

Many-Valued Modal Logic

Amir Karniel and Michael Kaminski

About me

- › Technion - Israel Institute of Technology
- › B.Sc. in Mathematics and Computer Science
- › (Almost) M.Sc. in Mathematics
- › Supervised by Prof. Michael Kaminski from the Department of Computer Science
- › This paper is a part of my master thesis

Table of Contents

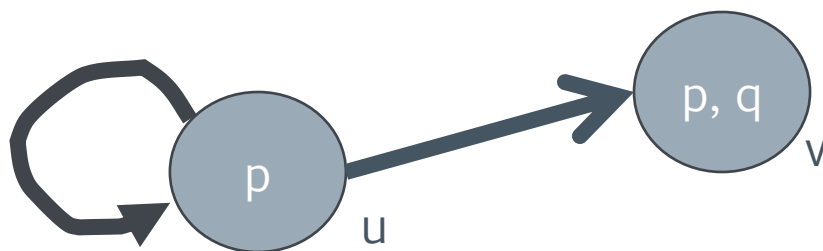
- › Background: Normal Modal Logics
- › Background: Many-Valued Logics and Sequents of Labeled Formulas
- › Semantics
- › Proof System and Sketch of Completeness Proof
- › Notes

Background: Normal Modal Logics

- › The set of formulas $\mathcal{F}$ is built recursively from:
 - Atomic propositions $\mathcal{P}$: $p_1, p_2, \dots$ or $p, q, \dots$
 - Connectives: $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\neg \varphi_1$ or, in general, $*$ ($\varphi_1, \dots, \varphi_\ell$)
 - Modal operators: $\Box$ (“necessarily”) and $\Diamond$ (“possibly”)

Background: Normal Modal Logics (2)

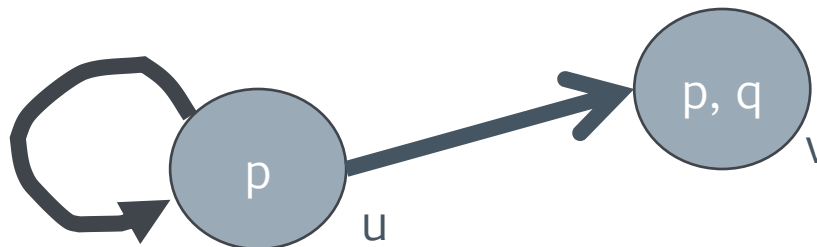
- › Semantically characterized by **Kripke models**
 $M = (W, R, I)$ where:
 - W is a set (of “worlds”)
 - R is a binary relation on W (“accessibility relation”)
 - I assigns a truth value (t or f) to any pair $(w, p) \in W \times \mathcal{P}$ (“valuation”)



Background: Normal Modal Logics (3)

- › The definition of I is extended to any $(w, \varphi) \in W \times \mathcal{F}$:
 - $I(w, *(\varphi_1, \dots, \varphi_\ell)) = * (I(w, \varphi_1), \dots, I(w, \varphi_\ell))$
 e.g., $p \wedge q$ is t in w iff both p and q are t in w
 - $I(w, \Box \varphi) = t$ iff $I(v, \varphi) = t$ for every v s.t. wRv
 i.e., $\Box \varphi$ is t in w iff φ is t in every world seen from w
 - $I(w, \Diamond \varphi) = t$ iff $I(v, \varphi) = t$ for some v s.t. wRv
 i.e., $\Diamond \varphi$ is t in w iff φ is t in some world seen from w

We say $M, w \models \varphi$ if $I(w, \varphi) = t$, and $M \models \varphi$ if for all $w \in W$, $M, w \models \varphi$.



