministic structures, that is structures where predicate and function symbols are interpreted non-deterministically. For instance, given a non-deterministic structure $S = \langle D, I \rangle$ for L_Q, for any n-ary predicate symbol p of L_Q, $I(p) \in P(P(D^n))$, and for any n-ary function symbol f of L_Q, $I(f) \in P(D^n \to D)$. A general theory combining non-deterministic structures with Nmatrices seems particularly useful for commonsense reasoning about fuzzy notions.

- Concrete Applications of Nmatrices:

Logical circuits: A promising concrete application of Nmatrices is for the representation, design and verification of circuits. The first steps to apply Nmatrices to represent non-deterministic behaviour of circuits were taken in Chapter 4. In general a logical gate manipulating boolean variables is an abstraction of a physical gate operating with a continuous range of electrical quantity. This electrical quantity is turned into a discrete variable by associating a whole range of electrical voltages with one logical value. Outside of these ranges there is a whole range of intermediate values for which the behavior of the gate cannot be predicted. However, even if a legal value is applied at the input of a gate, the output signal often deviates from the expected value, due to disturbing noise sources and other sources of deviations in circuit response. (See [114] for more details). Nmatrices seem particularly useful as a mathematical model of this non-deterministic I/O relation in a physical gate. Another potential application of Nmatrices in this area may be for designing efficient circuits for meeting *partial* specifications (e.g. when it is important to know the output only for a subset of the possible inputs).

It should be noted that in many cases the standard propositional language is not expressive enough to reason about non-deterministic behavior of circuits. To see this, consider the circuit on Figure 8.1, where $\diamond$ represents a gate with a non-deterministic behaviour. A natural representation of this circuit would

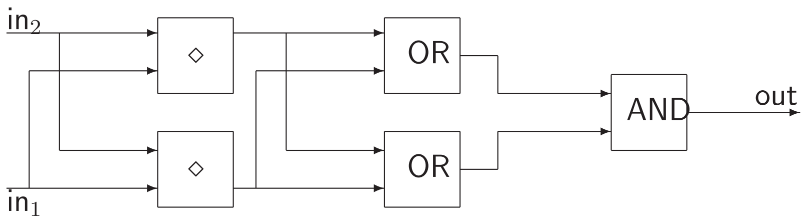

Figure 8.1: A circuit with shared lines

be using the formula:

$$\mathsf{out} \leftrightarrow \big((\mathsf{in}_1 \diamond \mathsf{in}_2) \vee (\mathsf{in}_1 \diamond \mathsf{in}_2)\big) \wedge \big((\mathsf{in}_1 \diamond \mathsf{in}_2) \vee (\mathsf{in}_1 \diamond \mathsf{in}_2)\big)$$

However, this representation in the framework of Nmatrices is not accurate, as any valuation in a given Nmatrix would assign the same truth-value to the first and the second occurrences of $(\mathsf{in}_1 \diamond \mathsf{in}_2)$ in this formula, while in reality they may have different values (because of the non-determinism of $\diamond$). This is opposed to the first and the third occurrences of $(\mathsf{in}_1 \diamond \mathsf{in}_2)$, which should always have the same value. A more appropriate formalism to describe such situation is the *Cirquent Calculus* of [93], which is designed to reason about sharing of resources. Combining it with the framework of Nmatrices might lead to a useful framework for reasoning about real-life circuits with shared lines.

Verification of Programs: Recently some work has been done on combining model checking with many-valued logic. This approach is particularly useful for analyzing models that contain uncertainty or inconsistency. For instance, [65, 64] use for this task Fitting's combination of Kripke structures with (deterministic) many-valued matrices ([74, 75]), where both propositions and transitions between states may take any of the truth values. It seems that the scope of this approach can be considerably extended by using Nmatrices rather than ordinary matrices. This can be done with the help of the methodology for combining Nmatrices with Kripke structures which was developed in [21, 22, 23].

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