Expander Random Walks: The General Case and Limitations

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Abstract

Cohen, Peri and Ta-Shma [CPT21] considered the following question: Assume the vertices of an expander graph are labelled by ± 1. What "test" functions $f : \{\pm 1\}^t \to \{\pm 1\}$ can or cannot distinguish t independent samples from those obtained by a random walk? [CPT21] considered only balanced labellings, and proved that for all symmetric functions the distinguishability goes down to zero with the spectral gap λ of the expander G. In addition, [CPT21] show that functions computable by AC^0 circuits are fooled by expanders with vanishing spectral expansion.

We continue the study of this question. We generalize the result to all labelling, not merely balanced ones. We also improve the upper bound on the error of symmetric functions. More importantly, we give a matching lower bound and show a symmetric function with distinguishability going down to zero with λ but not with t. Moreover, we prove a lower bound on the error of functions in AC^0 in particular, we prove that a random walk on expanders with constant spectral gap does not fool AC^0.

Contents

1 Introduction

Expanders are sparse undirected graphs that have many desirable pseudorandom properties. A formal definition can be given in several equivalent ways and here we consider the algebraic definition where an undirected graph $G = (V, E)$ is a λ-*spectral expander* if the second largest eigenvalue of its normalized adjacency matrix M is bounded above by λ. For simplicity, we only consider regular graphs, in which case M is also the random walk matrix of G. Expander graphs are among the most useful combinatorial objects in theoretical computer science, pivotal in derandomization [INW94, Rei05], complexity theory [Val76, AKS87, Din07] and coding theory [SS96, KMRZS17, TS17, DEL+21] to name a few. Many works in the literature have studied explicit constructions of expander graphs (see, e.g., [LPS88, Mar88, BL06, RVW00, BATS11, MOP20]) and utilized their pseudorandom properties. We refer the reader to the excellent expositions [HLW06, Tre17] and to Chapter 4 of [Vad12].

Expanders can be thought of as spectral sparsifiers of the clique. Let $\mathbf{J}$ be the normalized adjacency matrix of the n-vertex complete graph with self-loops, i.e., the $n \times n$ matrix with all entries equal to $\frac{1}{n}$. One can express the normalized adjacency matrix M of G as $M = (1-\lambda)\mathbf{J} + \lambda E$ for some operator E with spectral norm bounded by 1. As such, one can hope to substitute a sample of two *independent* vertices with the "cheaper" process of sampling an edge from an expander and using its two (highly correlated) end-points. This is captured, e.g., by the expander mixing lemma [AC88]. This idea also appears in many derandomization results, [INW94, AEL95, RRV99, Rei05, RV05, BCG20].

A useful generalization of the above is to consider not just an edge but rather a length $t-1$ random walk (where the length is measured in edges) on the expander as a replacement to t independent samples of vertices. For concreteness, consider a labelling $\mathsf{val} : V \to \{\pm 1\}$ of the vertices with mean $\mu = \mathbf{E}[\mathsf{val}(V)]$. Quite a lot is known about random walks on expanders. Next, we elaborate on the hitting property of expanders [AKS87, CW89, IZ89, BGG93] as well as the expander Chernoff bound [AKS87, CW89, IZ89, Gil98, Hea08].

The hitting property states that for every set $A \subset V$, a length $t-1$ random walk is contained in A with probability at most $(\mu+\lambda)^t$. For $\lambda \ll \mu$, this bound is close to μ^t - the probability of the event with respect to t independent samples. The expander hitting property corresponds to a random walk "fooling" the AND function, that is, for every λ-spectral expander and every labelling val as above, the AND function cannot distinguish with good probability labels obtained by t independent samples from labels obtained by taking a length $t-1$ random walk. The fundamental expander Chernoff bound states that the number of vertices in A visited by a random walk is highly concentrated around its measure $|A|/|V|$. The expander Chernoff bound corresponds to fooling functions indicating whether the normalized Hamming weight of the input is concentrated around some number μ. Perhaps surprisingly, it was shown that even the highly sensitive PARITY function is fooled by a random walk on expanders (this was noted independently by Alon in 1993 for arbitrarily long walks, Wigderson and Rozenman in 2004 for length 1 walks, and [TS17] where the result appears).

Sometimes a random walk *is not* a good replacement to independent samples. To see this, suppose G is a λ-spectral expander for some constant λ, that has a cut $A \subset V$ with $|A| = \frac{|V|}{2}$ and $|E(A, \overline{A})| \geqslant \mu|A|$ for $\mu \geqslant \frac{1}{2} + \widetilde{\Omega}(\lambda)$. Such graphs exist (see [GK21, Section 7]). If one samples t

independent vertices $(v_1, \ldots, v_t)$ from the graph, we expect (v_i, v_{i+1}) to cross the cut about half the time, and by the Chernoff bound the actual number of cut crossings is highly concentrated around the mean. In contrast, when we take a random walk on the graph we expect to cross the cut a μ-fraction of the time, and intuitively the number of cut crossings should be concentrated around μ.[1] Thus, the simple test function that counts the number of times we cross the cut and apply a threshold at $\frac{1}{2} + \tau$ for some $\tau = \widetilde{\Theta}(\lambda)$ should distinguish with probability close to 1 between a random walk and independent samples.

This brings to the forefront a natural question that was recently raised by [CPT21] (see also the work of Guruswami and Kumar [GK21] who considered a related question).

What test functions does a random walk on an expander fool?

Formally, we compare two distributions on the set $\{\pm 1\}^t$. The first "ideal" distribution is obtained by sampling independently and uniformly at random t vertices $v_1, \ldots, v_t$ and returning $(\mathsf{val}(v_1), \ldots, \mathsf{val}(v_t))$. If we let $\mu = \mathbf{E}[\mathsf{val}(V)]$, the latter induces the distribution U_t^μ in which the t bits are independent and each has mean μ. The second distribution, denoted by $\mathrm{RW}_{G,\mathsf{val}}$, is obtained by taking a length $t-1$ random walk on the graph, namely, sample v_1 uniformly at random from V, and then for $i = 2, 3, \ldots, t$, sample v_i uniformly at random from the set of neighbors of v_{i-1}, and return $(\mathsf{val}(v_1), \ldots, \mathsf{val}(v_t))$. Denote

$$\mathcal{E}_{G,\mathsf{val}}(f) = \left|\mathbf{E}\, f(\mathrm{RW}_{G,\mathsf{val}}) - \mathbf{E}\, f(U_t^\mu)\right|.$$

Informally, $\mathcal{E}_{G,\mathsf{val}}(f)$ measures the distinguishability between these two distributions as observed by the test function f on the graph G with respect to the labelling val. We wish to have a discussion that holds uniformly on all λ-spectral expanders (on any number of vertices) and for every labelling. The bound, however, is expected to depend on the expectation μ of the labelling. We denote by $\mathcal{E}_{\lambda,\mu}(f)$ the supremum of $\mathcal{E}_{G,\mathsf{val}}(f)$ over all λ-spectral expanders G, on any number of vertices, and all labelling functions $\mathsf{val} : V \to \{\pm 1\}$ with $\mathbf{E}[\mathsf{val}(V)] = \mu$.

The work [CPT21] focuses on the case $\mu = 0$. One result shows that

$$\mathcal{E}_{\lambda,0}(\mathsf{MAJ}) \leqslant O\left(\frac{\lambda^2}{\sqrt{t}}\right) \tag{1.1}$$

Their main result states that for each balanced labelling, for every symmetric function $f : \{\pm 1\}^t \to \{\pm 1\}$,

$$ef_{\lambda,0}(f) = O(\lambda \cdot \log^{3/2}(1/\lambda)). \tag{1.2}$$

This readily implies, for the specific case of balanced labelling, a central limit theorem with respect to the total variation distance, that vanishes as $\lambda \to 0$, thus strengthens previous results that considered the Kolmogorov distance [KV86, Lez01, Klo17] instead of the total variation distance.

To summarize the state of knowledge so far:

- Every symmetric function is fooled with error probability going down to zero with the spectral gap λ (see Equation (1.2)), where $\mu = 0$.

[1] To show such a concentration one needs to invoke a Chernoff bound for a walk on the corresponding directed line graph.

- The MAJ function is fooled with error probability going down to zero with t even when λ is fixed (see Equation (1.1)); and,
- The PARITY, AND, OR functions are fooled with error probability going down to zero *exponentially* fast with t even when λ is fixed.

Accordingly, let us say an error function vanishes with λ, if the error function is vanishing as $\lambda \to 0$. Similarly, we say an error function vanishes with t, if for some fixed $\lambda \geqslant 0$, it is going down to zero together with t.

[CPT21] further considers non-symmetric functions. In particular, they analyze test functions that are computable by AC^0 circuits and prove that if f is computable by a size-s depth-d circuit then

$$\mathcal{E}_{\lambda,0}(f) = O(\sqrt{\lambda} \cdot (\log s)^{2(d-1)}). \tag{1.3}$$

Thus, for balanced labelling, every test function in AC^0 cannot distinguish t independent labels from those obtained by a random walk on a λ-spectral expander provided λ is taken sufficiently small. This result can be thought of as an analog of Braverman's celebrated result [Bra10] (see also [Tal17]) that studies the pseudorandomness of k-wise independent distributions with respect to AC^0 test functions. However, for it to be meaningful, the spectral gap λ should be small.

1.1 Our contribution

The work of [CPT21] leaves several open problems. First, and foremost, while [CPT21] show the error function of any symmetric function vanishes with λ, it leaves open the possibility that a better convegence exists and, perhaps, the error function of any symmetric function vanishes with t, i.e., for some fixed λ, the error function goes down to zero together with the walk length t. Indeed, this is the case with the AND, OR and PARITY functions, where the error vanishes exponentially fast with t, and the MAJ function where the error goes down polynomially in t (see Equation (1.1)). Similarly, one may ask whether the error of AC^0 functions decays faster than Equation (1.3) and allows for larger spectral gaps λ then dictated by the above bound.

Our first result is that there exists a symmetric function for which the error function does not vanish with t:

Theorem 1.1. *There exists a family of symmetric functions $(f_t)_{t\in\mathbb{N}}$ where $f_t : \{\pm 1\}^t \to \{\pm 1\}$ such that for every λ there is a λ-spectral expander $G = (V, E)$, and a labelling $\mathsf{val} : V \to \{\pm 1\}$ with $\mathbf{E}[\mathsf{val}(V)] = 0$, such that for all t, $\mathcal{E}_{G,\mathsf{val}}(f_t) = \Omega(\lambda)$.*

To explain how we obtain such a lower bound on a function f, we first review how [CPT21] obtained their upper bound. The key idea in [CPT21] is to expand the test function f under consideration in the Fourier basis. The question of fooling general test functions then reduces to the study of test functions that are Fourier characters. Now, let G denote the adjacency matrix of the graph (i.e, $M = \frac{1}{d}G$). Also, for a labelling $\ell : V \to \{\pm 1\}$ let us denote by P the diagonal matrix with $\ell(i)$ in the i'th element on the diagonal. One can check that for the parity function $\chi_{[t]} : \{\pm 1\}^t \to \{\pm 1\}$, $\chi_{[t]}(x) = \prod_{i=1}^t x_i$, we get $\mathbf{E}[\chi_{[t]}(\mathrm{RW}_{G,\mathsf{val}})] = \mathbf{1}^T \left(\prod_{i=1}^t PG\right) \mathbf{1}$,, where $\mathbf{1} = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.