$M, u \models \Box p, \Diamond p \wedge \Diamond q, \Diamond(p \wedge q), \dots$
 $M, v \models p \wedge q, \Box p \wedge \Box \neg p, \dots$
 $M \models p, \Box p, \dots$

Background: Normal Modal Logics (4)

- › Duality: $\Diamond\varphi \equiv \neg\Box\neg\varphi$
- › Proof system (assuming standard connectives):
 - Same as for propositional logic, with one more axiom scheme **K** and one more inference rule **NEC**
 - **K**: $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$
 - **NEC**: $\frac{\varphi}{\Box\varphi}$
 - Sound and (strongly) complete w.r.t Kripke models
 - There is also a sequent-calculus proof system. We return to sequents later
- › Extensions:
 - **K**: no further axioms
 - **T**: $\Box\varphi \supset \varphi$. Sound & complete w.r.t reflexive models
 - **S4**: **T** + $\Box\varphi \supset \Box\Box\varphi$. Sound & complete w.r.t preordered models
 - **S5**: **T** + $\Diamond\varphi \supset \Box\Diamond\varphi$. Sound & complete w.r.t models whose R is an equivalence relation

Background: Many-Valued Logics and Sequents of Labeled Formulas

- › A formula can now be assigned with one of finitely-many truth values $\tau = \{v_1, v_2, \dots, v_n\}$
- › Correspondingly, an ℓ -ary connective is now a function from τ^ℓ to τ
 - e.g., in Kleene's 3-valued logic, with values $\{f, n, t\}$ ordered by $f < n < t$, $\wedge$ is defined as the minimum and $\vee$ as the maximum

$\bullet$ t	$\neg$		$\wedge$	f	n	t	$\vee$	f	n	t	$\supset$	f	n	t		
	f	t		f	f	f		f	f	n	t		f	t	t	
$\bullet$ n	n	n		n	f	n		n	n	n	t		n	n	t	
$\bullet$ f	t	f		t	f	n		t	t	t	t		t	f	n	t

Background: Many-Valued Logics and Sequents of Labeled Formulas (2)

- e.g., in Łukasiewicz's 3-valued logic:

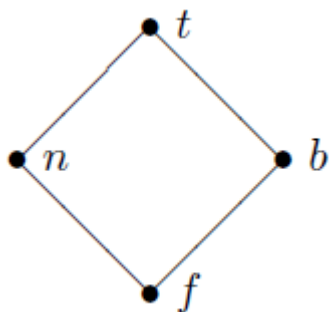
$\neg$		$\wedge$	f	n	t	$\vee$	f	n	t	$\supset$	f	n	t
f	t	f	f	f	f	f	f	n	t	f	t	t	t
n	n	n	f	n	n	n	n	n	t	n	n	t	t
t	f	t	f	n	t	t	t	t	t	t	f	n	t

- e.g., in Post's n -valued logic:

$\neg$		$\wedge$	$\dots$	v_j	$\dots$	$\vee$	$\dots$	v_j	$\dots$
v_0	v_1	v_0	$\dots$	v_0	$\dots$	v_0	$\dots$	v_j	$\dots$
v_1	v_2	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	v_i	$\dots$	$v_{\min\{i,j\}}$	$\dots$	v_i	$\dots$	$v_{\max\{i,j\}}$	$\dots$
v_{n-2}	v_{n-1}	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-1}	v_0	v_{n-1}	$\dots$	v_j	$\dots$	v_{n-1}	$\dots$	v_{n-1}	$\dots$

Background: Many-Valued Logics and Sequents of Labeled Formulas (3)

- e.g., in the Belnap-Dunn 4-valued logic, with values $\{f, n, b, t\}$ ordered as the diagram to the left, $\wedge$ is defined as the infimum and $\vee$ as the supremum



$\neg$	
f	t
n	n
b	b
t	f

$\wedge$	f	n	b	t
f	f	f	f	f
n	f	n	f	n
b	f	f	b	b
t	f	n	b	t

$\vee$	f	n	b	t
f	f	n	b	t
n	n	n	t	t
b	b	t	b	t
t	t	t	t	t

Background: Many-Valued Logics and Sequents of Labeled Formulas (4)

- › A **labeled formula** is a pair $(\varphi, k) \in \mathcal{F} \times \{1, \dots, n\}$
- › We no longer say that a valuation $v: \mathcal{P} \rightarrow \tau$ satisfies a formula φ , but a labeled formula (φ, k) , meaning that it assigns φ with the value v_k
- › Moreover, we want meta-expressions such as “ v satisfies either (φ, k_1) or (ψ, k_2) ”, “if v satisfies (φ, k_1) then it satisfies either (ψ, k_2) or (ψ, k_3) ”

Background: Many-Valued Logics and Sequents of Labeled Formulas (5)