In general,

$$\mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}})] = \mathbf{1}^T \left(\prod_{i=1}^{t} P^{\delta_S(i)} G \right) \mathbf{1}, \tag{1.4}$$

where $\delta_S(i)$ is 1 if $i \in S$ an 0 otherwise. In [CPT21] it is shown how to upper bound this expression for any λ-expander G and μ-biased function val.

For the proof of Theorem 1.1 we choose a λ-expander G and a labelling function val such that we can *exactly* express Equation (1.4) in terms of λ, S. To do so, we first choose G to be a Cayley graph over an Abelian group, and we use the fact that the eigenvectors of such a graph correspond to the characteristic functions of the underlying group, regardless of the set of generators used. One disadvantage in choosing a Cayley graph over an Abelian group is that it cannot give constant degree exapnders, though this is not a concern to us because with logarithmic degree we can have vanishing second eigenvalue. Next, we choose the underlying group to be $\mathbb{Z}_2^n$. This guarantees that the characteristic functions of $\mathbb{Z}_2^n$, and therefore also all the entries in all eigenvectors, are Boolean, i.e., either 1 or -1. Finally, we choose the labelling function val to correspond to the entries of the eigenvalue with the second largest eigenvalue.

The above choices guarantee that $P\mathbf{1} = v_2$ and $Pv_2 = \mathbf{1}$ (because $P^2 = I$). Also $G\mathbf{1} = \mathbf{1}$ and $Gv_2 = \lambda v_2$. It follows that no matter what S is, $\left(\prod_{i=1}^{t} P^{\delta_S(i)} G\right) \mathbf{1}$ belongs to the two dimensional subspace $\mathrm{Span}\,(\mathbf{1}, v_2)$ and, furthermore, has a closed expression as a function of t, λ and S.

We finally choose a function f for which we can estimate the expression we get. We choose f to have high mass on its second Fourier level. It turns out that we can take f to be, e.g., the threshold function that returns one if the number of ones exceeds the mean by one standard deviation, and this function has error function that is of the order λ, and, therefore, in particular, vanishes with λ but not with t. It is interesting to note that, in contrast, the MAJ function, that has threshold at the mean, vanishes with t.

Next, using the same graph and labelling we also prove that constant spectral expansion does not suffice to fool AC^0 circuits. In fact, the bound obtained by [CPT21] is tight up to a polynomial. Let $\mathsf{AC}(d)$ denote the class of all languages with polynomial size boolean circuit of depth at most d. Then:

Theorem 1.2. *There exists a constant $\varepsilon > 0$ such that the following holds. For every integer $d \geqslant 3$ there exist $t_d, c_d \in \mathbb{N}$, and a family of functions $(h_t)_{t_d \leqslant t \in \mathbb{N}} \subset \mathsf{AC}(d)$ such that the following holds. For every $\lambda \geqslant \frac{c_d}{\log^{d-2} t}$ there is a λ-spectral expander $G = (V, E)$ and a labelling $\mathsf{val} : V \to \{\pm 1\}$ with $\mathbf{E}[\mathsf{val}(V)] = 0$ such that $\mathcal{E}_{G,\mathsf{val}}(h_t) \geqslant \varepsilon$.*

The choice of function f here is more complicated. The key idea is that two adjacent bits obtained by such a random walk are λ correlated. Thus, evaluating a function f on the parity of consecutive bits obtained by a random walk is the same as applying the noise operator $T_\lambda(f)$ (see Claim 6.4 for an exact statement). Having this key fact, we construct small depth functions that are highly sensitive to small noise. We first start with the Tribes function composed with XOR on two adjacent bits. This gives a function in $\mathsf{AC}(3)$ with large distinguishability. We then give a recursive construction of a family of functions $h_d \in \mathsf{AC}(d+1)$ for every d, where in each step we increase the depth by one and the noise sensitivity of h_d by a logarithmic factor. This gives the desired dependence of $\mathcal{E}_{G,\mathsf{val}}(h_d)$ on d.

Finally, we also tighten and simplify the upper bounds given in [CPT21]. We prove:

Theorem 1.3. *For every symmetric function $f : \{\pm 1\}^t \to \{\pm 1\}$, all $\mu \in (-1, 1)$ and $0 < \lambda < \frac{1-|\mu|}{128e}$ it holds that*

$$\mathcal{E}_{\lambda,\mu}(f) \leqslant \frac{124}{\sqrt{1-|\mu|}} \cdot \lambda.$$

Theorem 1.3 improves upon the corresponding theorem in [CPT21] in two ways:

1. First, the results in [CPT21] are obtained only for balanced test functions f. In contrast, Theorem 1.3 holds for every test function f with arbitrary bias μ.

2. Second, the bound stated in Theorem 1.3 improves upon the bound in Equation (1.2) by removing the $\log^{3/2}(1/\lambda)$ factor.

The extension of the results of [CPT21] to arbitrary bias μ is obtained by modifying the Fourier basis we work with. For a given bias μ we choose a basis that consists of $\prod_{i \in S} \frac{x_i - \mu}{\sqrt{1-\mu^2}}$ for all $S \subseteq [t]$. The improvement of the poly-logarithmic factor is achieved by using a more direct Fourier anlysis argument. The proof strategy of [CPT21] is to bound the error of weight indicator functions, and use it to handle weights around the mean. Then the argument invokes the expander Chernoff bound for bounding the remaining weights. Our approach does not go through analyzing weight indicator functions nor it uses the expander Chernoff bound. Instead, we use a very simple bound on the Fourier mass of symmetric functions, which gives a simpler and better analysis.

1.2 Open problems

We conclude the introduction with several open problems that follow from our work.

1. Can one combine the distribution obtained by a random walk on an expander with another pseudorandom distribution to obtain stronger results for functions in AC^0. For example, does permuting the values of the random walk with a pairwise independent permutation yields a distribution that better fools AC^0?

2. As explained before, our lower bounds are obtained for a graph G that is a Caylay graph over an Abelian group. It is well-known that every such a Cayley graph with constant expansion gap, has degree that depends on the number of vertices. Thus, a natural question is whether we can give similar lower bounds for constant degree graphs.

3. Continuing this line of thought, it is still possible that there is a family of graphs that fools all symmetric functions with error going down to zero with t. I.e., that while for some graphs (like Cayley graphs over $\mathbb{Z}_2^n$) there are bad labelling functions, for some other expander graphs, no such bad labellings exist. Similarly, it is possible that for some specific expanders better bounds exist for test functions in AC^0. Finding such graphs is a compelling goal that might require studying additional properties of graphs beyond expansion.

4. Finally, there is still a polynomial gap between the value of λ that fools functions in AC^0 and the corresponding lower bound we obtain. Any progress towards closing this gap will be interesting.

1.3 Paper organization

In Section 2 we give some background, mainly on Fourier Analysis. In Section 3 we recall the basic framework of [CPT21], except that we do it for arbitrary bias μ rather than just bias $\mu = 0$. In Section 4 we choose the graph and labelling function that we use for the lower bounds, and for which we can compute exactly the error induced by characters. In Section 5 we prove Theorem 1.1 and show that threshold function at one standard deviation away from the mean has error that goes down to zero with λ but not with t. In Section 6 we prove a special case of Theorem 1.2 for the case of $d = 3$. The full proof of Theorem 1.2 is in Appendix B. Then we turn to give a better upper bound on the error function and in Appendix C show a better and tight upper bound with a simpler proof. Finally, in Appendix D we show the threshold function about the mean (if $\mu = 0$ it is MAJ) and weight indicator functions do vanish with t.

1.4 Acknowledgments

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2 Preliminaries

We let $[n] = \{1, \ldots, n\}$, $\mathbb{1} \in \mathbb{R}^n$ denote the all 1s vector, i.e., $\mathbb{1} = (1, \ldots, 1)^{\mathsf{T}} \in \mathbb{R}^n$. We let $\mathbf{1} \in \mathbb{R}^n$ denote the normalized vector of $\mathbb{1}$, i.e $\mathbf{1} = \frac{1}{\sqrt{n}} \cdot \mathbb{1}$. We denote with $\mathbf{J} = \mathbf{1}\mathbf{1}^{\mathsf{T}}$.When we write $\|\cdot\|$ we always refer to the L_2-norm. Unless stated otherwise, $\log x = \log_2 x$. Throughout the paper, we make use of the following well known inequalities about binomial coefficients.

Claim 2.1. *Let $a \geqslant b \geqslant 1$ be integers. Then,* $(\frac{a}{b})^b \leqslant \binom{a}{b} \leqslant (\frac{ea}{b})^b$.

2.1 Fourier analysis

Consider the space of functions $f : \{\pm 1\}^t \to \mathbb{R}$, along with the inner product

$$\langle f, g \rangle = 2^{-t} \sum_{x \in \{\pm 1\}^t} f(x)g(x).$$

It is a well-known fact that the set $\{\chi_S(x) \mid S \subseteq [t]\}$, where $\chi_S(x) = \prod_{i \in S} x_i$, forms an orthonormal basis with respect to this inner product, which is called the Fourier basis. Thus every function $f : \{\pm 1\}^t \to \mathbb{R}$ can be uniquely represented as $f(x) = \sum_{S \subseteq [t]} \widehat{f}(S)\chi_S(x)$, where $\widehat{f}(S) \in \mathbb{R}$.

In this work we consider other bases, with respect to a similar inner product. Let $\mu \in [-1,1]$, and denote by U_t^μ the distribution over $\{\pm 1\}^t$ where each bit is chosen independently with expectation μ. Define

$$\langle f, g\rangle_\mu = \mathop{\mathbf{E}}_{x\sim U_t^\mu}[f(x)g(x)].$$

Denote by $\sigma = \sqrt{1-\mu^2}$, and let

$$\chi_S^\mu(x) = \prod_{i\in S} \frac{x_i - \mu}{\sigma}.$$

It is easy to see that the set $\left\{\chi_S^\mu(x) \mid S \subseteq [t]\right\}$, forms an orthonormal basis with respect to this new inner product, which is called the *μ-biased Fourier basis*. To see this, note that, by design, for $S \neq \emptyset$, $\mathbf{E}[\chi_S^\mu] = 0$ and $\mathbf{E}[(\chi_S^\mu)^2] = 1$. Similarly to the standard Fourier basis, every function $f : \{\pm 1\}^t \to \mathbb{R}$ can be uniquely represented as

$$f(x) = \sum_{S\subseteq[t]} \widehat{f_\mu}(S)\chi_S^\mu(x),$$

where $\widehat{f_\mu}(S) \in \mathbb{R}$.

We say that a function $f : \{\pm 1\}^t \to \mathbb{R}$ is symmetric if for every permutation $\sigma \in S_t$, $f(x_1,\ldots,x_t) = f(x_{\sigma(1)},\ldots,x_{\sigma(t)})$. It is not hard to show that if f is symmetric, then for every $S_1, S_2 \subseteq [t]$, with $|S_1| = |S_2|$, $\widehat{f_\mu}(S_1) = \widehat{f_\mu}(S_2)$. This allows us to use the following definition for symmetric functions: $\widehat{f_\mu}(k) = \left|\widehat{f_\mu}([k])\right|$, which is the absolute value of the Fourier coefficients of any weight k character. For more details on biased Fourier analysis see Chapter 8 of [O'D14].

3 The basic framework extended to arbitrary balanced tests

[CPT21] reduced the analysis of the error function of a *balanced* test function f to the analysis of the error function of characters. In this section we restate this framework, but do it in a more general way that applies to any test function f, no matter how balanced it is.

Let $G = (V,E)$ be a regular λ-spectral expander, and let $\mathsf{val} : V \to \{\pm 1\}$ be a labelling of the vertices of G with $\mathbf{E}[\mathsf{val}(V)] = \mu$. Let $t \geqslant 1$ be an integer. We want to compare two distributions on $\{\pm 1\}^t$:

- The distribution obtained by sampling t vertices $v_1,\ldots,v_t$ uniformly and independently at random, and outputting the ordered tuple $(\mathsf{val}(v_1),\ldots,\mathsf{val}(v_t))$. Note that this is the same distribution as sampling a sequence of t elements in $\{\pm 1\}$ independently from a μ-biased distribution, that is a distribution with expectation μ. We denote this distribution by U_t^μ.
- $\mathrm{RW}_{G,\mathsf{val}}$ is the distribution obtained by sampling a random length $t-1$ path $v_1,\ldots,v_t$ in G and outputting the ordered tuple $(\mathsf{val}(v_1),\ldots,\mathsf{val}(v_t))$. Equivalently, sample v_1 uniformly at random from V. Then, for $i = 2,3,\ldots,t$, sample v_i uniformly at random from the neighbours of v_{i-1}.

Let $f : \{\pm 1\}^t \to \{\pm 1\}$ be a test function. Expand f in the μ-biased Fourier basis,

$$f(x) = \sum_{S \subseteq [t]} \widehat{f_\mu}(S) \chi_S^\mu(x).$$

Lemma 3.1. *Let $G = (V, E)$ be a regular λ-spectral expander, and let $\mathsf{val} : V \to \{\pm 1\}$ be a labelling of the vertices of G with $\mathbf{E}[\mathsf{val}(V)] = \mu$. Then, for every function $f : \{\pm 1\}^t \to \mathbb{R}$,*

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} |\widehat{f_\mu}(S)| \mathcal{E}_{G,\mathsf{val}}(\chi_S^\mu).$$

Proof. Since val has expectation μ, for $S \neq \emptyset$, $\mathbf{E}[\chi_S^\mu(U_t^\mu)] = 0$ and thus $\mathbf{E}[f(U_t^\mu)] = \widehat{f_\mu}(\emptyset)$. Hence,

$$\begin{aligned} \mathcal{E}_{G,\mathsf{val}}(f) &= |\mathbf{E}\, f(\mathrm{RW}_{G,\mathsf{val}}) - \mathbf{E}\, f(U_t^\mu)| \\ &= \Big| \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} \widehat{f_\mu}(S)\, \mathbf{E}[\chi_S^\mu(\mathrm{RW}_{G,\mathsf{val}})] \Big|. \end{aligned}$$

For $S \neq \emptyset$, $\mathcal{E}_{G,\mathsf{val}}(\chi_S^\mu) = |\,\mathbf{E}[\chi_S^\mu(\mathrm{RW}_{G,\mathsf{val}})]|$. The proof follows by the triangle inequality. $\square$

Lemma 3.1 motivates us to consider parity test functions which we do next. We start by introducing some notation. For an integer $k \geqslant 2$, we define the family $\mathcal{F}_k$ of subsets of $[k-1]$ that, informally, consists of all subsets for which at least one of every two consecutive elements participate in the set. We also require the "end points" $1, k-1$ to participate in the set. Formally, we define

$$\mathcal{F}_k = \{I \subseteq [k-1] \mid \{1, k-1\} \subseteq I \text{ and } \forall j \in [k-2] \ \ \{j, j+1\} \cap I \neq \emptyset\}. \tag{3.1}$$

So, for example, $\mathcal{F}_6$ consists of the elements $\{1,3,5\}$, $\{1,2,4,5\}$ as well as of all subsets of $[5]$ that has as a subset any one of these two elements, namely, $\{1,2,3,5\}$, $\{1,3,4,5\}$ and $\{1,2,3,4,5\}$. We extend the definition in the natural way to $k = 0, 1$ by setting $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$.

Definition 3.2. *For integers $t \geqslant 1$, $2 \leqslant k \leqslant t$ and $j \in [k-2]$ define the map*

$$\Delta_j : \binom{[t]}{k} \to \mathbb{N}$$

as follows. Let $S \subseteq [t]$ of size $k \geqslant 2$ and denote $S = \{s_1, \ldots, s_k\}$ where $s_1 < \cdots < s_k$. For $i \in [k-1]$ write $\delta_i = s_{i+1} - s_i$. Define

$$\Delta_j(S) = \min(\delta_j, \delta_{j+1}).$$

Definition 3.3. *For an integer $t \geqslant 1$ define the map $\Delta : \binom{[t]}{\geqslant 2} \to \mathbb{N}$ as follows. Let $S \subseteq [t]$ of size $k \geqslant 2$. For $k = 2$ we define $\Delta(S) = \Delta_1(S)$, and for $k \geqslant 3$,*

$$\Delta(S) = \sum_{i=1}^{k-2} \Delta_i(S). \tag{3.2}$$

We prove:

Proposition 3.4. *Let $G = (V, E)$ be a regular λ-spectral expander and* $\mathsf{val} : V \to \{\pm 1\}$ *a labelling of the vertices of G with $\mathbf{E}[\mathsf{val}(V)] = \mu$. Then, for every $1 \leqslant k \leqslant t$ and $S \subseteq [t]$ of size k,*

$$\mathcal{E}_{G,\mathsf{val}}(\chi_S^\mu) \leqslant \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} \cdot \sum_{I \in \mathcal{F}_k} \lambda^{\sum_{j\in I} \Delta_j(S)} \leqslant \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} 2^k \cdot \lambda^{\Delta(S)/2}.$$

We remark that for sets of size $|S| = 1$ the sum is taken over the empty index set $\mathcal{F}_1$ and so equals 0. We also note that when $|\mu| = 1$ the error is trivially zero, while our bound tends to infinity.

Proof. Consider any non-empty subset $S \subseteq [t]$ of size $|S| = k$. As $\mathbf{E}[\chi_S(U_t^\mu)] = 0$ we have that

$$\mathcal{E}_{G,\mathsf{val}}(\chi_S^\mu) = \left|\mathbf{E}[\chi_S^\mu(\mathrm{RW}_{G,\mathsf{val}})]\right|.$$

We wish to express the right hand side algebraically. Let $n = |V|$ and identify V with $[n]$ in an arbitrary way. Let P be the $n \times n$ diagonal matrix with

$$P_{v,v} = \frac{\mathsf{val}(v) - \mu}{\sqrt{1-\mu^2}}$$

for every $v \in [n]$. We slightly abuse notation and denote the random walk matrix (that is, the normalized adjacency matrix) of G also by G. Define $\delta_S(i) = 1$ if $i \in S$ and $\delta_S(i) = 0$ otherwise and observe that

$$\mathbf{E}[\chi_S^\mu(\mathrm{RW}_{G,\mathsf{val}})] = \mathbf{1}^T \left(\prod_{i=1}^{t} P^{\delta_S(i)} G\right) \mathbf{1},$$

where, recall, $\mathbf{1}$ is the all one vector normalized by $\frac{1}{\sqrt{n}}$. Indeed, informally, at the i'th step we take a random step using G and then, depending on i being an element of I or not, we multiply by P or by I, respectively. Thus, we can write

$$\mathbf{E}[\chi_S^\mu(\mathrm{RW}_{G,\mathsf{val}})] = \mathbf{1}^T G^{t-s_k} \left(\prod_{i=1}^{k-1} P G^{\Delta_i}\right) P G^{s_1} \mathbf{1} = \mathbf{1}^T \left(\prod_{i=1}^{k-1} P G^{\Delta_i}\right) P \mathbf{1}, \tag{3.3}$$

where we have used the regularity of G, namely, $G\mathbf{1} = \mathbf{1}$.

Next, we use the spectral decomposition of G. As G is a λ-spectral expander we know that $G = \mathbf{J} + \lambda E$ where $\| E \| \leqslant 1$. Similarly, As G^ℓ is a λ^ℓ-spectral expander we have that $G^\ell = \mathbf{J} + \lambda^\ell E_\ell$ for some operator E_ℓ with bounded norm $\| E_\ell \| \leqslant 1$. Thus,

$$\prod_{i=1}^{k-1} P G^{\Delta_i} = \sum_{I \subseteq [k-1]} \prod_{i=1}^{k-1} P B_i(I), \tag{3.4}$$

where

$$B_i(I) = \begin{cases} \lambda^{\Delta_i} E_{\Delta_i} & i \in I; \\ \mathbf{J} & \text{otherwise.} \end{cases}$$

For $I \subseteq [k-1]$ let

$$e_I = \mathbf{1}^T \left(\prod_{i=1}^{k-1} PB_i(I) \right) P\mathbf{1}.$$

Equations (3.3) and (3.4) imply that

$$\mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}})] = \sum_{I \subseteq [k-1]} e_I. \tag{3.5}$$

Not all subsets $I \subseteq [k-1]$ contribute non-zero values e_I to the sum. Indeed, if $k-1 \notin I$ then $B_{k-1}(I) = \mathbf{J}$ and so

$$\begin{aligned} e_I = \mathbf{1}^T \left(\prod_{i=1}^{k-2} PB_i(I) \right) (P\mathbf{J})P\mathbf{1} \ &= \mathbf{1}^T \left(\prod_{i=1}^{k-2} PB_i(I) \right) (P\mathbf{1}\mathbf{1}^T)P\mathbf{1} \\ &= \mathbf{1}^T \left(\prod_{i=1}^{k-2} PB_i(I) \right) P\mathbf{1}(\mathbf{1}^T P\mathbf{1}) = 0, \end{aligned}$$

because

$$\mathbf{1}^T P\mathbf{1} = \frac{1}{\sqrt{1-\mu^2}} \cdot \sum_{i \in [n]} \frac{\mathsf{val}(i) - \mu}{n} = \frac{\mathbf{E}[\mathsf{val}(V)] - \mu}{\sqrt{1-\mu^2}} = 0.$$

Similarly $e_I = 0$ for I not containing 1. Moreover, if $j, j+1$ are both not contained in I for some $j \in [k-2]$ then

$$\begin{aligned} e_I &= \mathbf{1}^T \left(\prod_{i=1}^{j-1} PB_i(I) \right) (PB_j(I))(PB_{j+1}(I)) \left(\prod_{i=j+2}^{k-2} PB_i(I) \right) P\mathbf{1} \\ &= \mathbf{1}^T \left(\prod_{i=1}^{j-1} PB_i(I) \right) (P\mathbf{J})(P\mathbf{J}) \left(\prod_{i=j+2}^{k-2} PB_i(I) \right) P\mathbf{1} = 0, \end{aligned}$$

Because

$$(P\mathbf{J})(P\mathbf{J}) = (P\mathbf{1}\mathbf{1}^T)(P\mathbf{1}\mathbf{1}^T) = P\mathbf{1}(\mathbf{1}^T P\mathbf{1})\mathbf{1}^T = 0.$$

Thus, any subset $I \subseteq [k-1]$ that may contribute to the sum in Equation (3.5) is contained in $\mathcal{F}_k$ as defined in Equation (3.1).