- › A **sequent** is an expression of the form $\Gamma \rightarrow \Delta$ where Γ, Δ are finite sets of labeled formulas
 - We say v satisfies $\Gamma \rightarrow \Delta$ if the following holds: if v satisfies every element of Γ then it satisfies an element of Δ
 - Intuitively:

$$\bigwedge \Gamma \supset \bigvee \Delta$$

- e.g., in Kleene's 3-valued logic, the following are satisfied by every valuation:
 - › $(\varphi, f) \rightarrow (\varphi \wedge \psi, f)$
 - › $(\varphi, n) \rightarrow (\varphi \vee \psi, n), (\varphi \vee \psi, t)$
 - › $(\varphi, t), (\varphi \vee \psi, n) \rightarrow$

Background: Many-Valued Logics and Sequents of Labeled Formulas (6)

- › Our proof systems deal with sequents, instead of just formulas
- › The following system (continued next slide) is a sound and complete proof system for many-valued logics with an arbitrary set of connectives

Basic Axiom

$$(\varphi, k) \rightarrow (\varphi, k)$$

Table axiom

$$(\varphi_1, k_1), \dots, (\varphi_\ell, k_\ell) \rightarrow (* (\varphi_1, \dots, \varphi_\ell), k) \\ \text{for } * (v_{k_1}, \dots, v_{k_\ell}) = v_k$$

π

Background: Many-Valued Logics and Sequents of Labeled Formulas (7)

Left-shift $\frac{\Gamma, (\varphi, k) \rightarrow \Delta}{\Gamma \rightarrow \Delta, \{\varphi\} \times \{k\}}$

Right-shift $\frac{\Gamma \rightarrow \Delta, (\varphi, k')}{\Gamma, (\varphi, k'') \rightarrow \Delta} \quad k' \neq k''$

Left-weakening $\frac{\Gamma \rightarrow \Delta}{\Gamma, (\varphi, k) \rightarrow \Delta}$

Right-weakening $\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, (\varphi, k)}$

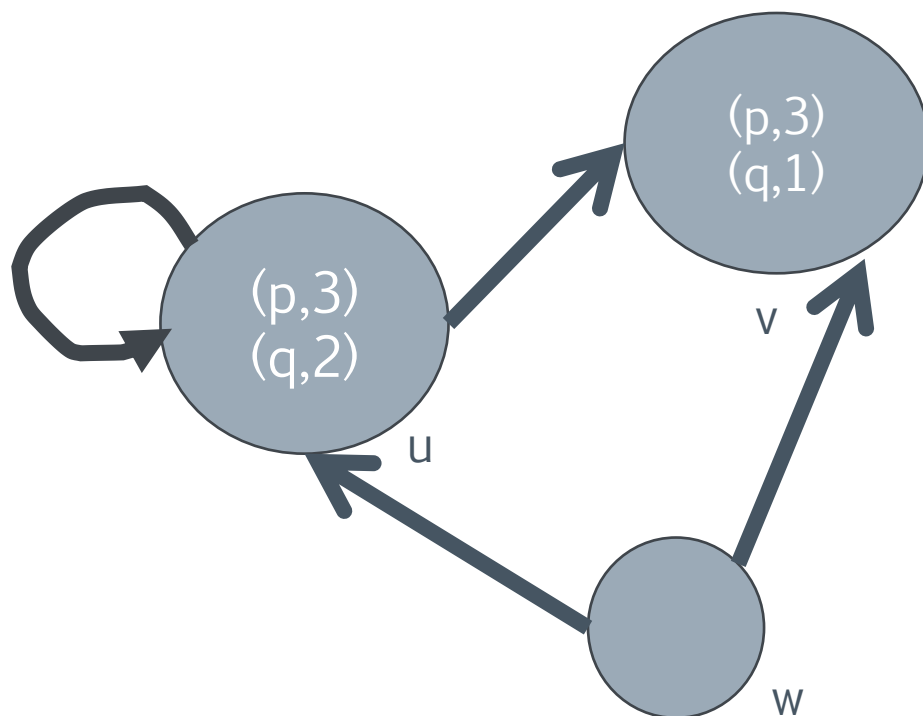
Cut $\frac{\Gamma' \rightarrow \Delta', (\varphi, k) \quad \Gamma'', (\varphi, k) \rightarrow \Delta''}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''}$

Resolution $\frac{\Gamma' \rightarrow \Delta', (\varphi, k') \quad \Gamma'' \rightarrow \Delta'', (\varphi, k'')}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''} \quad k' \neq k''$

mv-K: Semantics

- › **mv-K** is the many-valued counterpart of **K**.
 - Assume $\tau = \{v_1, \dots, v_n\}$ is linearly ordered
- › Again, characterized by Kripke models
 $M = (W, R, I)$ with $I: W \times \mathcal{P} \rightarrow \tau$
- › Again, $I(w, * (\varphi_1, \dots, \varphi_\ell)) = * (I(w, \varphi_1), \dots, I(w, \varphi_\ell))$
- › $I(w, \Box \varphi)$ is now the *infimum* of all $I(v, \varphi)$ s.t. wRv
 - $\inf(\emptyset) = v_n$
- › $I(w, \Diamond \varphi)$ is now the *supremum* of all $I(v, \varphi)$ s.t. wRv
 - $\sup(\emptyset) = v_1$
- › $M, w \models (\varphi, k)$ if $I(w, \varphi) = v_k$
 - $M, w \models \Gamma \rightarrow \Delta$ is defined as before
 - $M \models \Gamma \rightarrow \Delta$ if $M, w \models \Gamma \rightarrow \Delta$ for all $w \in W$

mv-K: Semantics (2)

 $M, u \models (\Box p, 3), (\Diamond p, 3), (\Diamond q, 2), (\Box(p \wedge q), 1), \dots$ $M, v \models (\Box \varphi, 3), (\Diamond \varphi, 1), \dots$ $M, w \models (\Box q, 1), (\Diamond q, 2), (\Box(p \wedge q), 1), \dots$ $(n=3, \text{Kleene's connectives})$

mv-K: Proof System and Sketch of Completeness Proof

- › Define $\Gamma^\times = \bigcup \{ \{ \psi \} \times \overline{[i_\psi, j_\psi]} \mid (\Box \psi, i_\psi), (\Diamond \psi, j_\psi) \in \Gamma \}$
 - That is, for all ψ that satisfies $(\Box \psi, i_\psi), (\Diamond \psi, j_\psi) \in \Gamma$, $\Gamma^\times$ contains the set $\{ \psi \} \times \overline{[i_\psi, j_\psi]}$; and $\Gamma^\times$ contains only these sets
 - Idea: if uRv and u satisfies all Γ , v satisfies none of $\Gamma^\times$
- › Add to the proof system we saw above two (dual) inference rules:

$$\Box\text{-introduction} \quad \frac{(\varphi, k) \rightarrow \Gamma^\times}{(\Box \varphi, k), \Gamma \rightarrow} \quad k \neq n$$

$$\Diamond\text{-introduction} \quad \frac{(\varphi, k) \rightarrow \Gamma^\times}{(\Diamond \varphi, k), \Gamma \rightarrow} \quad k \neq 1$$

mv-K: Proof System and Sketch of Completeness Proof (2)

- › Γ is **Σ -consistent** if $\Sigma \not\vdash \Gamma \rightarrow$
 - Definition of $\Sigma \vdash \Gamma \rightarrow \Delta$ extended to infinite Γ 's
- › Two basic lemmas
 - Every Σ -consistent set is contained in a maximal one
 - If Γ is maximal consistent, for all φ there is a unique k s.t. $(\varphi, k) \in \Gamma$

mv-K: Proof System and Sketch of Completeness Proof (3)

- › Enumerate $\mathcal{F}$ as $\psi_1, \psi_2, \dots$
- › The **Σ -canonical model** M is defined as follows
 - W is the set of all maximal Σ -consistent sets
 - Assume $(\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \dots \in \Gamma_1$ and $(\psi_1, k_1), (\psi_2, k_2), \dots \in \Gamma_2$. Let $\Gamma_1 R \Gamma_2$ iff $k_m \in [i_m, j_m]$ for all m
 - › Equivalently, $\Gamma_1 R \Gamma_2$ iff $\Gamma_1^\times \cap \Gamma_2 = \emptyset$
 - $I(\Gamma, p)$ is the only v_k s.t. $(p, k) \in \Gamma$

mv-K: Proof System and Sketch of Completeness Proof (4)