Next, we look at $I \in \mathcal{F}_k$. We have that

$$e_I = \mathbf{1}^T \left(\prod_{i=1}^{k-1} PB_i(I) \right) P\mathbf{1} \ \leqslant \prod_{i=1}^{k-1} \|PB_i(I)\| \ \leqslant \|P\|^{k-1} \prod_{i \in I} \|B_i(I)\|. \tag{3.6}$$

Recall that for every $i \in I$, $B_i(I) = \lambda^{\Delta_i} E_{\Delta_i}$ and that $\|E_{\Delta_i}\| \leqslant 1$. Thus, $\prod_{i \in I} \|B_i(I)\| \leqslant \prod_{i \in I} \lambda^{\Delta_i}$. Also, Let M be the $n \times n$ diagonal matrix defined by $M_{v,v} = \mathsf{val}(v)$ for all $v \in [n]$. Note that $P = \frac{1}{\sqrt{1-\mu^2}}(M - \mu I)$. As $\|M\| = 1$, using the triangle inequality we get

$$\|P\| \leqslant \frac{\|M\| + \|\mu I\|}{\sqrt{1-\mu^2}} \leqslant \frac{1+|\mu|}{\sqrt{1-\mu^2}} = \sqrt{\frac{1+|\mu|}{1-|\mu|}}. \tag{3.7}$$

Equation (3.6) and Equation (3.7) together imply that $e_I \leqslant \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} \prod_{i\in I} \lambda^{\Delta_i}$. This proves the first inequality in the proposition.

To prove the second inequality consider $I \in \mathcal{F}_k$, and notice that

$$2\sum_{i\in I} \Delta_i \geqslant \sum_{i=1}^{k-2} \delta_i \Delta_i + \delta_{i+1}\Delta_{i+1} \geqslant \sum_{i=1}^{k-2} \min(\Delta_i, \Delta_{i+1}),$$

because for every $i \in [k-2]$, at least one of $i, i+1$ is in I. To complete the proof of the second inequality notice also that $|\mathcal{F}_k| \leqslant 2^{k-1}$. □

4 Choosing the graph

In this section we choose an expander graph for which we obtain a precise analytic formula for the expectation of characters under the input distribution given by the random walk.

Claim 4.1. *Let $G = ([n], E)$ be a regular graph with second largest eigenvalue λ_2 and corresponding eigenvector v_2. Further assume all coordinates in v_2 have ± 1 values. Define $\mathsf{val} : [n] \to \{\pm 1\}$ by $\mathsf{val}_2(i) = v_2(i)$. Let $S \subseteq [n]$, $|S| = k$. Let P be the diagonal matrix corresponding to val_2, that is, $P_{i,i} = \mathsf{val}_2(i) = v_2(i)$. Then,*

$$\left(\prod_{i=1}^{k-1} PG^{\Delta_i}\right) P\mathbb{1} = \begin{cases} \lambda^{\sum_{i=1}^{k-2/2} \Delta_{2i+1}} \mathbb{1} & k \in \mathbb{N}_{even}, \\ \lambda^{\sum_{i=1}^{k-1/2} \Delta_{2i+1}} v_2 & k \in \mathbb{N}_{odd}. \end{cases}$$

Proof. We will prove the claim by induction. For the base case $k = 1$ it holds that $\prod_{i=1}^{k-1} PG^{\Delta_i} = I$, and the statement follows as $IP\mathbb{1} = v_2 = \lambda^0 v_2$. For the induction step, note that

$$\left(\prod_{i=1}^{k} PG^{\Delta_i}\right) P\mathbb{1} = PG^{\Delta_k} \left(\prod_{i=1}^{k-1} PG^{\Delta_i}\right) P\mathbb{1}.$$

If $k \in \mathbb{N}_{even}$ than $k-1 \in \mathbb{N}_{odd}$ and, using the induction hypothesis we get that

$$PG^{\Delta_k} \left(\prod_{i=1}^{k-1} PG^{\Delta_i}\right) P\mathbb{1} = PG^{\Delta_k} \lambda^{\sum_{i=1}^{(k-2)/2} \Delta_{2i+1}} v_2 \ = \lambda^{\sum_{i=1}^{k/2} \Delta_{2i+1}} \mathbb{1},$$

which is what we wanted to prove. The proof in the case that $k \in \mathbb{N}_{odd}$ is similar. □

Definition 4.2. *For $S \subseteq [t]$ denote $\Delta_{odd}(S) = \sum_{i=1}^{\lfloor(|S|-1)/2\rfloor} \Delta_{2i+1}(S)$.*

Corollary 4.3. *Let $G = ([n], E)$ and $\mathsf{val}_2 : [n] \to \{\pm 1\}$ be as above. Then,*

$$\mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}_2})] = \begin{cases} \lambda^{\Delta_{odd}(S)} & |S| \in \mathbb{N}_{even}, \\ 0 & |S| \in \mathbb{N}_{odd}. \end{cases}$$

Proof. Note that $Pv_2 = \mathbb{1}$ and $P\mathbb{1} = v_2$. As before, it holds that

$$\mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}_2})] = \mathbf{1}^T \left(\prod_{i=1}^{k-1} PG^{\Delta_i}\right) P\mathbf{1} = \frac{1}{n}\mathbb{1}^\mathsf{T} \left(\prod_{i=1}^{k-1} PG^{\Delta_i}\right) P\mathbb{1}.$$

Using Claim 4.1, we conclude that,

$$\mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}_2})] = \begin{cases} \lambda^{\Delta_{odd}(S)} \cdot \frac{1}{n}\mathbb{1}^\mathsf{T}\mathbb{1} & k \in \mathbb{N}_{even}, \\ \lambda^{\sum_{i=1}^{k-1/2} \Delta_{2i+1}} \cdot \frac{1}{n}\mathbb{1}^\mathsf{T} v_2 & k \in \mathbb{N}_{odd}. \end{cases}$$

The fact that G is regular implies that $\mathbb{1}^\mathsf{T} v_2 = 0$, which finishes the case that k is odd; the case that k is even is handled similarly by noting that $\mathbb{1}^\mathsf{T}\mathbb{1} = n$. □

We now give an example to such a Graph G. Cayley graphs over an Abelian group commute and share an orthonormal basis of eigenvectors, which is known to be the set of all characters of the group. If the group is $\mathbb{Z}_2^n$, the eigenvectors have entries that are 2nd roots of unity, i.e., have ± 1 entries as desired. The eigenvalues have a direct correspondence to the set of generators of the Cayley graph. Building on that, [AR94] proved that for every $0 < \lambda < 1$ of the form $\frac{1}{m}$ for $m \in \mathbb{N}$ and $m \leqslant n \in \mathbb{N}$, there is a Cayley graph on the n dimensional boolean cube, with $\lambda_2 = \lambda$. The degree of this graph depends both on n and λ.

From now on we let G be a regular exapnder with second largest eigenvalue λ and corresponding eigenvector with ± 1 entries, and we let val_2 reflect that eigenvector.

5 A lower bound for symmetric functions

In this section we prove the following theorem.

Theorem 5.1. *Let $0 < c_0 \leqslant 1$, and let G, val_2 be as in the previous section, for $0 < \lambda < \frac{c_0^2}{12800 \cdot e}$. Let $f \colon \{\pm 1\}^t \to \{\pm 1\}$ be a symmetric function with $\left|\widehat{f}(2)\right| \geqslant \frac{c_0}{\sqrt{\binom{t}{2}}}$. Then,*

$$\mathcal{E}_{G,\mathsf{val}_2}(f) \geqslant 0.001 c_0 \lambda.$$

The idea behind the proof is to show that when choosing G, val_2 as in Section 4, the upper bound given by [CPT21] is tight (up to the redundant poly logarithmic factor). We will use the following claim from [CPT21].

Lemma 5.2 ([CPT21], Lemma 4.4)**.** *Denote*

$$\beta_k = \sum_{\substack{S \subseteq [t] \\ |S| = k}} \mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}})]. \tag{5.1}$$

Then,

$$\beta_k \leqslant 2^k \binom{t-1}{\lfloor \frac{k}{2} \rfloor} \left(\frac{\lambda}{1-\lambda}\right)^{\lceil \frac{k}{2} \rceil}. \tag{5.2}$$

Using these notations we are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Denote by $\mathcal{B}_2 = \{\{i, i+1\} \mid i \in [t-1]\}$. Note that $|\mathcal{B}_2| = t - 1$ and that for every $S \in \mathcal{B}_2$ it holds that $\Delta_{odd}(S) = 1$. Recall that $\mathcal{E}_{G,\mathsf{val}_2}(\chi_S) = 0$ if $|S| = 1$, therefore,

$$\mathcal{E}_{G,\mathsf{val}_2}(f) = \left| \sum_{S\subseteq[t],|S|\geqslant 2} \widehat{f}(|S|)\, \mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}})] \right| \tag{5.3}$$

$$\geqslant \left|\widehat{f}(2)\right| \left| \sum_{S\subseteq[t],|S|=2} \mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}})] \right| - \left| \sum_{S\subseteq[t],|S|>2} \widehat{f}(|S|)\mathcal{E}_{G,\mathsf{val}_2}(\chi_S) \right|. \tag{5.4}$$

However, by Corollary 4.3,

$$\left|\widehat{f}(2)\right| \left| \sum_{S\subseteq[t],|S|=2} \mathbf{E}[\chi_S(\mathrm{RW}_{G,\mathsf{val}})] \right| \geqslant \left|\widehat{f}(2)\right| \sum_{S\subseteq\mathcal{B}_2} \lambda \geqslant c_0\sqrt{2}\sqrt{\frac{t-1}{t}}\lambda \geqslant \frac{c_0}{\sqrt{2}}\lambda.$$

Furthermore,

$$\left| \sum_{S\subseteq[t],|S|>2} \widehat{f}(S)\mathcal{E}_{G,\mathsf{val}_2}(\chi_S) \right| \leqslant \sum_{k\geqslant 3} \left|\widehat{f}(k)\right| \beta_k \leqslant \sum_{k\geqslant 3} \frac{1}{\sqrt{\binom{t}{k}}} 2^k \binom{t-1}{\lfloor \frac{k}{2} \rfloor} \left(\frac{\lambda}{1-\lambda}\right)^{\lceil \frac{k}{2} \rceil},$$

where in the last inequality we used Lemma 5.2 and Claim C.3. The right hand side of the above equation is bounded above by

$$\sum_{k\geqslant 3} (16e)^{k/2}\lambda^{k/2} \leqslant 124\lambda^{1.5}.$$

We omit the calculations (a similar calculation appears in Theorem C.1). Assume that $\lambda \leqslant \frac{c_0^2}{128e\cdot 100}$. Then Equation (5.3) yields

$$\mathcal{E}_{G,\mathsf{val}_2}(f) \geqslant \frac{c_0}{\sqrt{2}}\lambda - 124\lambda^{1.5} \geqslant 0.04c_0\lambda. \qquad \square$$

In order to prove Theorem 1.1, we are left with providing a function f that satisfy the conditions of Theorem 5.1. In fact, we show that the threshold function at one standard deviation distance from the mean has non-vanishing error in t. This is done in **??**.

6 A lower bound for AC^0 tests

In this section we use the noise operator. The following definitions and claims appear in [O'D14].

Definition 6.1. *Let $\rho \in [-1,1]$. For a fixed $x \in \{\pm 1\}^t$ we write $y \sim N_\rho(x)$ to denote the random string y that is drawn as follows: for each $i \in [t]$ independently,*

$$y_i = \begin{cases} x_i & \text{with probability } \frac{1+\rho}{2}, \\ -x_i & \text{with probability } \frac{1-\rho}{2}. \end{cases}$$

Definition 6.2. *Let $\rho \in [-1,1]$. The noise operator T_ρ is the linear operator on functions $\{\pm 1\}^t \to \mathbb{R}$ defined by $T_\rho f(x) = \mathbf{E}_{y\sim N_\rho(x)} f(y)$. The fact that the operator is linear follows directly from the linearity of the expectation.*

Notice that $T_1(f) = f$ whereas $T_0(f)$ is the constant function $T_0(f) = \mathbf{E}\, f$. We make use of the following lemma.

Lemma 6.3. *For every function $f : \{\pm 1\}^t \to \mathbb{R}$ it holds that: $\widehat{T_\rho f}(S) = \widehat{f}(S)\rho^{|S|}$.*

The starting point of this section is to connect the expectation of f under a random walk and the noise function $T_\lambda(f)$, we prove this claim in Appendix B.

Claim 6.4. *For $f\colon \{\pm 1\}^t \to \mathbb{R}$ define $\tilde{f}\colon \{\pm 1\}^{2t} \to \mathbb{R}$ by*

$$\tilde{f}(x_1, x_2, \ldots, x_{2t-1}, x_{2t}) = f(x_1 \cdot x_2, \ldots, x_{2t-1} \cdot x_{2t}).$$

Then, $\mathbf{E}[\tilde{f}(\mathrm{RW}_{G,\mathsf{val}_2})] = (T_\lambda f)(\mathbb{1})$.

6.1 A lower bound for the Tribes function composed with IP

We now construct a two function in $\mathsf{AC}(2)$, that satisfy Theorem 1.2. Later on we extend the construction inductively to obtain the general theorem. The idea behind the depth-3 construction is the following. We look for a function $f = f(x_1, \ldots, x_t) \in \mathsf{AC}(2)$ such that

$$|\,\mathbf{E}[f(U_t)] - T_\lambda(f)(\mathbf{1})| \geqslant \lambda \cdot \log t. \tag{6.1}$$

We then look at $\tilde{f}(y_1, \ldots, y_{2t}) = f(y_1 \cdot y_2, \ldots, y_{2t-1} \cdot y_{2t}) \in \mathsf{AC}(3)$ and note that:

- $\mathbf{E}[\tilde{f}(U_t)] = \mathbf{E}[f(U_t)]$ as the product of two uniform ± 1 bits is uniform; However,
- by Theorem 6.4, $\mathbf{E}[\tilde{f}(\mathrm{RW}_{G,\mathsf{val}_2})] = T_\lambda(f)(\mathbf{1})$.

Together,

$$\mathcal{E}_\lambda(\tilde{f}) \geqslant \mathcal{E}_{G,\mathsf{val}_2}(\tilde{f}) = |\,\mathbf{E}[\tilde{f}(U_t)] - \mathbf{E}[\tilde{f}(\mathrm{RW}_{G,\mathsf{val}_2})]| = |\,\mathbf{E}[f(U_t)] - T_\lambda(f)(\mathbf{1})] \geqslant \lambda \cdot \log t,$$

which in turns implies Theorem 1.2, for $d = 3$.

We take f to be the Tribes function. Fix t; we choose parameters r, h such that $r \cdot h \leqslant t$ by taking $h = \log(t) - \log\log(t)$[2] and $r = \lfloor \frac{t}{\log t} \ln(2) \rfloor$. Partition $[t]$ into disjoint sets $I_1, \ldots, I_r$, each of size h. We define $f : \{\pm 1\}^t \to \{0, 1\}$ to be the Tribes function on t bits and define g to be the related function

$$f(z_1, \ldots, z_t) = \bigvee_{i \in [r]} \bigwedge_{j \in I_i} z_j, \qquad\qquad g(z_1, \ldots, z_t) = \bigwedge_{i \in [r]} \bigvee_{j \in I_i} z_j.$$

Here, -1 is interpreted as "true", 1 is interpreted as "false". Note that $f, g \in \mathsf{AC}(2)$.

As before, we choose G to be a Cayley graph on the boolean hypercube with $\lambda_2 = \lambda$ and $\mathsf{val} = \mathsf{val}_2$.

Claim 6.5. *The functions f and g are almost balanced with respect to the uniform distribution. Quantitatively, $\mathbf{E}[f], \mathbf{E}[g] \in \left[\frac{1}{2} - O\left(\frac{\log t}{t}\right), \frac{1}{2} + O\left(\frac{\log t}{t}\right)\right]$.*

[2] Recall that $\log t = \log_2 t$

Proof. From De Morgan's identity we have $g(x_1, \dots, x_t) = 1 - f(\overline{x_1}, \dots, \overline{x_t})$, so $\mathbf{E}[g] = 1 - \mathbf{E}[f]$ and so it is enough to prove the statement for f. To this end write

$$\mathbf{E}[f] = \mathbf{Pr}[f = 1] = 1 - \prod_{i=1}^{r} \mathbf{Pr}\left[\bigwedge_{j \in I_i} z_j = 0\right]$$
$$= 1 - \prod_{i=1}^{r} \left(1 - \mathbf{Pr}\left[\bigwedge_{j \in I_i} z_j = 1\right]\right) = 1 - \left(1 - \frac{1}{2^h}\right)^r.$$

Using the fact that $1 - \varepsilon = e^{-\varepsilon + O(\varepsilon^2)}$ we obtain that

$$1 - \left(1 - \frac{1}{2^h}\right)^r = 1 - e^{-2^{-h} r + O(2^{-2h} r)}$$
$$= 1 - e^{-\ln 2 + O(\frac{\log t}{t})} = \frac{1}{2} + \Theta\left(\frac{\log t}{t}\right),$$

as desired. □

Denote by μ_p the product distribution over $\{\pm 1\}^t$, wherein for each $i \in [t]$ we have that $\mathbf{Pr}[z_i = -1] = p$. Abusing notation denote $\mu_p(f) = \mathbf{E}_{x \sim \mu_p}[f(x)]$.

Claim 6.6. *Let $p = \frac{1-\varepsilon}{2}$ and assume $\varepsilon \geqslant \frac{k}{\log(t)}$. Then,*

$$\mu_p(f), \mu_p(g) \leqslant e^{-k/10}.$$

Proof. First, we analyze $\mu_p(f)$. By definition it is equal to

$$\mathop{\mathbf{Pr}}_{\mu_p}[f = 1] = 1 - (1 - p^h)^r = 1 - (1 - 2^{-h}(1 - \varepsilon)^h)^r$$
$$= 1 - \left(1 - 2^{-h}\left(1 - \frac{k}{\log t}\right)^h\right)^r \leqslant 1 - \left(1 - 2^{-h} e^{-k}\right)^r.$$

Using $(1 - \delta)^r \geqslant 1 - r\delta$, we get that the above expression is bounded by $r 2^{-h} e^{-k} \leqslant e^{-k}$. Next, we upper bound $\mu_p(g)$. By definition, it is equal to

$$\mathop{\mathbf{Pr}}_{\mu_p}[g = 1] \leqslant (1 - (1 - p)^h)^r = (1 - 2^{-h}(1 + \varepsilon)^h)^r = \left(1 - 2^{-h}\left(1 + \frac{k}{\log t}\right)^h\right)^r.$$

Using $(1 + \delta)^r \geqslant \delta r$ for $\delta > 0$, we get that this is at most

$$(1 - 2^{-h} k)^r \leqslant e^{-r 2^{-h} k} \leqslant e^{-k/10}. \quad \square$$

We now prove Theorem 1.2 for $d = 3$. We take $h(x_1, y_1, \dots, x_t, y_t) = f(x_1 \cdot y_1, \dots, x_t \cdot y_t)$. $h \in \mathsf{AC}(3)$ because $f \in \mathsf{AC}(2)$.

- On the one hand, by Claim 6.4, $\mathbf{E}[h(\mathrm{RW}_{G, \mathsf{val}_2})] = T_\lambda(f)(\mathbb{1}) = \mu_{\frac{1-\lambda}{2}}(f)$. By Claim 6.6, and using $\lambda \geqslant \frac{k}{\log t}$, we get that $\mathbf{E}[h(\mathrm{RW}_{G, \mathsf{val}_2})] < e^{-k/10}$.
- On the other hand, By Claim 6.5, $\mathbf{E}[h] = \mathbf{E}[f] \geqslant \frac{1}{2} - O(\frac{\log t}{t})$.

Together, h is as desired.

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A Missing proof from Section 5

We use the following definitions and claim. For integers t and $w \in \{0, 1, \ldots, t\}$ let $\mathbf{1}_w : \{\pm 1\}^t \to \{0, 1\}$ be the function indicating whether the weight of the input is w. That is, $\mathbf{1}_w(x_1, \ldots, x_t) = 1$ if $|\{i \in [t] \mid x_i = 1\}| = w$ and $\mathbf{1}_w(x_1, \ldots, x_t) = 0$ otherwise. We also define $\mathbf{1}_{>w} : \{\pm 1\}^{t+1} \to \{0, 1\}$ be the function indicating whether the weight of the input is greater w. That is, $\mathbf{1}_w(x_1, \ldots, x_t) = 1$ if $\sum_i x_i > w$ and $\mathbf{1}_w(x_1, \ldots, x_t) = 0$ otherwise.

Claim A.1. *For every $S \subseteq [t]$, it holds that*

$$\widehat{(\mathbf{1}_w)_\mu}(S) = \frac{\widehat{(\mathbf{1}_{>w})_\mu}(S \cup \{0\})}{\sqrt{1 - \mu^2}}.$$

Proof.

$$
\begin{aligned}
\mathbf{1}_w(x_1,\ldots,x_t) &= \mathbf{1}_{>w}(1,x_1,\ldots,x_t) - \mathbf{1}_{>w}(0,x_1,\ldots,x_t)\\
&= \sum_{S\subseteq\{0,\ldots,t\}} \widehat{(\mathbf{1}_{>w})}_\mu(S)\chi^\mu_S(1,x_1,\ldots,x_t) - \sum_{S\subseteq\{0,\ldots,t\}} \widehat{(\mathbf{1}_{>w})}_\mu(S)\chi^\mu_S(0,x_1,\ldots,x_t)\\
&= \sum_{S\subseteq\{0,\ldots,t\}} \widehat{(\mathbf{1}_{>w})}_\mu(S)(\chi^\mu_S(1,x_1,\ldots,x_t) - \chi^\mu_S(0,x_1,\ldots,x_t))\\
&\sum_{\substack{S\subseteq\{0,\ldots,t\}\\ 0\in S}} \widehat{(\mathbf{1}_{>w})}_\mu(S)\frac{1}{\sqrt{1-\mu^2}}\chi^\mu_{S\setminus\{0\}}(x_1,\ldots,x_t),
\end{aligned}
$$

and the claim follows. $\square$

Proof of Theorem 1.1. Take $f = \mathbf{1}_{>w}$ for $w = \frac{t-\sqrt{t}}{2}$. We claim that $\left|\widehat{f}(2)\right| > \frac{c_0}{\sqrt{\binom{t}{2}}}$, for some absolute constant $c_0 > 0$ and therefore by Theorem 5.1, $\mathcal{E}_{G,\mathsf{val}}(f_t) \geqslant c\cdot\lambda$ for some constant c. Indeed, by Claim A.1 we have $\widehat{f}(2) = \widehat{\mathbf{1}_w}(1)$ for $\mathbf{1}_w : \{\pm1\}^{t-1} \to \{0,1\}$. To compute $\widehat{\mathbf{1}_w}(1)$ we apply [CPT21, Claim 4.9] for $w = \frac{t-\sqrt{t}}{2}$ and get

$$
\left|\widehat{\mathbf{1}_w}(1)\right| = \left|\frac{1}{2^{t-1}}\frac{\binom{t-1}{w}}{\binom{t-1}{1}}\sum_{\ell=0}^{\lfloor\frac{1}{2}\rfloor}(-1)^{1-\ell}\binom{w}{\ell}\binom{t-2w-1}{1-2\ell}\right| = \frac{1}{2^{t-1}}\frac{\binom{t-1}{w}}{t-1}(t-1-2w).
$$

Substituting $w = \frac{t-\sqrt{t}}{2}$, together with the fact that $\binom{t-1}{\frac{t-\sqrt{t}}{2}} \geqslant \Omega\left(\frac{1}{\sqrt{t}}2^t\right)$, concludes the proof. $\square$

B Appendix for Section 6

Proof: [Proof of Claim 6.4] For $\{s_1,\ldots,s_k\} = S \subseteq [t]$ denote $2S\colon = \{2s_1-1, 2s_1,\ldots,2s_k-1, 2s_k\} \subseteq [2t]$. Note that $\Delta_{odd}(2S) = |S|$.

$$
\begin{aligned}
\tilde{f}(x_1,x_2,\ldots,x_{2t-1},x_{2t}) &= f(x_1\cdot x_2,\ldots,x_{2t-1}\cdot x_{2t})\\
&= \sum_{S\subseteq[t]} \widehat{f}(S)\chi_S(x_1\cdot x_2,\ldots,x_{2t-1}\cdot x_{2t})\\
&= \sum_{S\subseteq[t]} \widehat{f}(S)\chi_{2S}(x_1,x_2,\ldots,x_{2t-1},x_{2t}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}[\tilde{f}(\mathrm{RW}_{G,\mathsf{val}_2})] &= \sum_{S\subseteq[t]} \widehat{f}(S)\,\mathbf{E}[\chi_{2S}(\mathrm{RW}_{G,\mathsf{val}_2})]\\
&= \sum_{S\subseteq[t]} \widehat{f}(S)\lambda^{|S|}\\
&= \sum_{S\subseteq[t]} \widehat{f}(S)\lambda^{|S|}\chi_S(\mathbb{1}),
\end{aligned}
$$

which is equal to $T_\lambda(f)(\mathbb{1})$ by Lemma 6.3. For the second equality we used Corollary 4.3. $\square$

B.1 Upper and lower bounds on $\mu_p(f), \mu_p(g)$

Our construction of the functions h_d is iterative, building on f and g from the previous section. Before we do so we need better upper *and lower* bounds on the noise sensitivity of f and g, as defined in the previous section.

Claim B.1. *There is a sufficiently large constant $k > 0$ such that the following holds. Suppose $p = \frac{1-\varepsilon}{2}$ is such that $\frac{k^2}{t} \leqslant \varepsilon \leqslant \frac{k}{\log t}$. Then,*

$$\mu_p(f), \mu_p(g) \leqslant \frac{1 - \frac{h\varepsilon}{20k}}{2}.$$

Proof. First, we analyze $\mu_p(f)$. By definition, it is equal to

$$\begin{aligned}\Pr_{\mu_p}[f = 1] &\leqslant 1 - (1 - p^h)^r \\ &= 1 - (1 - 2^{-h}(1 - \varepsilon)^h)^r = 1 - e^{-r \cdot 2^{-h}(1-\varepsilon)^h + O(2^{-2h} r (1-\varepsilon)^{2h})}\end{aligned}$$

As $\varepsilon h \leqslant k$ we have $(1 - \varepsilon)^h \leqslant 1 - \frac{\varepsilon h}{10k}$. Therefore we may upper bound the previous expression by

$$1 - e^{-r \cdot 2^{-h}\left(1 - \frac{h\varepsilon}{10k}\right) + O(2^{-2h} r (1-\varepsilon)^{2h})} \leqslant 1 - e^{-\ln 2 + \frac{h\varepsilon \ln 2}{10k} + O\left(\frac{\log t}{t}\right)} = 1 - \frac{1}{2} e^{\frac{h\varepsilon \ln 2}{10k} + O\left(\frac{\log t}{t}\right)}.$$

which is at most $\frac{1}{2} - \frac{h\varepsilon}{40k}$ provided that k is sufficiently large. Here, we used the bound $e^z \geqslant 1 + z$. Now we analyze $\mu_p(g)$. By definition,

$$\Pr_{\mu_p}[g = 1] = (1 - (1 - p)^h)^r = (1 - 2^{-h}(1 + \varepsilon)^h)^r \leqslant e^{-r \cdot 2^{-h}(1+\varepsilon)^h + O(2^{-2h} r (1+\varepsilon)^{2h})}.$$

Using the fact that $(1 + \varepsilon)^h \leqslant 1 + 10h\varepsilon$, as $h\varepsilon \leqslant 1$ we may further upper bound this by

$$\leqslant e^{-r \cdot 2^{-h}(1 + 10h \cdot \varepsilon) + O\left(\frac{\log t}{t}\right)} \leqslant e^{-\ln 2 - 10h\varepsilon \ln 2 + O\left(\frac{\log t}{t}\right)} \leqslant \frac{1}{2} e^{-10h\varepsilon \ln 2 + O\left(\frac{\log t}{t}\right)}$$

Using $e^{-z} \leqslant 1 - \frac{z}{10k}$ that holds for $0 \leqslant z \leqslant k$, we may upper bound this by

$$\frac{1}{2}\left(1 - \frac{10h\varepsilon \ln 2}{20k} + O\left(\frac{\log t}{t}\right)\right) \leqslant \frac{1}{2} - \frac{h\varepsilon}{40k},$$

where the last inequality holds provided that k is sufficiently large. $\square$

Next we give a *lower bound* on $\mu_p(f)$ and $\mu_p(g)$:

Claim B.2. *Let $p = \frac{1-\varepsilon}{2}$ such that there exists $k \in \mathbb{N}$ satisfies $\frac{k^2}{t} \leqslant \varepsilon \leqslant \frac{1}{\sqrt{t}}$. Then,*

$$\mu_p(f), \mu_p(g) \geqslant \frac{1 - 20h\varepsilon}{2}.$$

Proof. First, we analyze $\mu_p(f)$. By definition, it is equal to

$$\Pr_{\mu_p}[f = 1] = 1 - (1 - p^h)^r = 1 - (1 - 2^{-h}(1 - \varepsilon)^h)^r = 1 - e^{-r \cdot 2^{-h}(1-\varepsilon)^h + O(2^{-2h} r (1-\varepsilon)^{2h})}.$$

Using $(1-\varepsilon)^h \geqslant 1-\varepsilon h$, we may lower bound the previous expression by

$$1 - e^{-r\cdot 2^{-h}(1-h\varepsilon)+O(2^{-2h}r(1-\varepsilon)^{2h})} \geqslant 1 - e^{-\ln 2 + h\varepsilon \ln 2 + O\left(\frac{\log t}{t}\right)} = 1 - \frac{1}{2}e^{h\varepsilon \ln 2 + O\left(\frac{\log t}{t}\right)}.$$

which is at least $\frac{1}{2} - 20h\varepsilon$. Here, we used the bound $e^z \leqslant 1+2z$, provided $0 \leqslant z \leqslant 1$. Now we analyze $\mu_p(g)$. By definition,

$$\Pr_{\mu_p}[g=1] = (1-(1-p)^h)^r = (1-2^{-h}(1+\varepsilon)^h)^r \geqslant e^{-r\cdot 2^{-h}(1+\varepsilon)^h + O(2^{-2h}r(1+\varepsilon)^{2h})}.$$

Using the fact that $(1+\varepsilon)^h \leqslant 1 + 10\cdot h\varepsilon$ as $h\varepsilon \leqslant 1$, we may lower bound this by

$$\geqslant e^{-r\cdot 2^{-h}(1+10h\cdot\varepsilon)+O\left(\frac{\log t}{t}\right)} = e^{-\ln 2 - 10h\cdot\varepsilon \ln 2 + O\left(\frac{\log t}{t}\right)} \geqslant \frac{1}{2}e^{-h\cdot\varepsilon\cdot 10\ln 2 + O\left(\frac{\log t}{t}\right)}.$$

Using $e^{-z} \geqslant 1-z$, we may lower bound the above expression by

$$\frac{1}{2}\left(1 - 10\ln 2h\cdot\varepsilon + O\left(\frac{\log t}{t}\right)\right) \geqslant \frac{1-20h\cdot\varepsilon}{2}.$$

□

B.2 A lower bound for a function in $\mathsf{AC}(d)$

We are now ready to prove Theorem 1.2, restated below. We continue with the choice of G and val as in Section 4.

Theorem B.3 (Theorem 1.2; *restated*)**.** *There are universal constants $\varepsilon > 0$, $k \in \mathbb{N}$, satisfying the following. For every $3 \leqslant d \in \mathbb{N}$, there exists a constant t_d, and a family of functions $(h_t)_{t_d \leqslant t \in \mathbb{N}} \subset \mathsf{AC}(d)$, such that for every*

$$\lambda \geqslant \frac{(40(d-2)k)^{d-2}}{\log^{d-2} t}$$

there is a λ-spectral expander $G=(V,E)$ and a labelling $\mathsf{val}: V \to \{\pm 1\}$ with $\mathbf{E}[\mathsf{val}(V)] = 0$ such that

$\mathcal{E}_{G,\mathsf{val}}(h_t) \geqslant \varepsilon.$

For the proof we define a sequence of functions $\{h_d\}_{d\in\mathbb{N}}$ as follows. The function h_1 is defined to be g; the function h_2 is given t^2 input bits. We group the t^2 inputs into t blocks $x^1,\ldots,x^t \in \{\pm 1\}^t$ and define

$$h_2 = g(f(x^1),\ldots,f(x^t)).$$

Iteratively, once h_d has been defined, we view the input to $h_{d+1}\colon \{\pm 1\}^{t^{d+1}} \to \{0,1\}$ as $y^1,\ldots,y^t \in \{\pm 1\}^{t^d}$ and define

$$h_{d+1}(y^1,\ldots,y^t) = \begin{cases} h_d(g(y^1),\ldots,g(y^t)) & 1 < d \in \mathbb{N}_{even} \\ h_d(f(y^1),\ldots,f(y^t)) & 2 < d \in \mathbb{N}_{odd} \end{cases}$$

Observe that $h_d \in \mathsf{AC}(d+1)$.

Claim B.4. *There is $c \in \mathbb{N}$, such that for every $\frac{c^2}{t} \leqslant \varepsilon \leqslant \frac{1}{20^d \log^d t\sqrt{t}}$, denoting $p = \frac{1-\varepsilon}{2}$, it holds that*

$$\mu_p(h_d) \geqslant \frac{1-(20h)^d\varepsilon}{2}.$$

Proof. By induction on d. The base case $d = 1$ is the content of Claim B.2. Assume that the claim holds for $d' < d$, and that $d \in \mathbb{N}_{even}$. Then,

$$\mu_p(h_d) = \mu_{\mu_p(f)}(h_{d-1}) \geqslant \mu_{\frac{1-20h\varepsilon}{2}}(h_{d-1}),$$

where the last inequality holds as h_{d-1} is monotone, and $\mu_p(f) \geqslant \frac{1-20h\varepsilon}{2}$ by Claim B.2.

As $\varepsilon \leqslant \frac{1}{20^d \log^d t\sqrt{t}}$, using the induction hypothesis, it holds that

$$\mu_p(h_d) \geqslant \mu_{\frac{1-20h\varepsilon}{2}}(h_{d-1}) \geqslant \frac{1-(20h)^d\varepsilon}{2}.$$

When $d \in \mathbb{N}_{odd}$ the analysis is analogous. □

Corollary B.5. *For every $\delta > 0$, $d \in \mathbb{N}$ there is $t_d \in \mathbb{N}$ such that for every $t_d \leqslant t \in \mathbb{N}$.*

$$\mathbf{E}[h_d] \geqslant \frac{1}{2} - \delta.$$

Proof. Follows immediately from Claim B.4 and Claim 6.5. □

Claim B.6. *There is a sufficiently large constant $k > 0$ such that the following holds. If $\frac{k^2}{t} \leqslant \lambda \leqslant \frac{k}{\log t}$ then*

$$\mu_{\frac{1-\lambda}{2}}(h_d) \leqslant \mu_{\frac{1-(h/20k)\cdot\lambda}{2}}(h_{d-1}).$$

Proof. Denote $p = \frac{1-\lambda}{2}$. If $2 < d \in \mathbb{N}_{odd}$, then

$$\mu_p(h_d) = \mu_p(h_{d-1}(g_1, \ldots, g_t)) = \mu_q(h_{d-1}(x_1, \ldots, x_t)),$$

where $q = \mathbf{Pr}_{\mu_p}[g = 1]$. Using Claim B.1, we get that $q \leqslant \frac{1-(h/20k)\cdot\lambda}{2}$ and from the monotonicity we get that

$$\mu_{\frac{1-\lambda}{2}}(h_d) \leqslant \mu_{\frac{1-(h/20k)\cdot\lambda}{2}}(h_{d-1}).$$

When $2 < d \in \mathbb{N}_{even}$ the proof is identical. □

Claim B.7. *For every $\varepsilon \geqslant \frac{k}{\log t}$ and for every $d \in \mathbb{N}$, denoting $p = \frac{1-\varepsilon}{2}$, it holds that*

$$\mu_p(h_d) \leqslant e^{-k/10}.$$

Proof. The proof is by induction. The case $d = 1$ implies $h_r = g$ and the claim follows from Claim 6.6. Assuming the claim holds for $d' \leqslant d$, then, if $d+1 \in \mathbb{N}_{odd}$

$$\mu_p(h_{d+1}) = \mu_p(h_d(g_1, \dots, g_t)) = \mu_{\mu_p(g)}[h_d(x_1, \dots, x_t)].$$

By Claim 6.6 $m_p(g) \leqslant e^{-k/10}$, which means that

$$\varepsilon = 1 - 2p \geqslant 1 - 2e^{-k/10} \geqslant \frac{k}{\log t},$$

and thus we can apply the induction hypothesis to conclude that

$$\mu_p(h_{d+1}) \leqslant e^{-k}.$$

The induction step when $r \in \mathbb{N}_{even}$ is identical. $\square$

Claim B.8. *For every $k \in \mathbb{N}$, define $c_d = k(20k)^d$. Then, for every $\lambda \geqslant \frac{c_d}{\log^d t}$ it holds that*

$$\mu_{\frac{1-\lambda}{2}}(h_d) \leqslant e^{-k/10}.$$

Proof. By induction, if $\lambda \geqslant \frac{k}{\log t}$, the claim follows from Claim B.7.

For $0 \leqslant j < d$, denote $\lambda_j = (\frac{h}{20k})^j \lambda$. While $\lambda_j \leqslant \frac{k}{\log t}$, we can use Claim B.6, to obtain that

$$\mu_{\frac{1-\lambda_j}{2}}(h_{d-j}) \leqslant \mu_{\frac{1-\lambda_{j+1}}{2}}(h_{d-j-1}).$$

The assumption that $\lambda > \frac{c_d}{\log^d t}$ implies that

$$\lambda_{d-1} = (h/20k)^{d-1}\lambda \geqslant \frac{\log^{d-1} t}{(20k)^{d-1}} \frac{c_d}{\log^d t} \geqslant k/\log t.$$

Thus, we may consider the minimal j such that $\lambda_j \geqslant k/\log t$ and get that

$$\mu_{\frac{1-\lambda}{2}}(h_d) \leqslant \mu_{\frac{1-\lambda_j}{2}}(h_{d-j}) \leqslant e^{-k/10},$$

where for the last inequality we used Claim B.7. $\square$

Proof of Theorem B.3. Let $c'_d = (2d)^{d+1} c_d$. As before consider,

$$\tilde{h}_d(x_1, y_1, \dots, x_t, y_t) = h_d(x_1 \cdot y_1, \dots, x_t \cdot y_t).$$

Note that $\tilde{h}_d \colon \{\pm 1\}^{t'_d} \to \{0, 1\}$ where $t'_d = 2t^d$. Thus $\log t'_d / 2d \leqslant \log t$, and $\frac{c_d}{\log^d t} \leqslant \frac{c'_d}{\log^d t'_d}$ It holds that $\tilde{h}_d \in \mathsf{AC}(d+2)$. Using Claim 6.4 we obtain that

$$\mathbf{E}[\tilde{h}_d(\mathrm{RW}_{G,\mathsf{val}_2})] = T_\lambda(h_d)(\mathbb{1}) = \mu_{\frac{1-\lambda}{2}}(h_d).$$

From Claim B.8 we obtain that if $\lambda \geqslant \frac{c'_d}{\log^d t'_d} \geqslant \frac{c_d}{\log^d t}$ then

$$\mathbf{E}[\tilde{h}_r(\mathrm{RW}_{G,\mathsf{val}_2})] = \mu_{\frac{1-\lambda}{2}}(h) \leqslant e^{-k/10}.$$

Corollary B.5 implies that

$$\mathbf{E}[\tilde{h}_d(\mathrm{Ind}_{2t})] = \mathbf{E}[h_d(\mathrm{Ind}_t)] = \mu_{\frac{1}{2}}(h_d) \geqslant \frac{1}{2} - o(1).$$

This implies that $\tilde{h}_d$ distinguishes between the distribution obtained by a random walk on G with $\lambda \geqslant \frac{c'_d}{\log^d t'_d}$ and t'_d independent bits, as stated. $\square$

C A better upper bound for symmetric functions

In this section we prove Theorem 1.3. For convenience we restate it here.

Theorem C.1. *For every symmetric function $f : \{\pm 1\} \to \mathbb{R}$, $\mu \in (-1,1)$ and $0 < \lambda < \frac{1-|\mu|}{128e}$ it holds that*

$$\mathcal{E}_{\lambda,\mu}(f) \leqslant \frac{124}{\sqrt{1-|\mu|}} \cdot \lambda.$$

Given a symmetric function $f : \{\pm 1\}^t \to \mathbb{R}$ and $k \in [t]$. For analyzing the random walk with respect to symmetric test functions, we define for every integer $k \in \{0, 1, \ldots, t\}$,

$$\beta_k^{\mu} = \sum_{\substack{S \subseteq [t] \\ |S|=k}} \mathbf{E}[\chi_S^{\mu}(\mathrm{RW}_{G,\mathsf{val}})]. \tag{C.1}$$

Note that β_k is independent of the choice of test function. However, for symmetric tests functions, these quantities will appear in the analysis, and so we begin by analysing them. By Proposition 3.4,

$$\beta_k^{\mu} \leqslant \sum_{\substack{S \subseteq [t] \\ |S|=k}} \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} \cdot \sum_{I \in \mathcal{F}_k} \lambda^{\sum_{j \in I} \Delta_j(S)} = \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} \cdot \sum_{\substack{S \subseteq [t] \\ |S|=k}} \sum_{I \in \mathcal{F}_k} \lambda^{\sum_{j \in I} \Delta_j(S)}.$$

Denote

$$\beta_k = \sum_{\substack{S \subseteq [t] \\ |S|=k}} \sum_{I \in \mathcal{F}_k} \lambda^{\sum_{j \in I} \Delta_j(S)}.$$

We have that $\beta_k^{\mu} \leqslant \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} \beta_k$. A straightforward corollary of Lemma 3.1 is the following

Corollary C.2. *Let $G = (V, E)$ be a regular λ-spectral expander, and $\mathsf{val} : V \to \{\pm 1\}$ a labelling of the vertices of G with $\mathbf{E}[\mathsf{val}(V)] = \mu$. Then, for every symmetric function $f : \{\pm 1\}^t \to \mathbb{R}$,*

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=2}^{t} \widehat{f}_{\mu}(k) \beta_k^{\mu} \leqslant \sum_{k=2}^{t} \widehat{f}_{\mu}(k) \left(\frac{1+|\mu|}{1-|\mu|}\right)^{\frac{k-1}{2}} \beta_k.$$

Next we recall a simple and standard fact from Fourier analysis.

Claim C.3. *Let $f : \{\pm 1\}^t \to \{\pm 1\}$ be a symmetric function, then for every $\mu \in (0,1)$ and $S \subseteq [t]$ it holds that*

$$\left|\widehat{f}_{\mu}(S)\right| \leqslant \frac{1}{\sqrt{\binom{t}{|S|}}}.$$

Proof. By Parseval's equality,

$$1 = \mathbf{E}[f^2] = \sum_{S \subset [t]} \widehat{f}_{\mu}(S)^2.$$

For a symmetric function, every $S_1, S_2 \subseteq [t]$ with $|S_1| = |S_2|$ satisfy $\widehat{f}_\mu(S_1) = \widehat{f}_\mu(S_2)$ and thus we can write

$$1 = \mathbf{E}[f^2] = \sum_{k \leqslant t} \widehat{f}_\mu(k)^2 \binom{t}{k}$$

which implies that for every $k \leqslant t$, $\left|\widehat{f}_\mu(k)\right| \leqslant \frac{1}{\sqrt{\binom{t}{k}}}$.

□

Lemma 5.2 and Claim C.3 give a simple proof of Theorem C.1:

Proof of Theorem C.1. Let $G = (V, E)$ be a regular λ-spectral expander, and $\mathsf{val} : V \to \{\pm 1\}$ a labelling of V with $\mathbf{E}(\mathsf{val}) = \mu$. Write

$$f(x_1, \ldots, x_t) = \sum_{S \subseteq [t]} \widehat{f}_\mu(|S|) \chi_S^\mu(x_1, \ldots, x_t).$$

Denote $\nu = \sqrt{\frac{1+|\mu|}{1-|\mu|}}$. By Corollary C.2,

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=2}^{t} \widehat{f}_\mu(k) \nu^{k-1} \beta_k.$$

Claim C.3 and Lemma 5.2 then imply that

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=2}^{t} \frac{1}{\sqrt{\binom{t}{k}}} \nu^{k-1} 2^k \binom{t-1}{\lfloor \frac{k}{2} \rfloor} \left(\frac{\lambda}{1-\lambda}\right)^{\lceil \frac{k}{2} \rceil}.$$

A straightforward calculation implies that

$$\frac{1}{\sqrt{\binom{t}{k}}} \binom{t-1}{\lfloor \frac{k}{2} \rfloor} \leqslant (2e)^{k/2},$$

and so

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=2}^{t} \nu^{k-1} (8e)^{k/2} \left(\frac{\lambda}{1-\lambda}\right)^{\lceil \frac{k}{2} \rceil} \leqslant \sum_{k=2}^{t} \nu^{k-1} (16e)^{k/2} \lambda^{\frac{k}{2}} = \nu^{-1} \sum_{k=2}^{t} \alpha^k \leqslant \frac{\nu^{-1}\alpha^2}{1-\alpha},$$

where $\alpha = \sqrt{16e\nu^2\lambda}$. Per our assumption that $\lambda < \frac{1-|\mu|}{128e}$ we have $\alpha \leqslant \frac{1}{2}$ and so

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant 2\nu^{-1}\alpha^2 = 32e\nu\lambda \leqslant \frac{124\lambda}{\sqrt{1-|\mu|}}.$$

□

D MAJ and $\mathbf{1}_w$ vanish with t

In this section we prove better error bounds for the MAJ and weight indicator functions, $\mathbf{1}_w$ that vanish with t.

D.1 Bounds for the MAJ function

For $w \in [t]$ we define $\mathsf{Th}_w : \{\pm 1\}^t \to \{\pm 1, 0\}$ by $\mathsf{Th}_w(x) = 1$ if $|\{i \in [t] \mid x_i = 1\}| \geqslant w$ and $\mathsf{Th}_w(x) = -1$ otherwise. Put differently,

$$\mathsf{Th}_w(x_1, \ldots, x_t) = \mathsf{Sign}(x_1 + \cdots + x_t - 2w + t),$$

with the understanding that $\mathsf{Sign}(0) = 0$. Note that for odd t, the function $\mathsf{Th}_{t/2}$ is the majority function.

Theorem D.1. *For every $\mu \in (-1, 1)$, $0 < \lambda < \frac{1-|\mu|}{192e}$ and every $t \in \mathbb{N}$*

$$\mathcal{E}_{\lambda,\mu}(\mathsf{Th}_{\frac{t+\mu t}{2}}) \leqslant \frac{1}{\sqrt{t}} \cdot \frac{(96e)^2\lambda^2}{1-|\mu|}.$$

From here on, for ease of readability, we denote $\mathsf{Th}_{\frac{t+\mu t}{2}}$ by f. Note that $f = \mathsf{Sign}(\sum_i x_i - \mu t)$.

Claim D.2. *For $|S|$ even, $\widehat{f_\mu}(S) = 0$.*

Proof. Consider the transformation $y_i = 2\mu - x_i$. It holds that

$$\begin{aligned} f(y_1, \ldots, y_t) &= \mathsf{Sign}\left(\sum_{i=1}^{t}(y_i - \mu)\right) \\ &= -\mathsf{Sign}\left(-\sum_{i=1}^{t}(\mu - x_i)\right) \\ &= -\mathsf{Sign}\left(\sum_{i=1}^{t} x_i - \mu t\right) \\ &= -f(x_1, \ldots, x_t). \end{aligned}$$

Expanding f in the respective Fourier basis we get

$$f(x_1, \ldots, x_t) = \sum_{S \subseteq [t]} \widehat{f_\mu}(S)\chi_S^\mu(x_1, \ldots, x_t).$$

Note that $\chi_S^\mu(y_1, \ldots, y_t) = (-1)^{|S|}\chi_S^\mu(x_1, \ldots, x_t)$, and so

$$\begin{aligned} \sum_{S \subseteq [t]} -(\widehat{f_\mu}(S))\chi_S^\mu(x_1, \ldots, x_t) &= -f(x_1, \ldots, x_t) \\ &= f(y_1, \ldots, y_t) \\ &= \sum_{S \subseteq [t]} \widehat{f_\mu}(S)\chi_S^\mu(y_1, \ldots, y_t) \\ &= \sum_{S \subseteq [t]} (-1)^{|S|}\widehat{f_\mu}(S)\chi_S^\mu(x_1, \ldots, x_t). \end{aligned}$$

By comparing both sides we get that $\widehat{f_\mu}(S) = 0$ for all S of even size. $\square$

Proof of Theorem D.1. Let $G = (V, E)$ be a regular λ-spectral expander, and $\mathsf{val} : V \to \{\pm 1\}$ a labelling of V with $\mathbf{E}[\mathsf{val}(V)] = \mu$. Expand

$$f(x_1, \ldots, x_t) = \sum_{S \subseteq [t]} \widehat{f_\mu}(|S|) \chi_S^\mu(x_1, \ldots, x_t).$$

By Corollary C.2,

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=2}^{t} \widehat{f_\mu}(k) \nu^{k-1} \beta_k.$$

where, recall, $\nu = \sqrt{\frac{1+|\mu|}{1-|\mu|}}$. Claim D.2 then implies that

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=1}^{\lfloor \frac{t-1}{2} \rfloor} \widehat{f_\mu}(2k+1) \nu^{2k} \beta_{2k+1}.$$

Applying Claim C.3 and Lemma 5.2

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \sum_{k=1}^{\lfloor \frac{t-1}{2} \rfloor} \frac{1}{\sqrt{\binom{t}{2k+1}}} \nu^{2k} 2^{2k+1} \binom{t-1}{k} \left(\frac{\lambda}{1-\lambda} \right)^{k+1}.$$

A straightforward calculation using Claim 2.1 implies that

$$\frac{1}{\sqrt{\binom{t}{2k+1}}} \binom{t-1}{k} \leqslant \frac{(6e)^k}{\sqrt{t}}.$$

Thus,

$$\begin{aligned}
\mathcal{E}_{G,\mathsf{val}}(f) &\leqslant \frac{1}{\sqrt{t}} \sum_{k=1}^{\lfloor \frac{t-1}{2} \rfloor} \nu^{2k} (6e)^k 2^{2k+1} \left(\frac{\lambda}{1-\lambda} \right)^{k+1} \leqslant \frac{\nu^{-2}}{\sqrt{t}} \sum_{k=1}^{\lfloor \frac{t-1}{2} \rfloor} \left(48 e \nu^2 \lambda \right)^{k+1} \\
&\leqslant \frac{\nu^{-2}}{\sqrt{t}} \sum_{k=2}^{\infty} \alpha^k \leqslant \frac{\alpha^2 \nu^{-2}}{\sqrt{t}(1-\alpha)},
\end{aligned}$$

where $\alpha = 48 e \nu^2 \lambda$. Per our assumption that $\lambda < \frac{1-|\mu|}{192e}$ we have $\alpha \leqslant \frac{1}{2}$ and so

$$\mathcal{E}_{G,\mathsf{val}}(f) \leqslant \frac{2\alpha^2 \nu^{-2}}{\sqrt{t}} \leqslant \frac{1}{\sqrt{t}} \cdot \frac{(96e)^2 \lambda^2}{1-|\mu|}.$$

□

D.2 Bounds for weight indicators

Recall, that for integers t and $w \in \{0, 1, \ldots, t\}$ we defined $\mathbf{1}_w : \{\pm 1\}^t \to \{0, 1\}$ be the function indicating whether the weight of the input is w. That is, $\mathbf{1}_w(x_1, \ldots, x_t) = 1$ if $|\{i \in [t] \mid x_i = 1\}| = w$ and $\mathbf{1}_w(x_1, \ldots, x_t) = 0$ otherwise. We also define $\mathbf{1}_{>w} : \{\pm 1\}^{t+1} \to \{0, 1\}$ be the function indicating whether the weight of the input is greater w. That is, $\mathbf{1}_w(x_1, \ldots, x_t) = 1$ if $\sum_i x_i > w$ and $\mathbf{1}_w(x_1, \ldots, x_t) = 0$ otherwise.

In this section we prove

Theorem D.3. *For every $\mu \in (-1,1)$, $0 < \lambda \leqslant \frac{1-|\mu|}{768e}$, every $t \in \mathbb{N}$ and $0 \leqslant w \leqslant t$, it holds that*

$$\mathcal{E}_\lambda(\mathbf{1}_w) \leqslant \frac{1}{\sqrt{t}} \cdot \frac{192e\lambda}{1-|\mu|}.$$

We analyze the weight indicator function in a similar way to the majority function, except that we need a new argument in order to bound its Fourier coefficients. The analysis is more delicate as the weight indicator function is not anti-symmetric and therefore has Fourier mass on even layers. Note that,

$$\mathbf{1}_w(x_1,\dots,x_t) = \mathbf{1}_{>w}(1,x_1,\dots,x_t) - \mathbf{1}_{>w}(0,x_1,\dots,x_t). \tag{D.1}$$

Claim D.4. *For every $k \leqslant t$ it holds that*

$$\widehat{(\mathbf{1}_w)}_\mu(k)^2 \leqslant \frac{k+1}{(1-\mu^2)\binom{t}{k}(t+1)}.$$

Proof. As $\mathbf{1}_{>w}$ is symmetric with range $\{0,1\}$,

$$\sum_{k=0}^{t+1} \binom{t+1}{k} \widehat{(\mathbf{1}_{>w})}_\mu^2(k) \leqslant 1.$$

In particular, for every $k \leqslant t+1$, $\widehat{(\mathbf{1}_{>w})}_\mu^2(k) \leqslant \binom{t+1}{k}^{-1}$. By Claim A.1, for every $k \leqslant t$,

$$\widehat{(\mathbf{1}_w)}_\mu^2(k) = \frac{\widehat{(\mathbf{1}_{>w})}_\mu^2(k+1)}{1-\mu^2}.$$

Hence,

$$\widehat{(\mathbf{1}_w)}_\mu^2(k) \leqslant \frac{1}{(1-\mu^2)\binom{t+1}{k+1}},$$

which concludes the proof. ☐

Proof of Theorem D.3. Let $G=(V,E)$ be a regular λ-spectral expander, and $\mathsf{val} : V \to \{\pm 1\}$ a labelling of V with $\mathbf{E}[\mathsf{val}(V)] = \mu$. Expand

$$f(x_1,\dots,x_t) = \sum_{S \subseteq [t]} \widehat{f}_\mu(|S|)\chi_S^\mu(x_1,\dots,x_t).$$

By Corollary C.2,

$$\mathcal{E}_{G,\mathsf{val}}(\mathbf{1}_w) \leqslant \sum_{k=2}^{t} \widehat{(\mathbf{1}_w)}_\mu(k)\nu^{k-1}\beta_k,$$

where, recall, $\nu = \sqrt{\frac{1+|\mu|}{1-|\mu|}}$. Using Claim D.4 to upper bound $\widehat{(\mathbf{1}_w)}_\mu$ and Lemma 5.2 to bound β_k we get,

$$\mathcal{E}_{G,\mathsf{val}}(\mathbf{1}_w) \leqslant \sum_{k=2}^{t} \sqrt{\frac{k+1}{(1-\mu^2)(t+1)}} \frac{1}{\sqrt{\binom{t}{k}}} \cdot \nu^{k-1} 2^k \binom{t-1}{\lfloor \frac{k}{2} \rfloor} \left(\frac{\lambda}{1-\lambda}\right)^{\lceil \frac{k}{2} \rceil}.$$

As in the calculation in Theorem D.1, it is not hard to verify that

$$\frac{1}{\sqrt{\binom{t}{k}}} \cdot \binom{t-1}{\lfloor \frac{k}{2} \rfloor} \leqslant (3e)^{\frac{k}{2}}.$$

Thus,

$$\begin{aligned}
\mathcal{E}_{G,\mathsf{val}}(\mathbf{1}_w) &\leqslant \sum_{k=2}^{t} \sqrt{\frac{k+1}{(1-\mu^2)(t+1)}} \cdot \nu^{k-1}(3e)^{\frac{k}{2}} 2^k \left(\frac{\lambda}{1-\lambda}\right)^{\lceil \frac{k}{2} \rceil} \\
&\leqslant \frac{1}{\sqrt{t}} \cdot \frac{\nu^{-1}}{\sqrt{1-\mu^2}} \sum_{k=2}^{t} (\nu^2)^{\frac{k}{2}} (96e)^{\frac{k}{2}} \lambda^{\frac{k}{2}} \;= \frac{1}{\sqrt{t}} \cdot \frac{\nu^{-1}}{\sqrt{1-\mu^2}} \sum_{k=2}^{t} \alpha^k \\
&\leqslant \frac{1}{\sqrt{t}} \cdot \frac{\nu^{-1}}{\sqrt{1-\mu^2}} \cdot \frac{\alpha^2}{1-\alpha},
\end{aligned}$$

where $\alpha = \sqrt{96e\nu^2\lambda}$. Per our assumption that $\lambda < \frac{1-|\mu|}{768e}$ we have $\alpha \leqslant \frac{1}{2}$ and so

$$\mathcal{E}_{G,\mathsf{val}}(\mathbf{1}_w) \leqslant \frac{1}{\sqrt{t}} \cdot \frac{192e\lambda}{1-|\mu|}.$$

$\square$