- › Main theorem: $(\varphi, k) \in \Gamma$ iff $M, \Gamma \models (\varphi, k)$ (for all $\Gamma \in W$)
 - Assume $(\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \dots \in \Gamma$
 - Main step: if $k \neq n$ and $(\Box\varphi', k) \in \Gamma$, we need to show there is a maximal consistent set containing $(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots$ for some choice of $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2] \dots$ (so that Γ “sees” it in M)
 - Indeed, if $\Sigma \vdash (\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots \rightarrow$ for every such choice, then we can show by *König’s lemma* that there exists $L \in \mathbb{N}$ s.t. $\Sigma \vdash (\varphi', k), (\psi_1, k_1), \dots, (\psi_L, k_L) \rightarrow$ for all $k_1 \in [i_1, j_1], \dots, k_L \in [i_L, j_L]$. Applying some structural rules,

$$\Sigma \vdash (\varphi', k) \rightarrow \{\psi_1\} \times \overline{[i_1, j_1]}, \dots, \{\psi_L\} \times \overline{[i_L, j_L]}$$
 thus $\Sigma \vdash (\Box\varphi', k), (\Box\psi_1, i_1), (\Diamond\psi_1, j_1), \dots, (\Box\psi_L, i_L), (\Diamond\psi_L, j_L) \rightarrow$
 contradicting consistency

Notes

π

Logic Extensions

Adding the axiom(s)...	to...	results in a sound & complete system with respect to...	and we call the new logic...
$(\Box\varphi, n) \rightarrow (\Diamond\varphi, n)$	$mv - K$	serial models	$mv - D$
$(\Box\varphi, k) \rightarrow (\varphi, k)^+$ $(\varphi, k) \rightarrow (\Diamond\varphi, k)^+$	$mv - K$	reflexive models	$mv - T$
$(\Box\varphi, k) \rightarrow (\Box\Box\varphi, k)^+$ $(\Diamond\Diamond\varphi, k) \rightarrow (\Diamond\varphi, k)^+$	$mv - K$	transitive models	$mv - K4$
$(\Box\varphi, k) \rightarrow (\Box\Box\varphi, k)^+$ $(\Diamond\Diamond\varphi, k) \rightarrow (\Diamond\varphi, k)^+$	$mv - T$	preordered models	$mv - S4$
$(\varphi, k) \rightarrow (\Box\Diamond\varphi, k)^+$ $(\Diamond\Box\varphi, k) \rightarrow (\varphi, k)^+$	$mv - K$	symmetric models	$mv - B$
$(\Diamond\varphi, k) \rightarrow (\Box\Diamond\varphi, k)^+$ $(\Diamond\Box\varphi, k) \rightarrow (\Box\varphi, k)^+$	$mv - K$	Euclidean models	
$(\Diamond\varphi, k) \rightarrow (\Box\Diamond\varphi, k)^+$ $(\Diamond\Box\varphi, k) \rightarrow (\Box\varphi, k)^+$	$mv - T$	models whose R is an equivalence relation	$mv - S5$

Connection to Intuitionism

- › Just like intuitionistic logic can be embedded in $S4$, we can embed *many-valued intuitionistic logic*, $mvIL$, in $mv - S4$
- › The language of $mvIL$ is the same as classical many-valued logic but semantically characterized by preordered Kripke models with a monotonic valuation: $I(u, p) \leq I(v, p)$ whenever uRv
 - $I(u, * (\varphi_1, \dots, \varphi_\ell))$ is the infimum of all $* (I(v, \varphi_1), \dots, I(v, \varphi_\ell))$ s.t. uRv
 - So I is, actually, monotonic for every φ

Connection to Intuitionism (2)

- › Translation from $mvIL$ to $mv - S4$: φ^t is obtained by inserting $\Box$ before every subformula of φ
- › Theorem: $\Sigma \models_{mvIL} \Gamma \rightarrow \Delta$ iff $\Sigma^t \models_{mv-S4} \Gamma^t \rightarrow \Delta^t$
- › Main idea: given an $mv-S4$ -model $M = (W, R, I)$, define an $mvIL$ -model $M' = (W, R, I')$ by letting $I'(u, p)$ be $I(u, \Box p)$
 - By induction: $M', w \models_{mvIL} (\varphi, k) \Leftrightarrow M, w \models_{mv-S4} (\varphi^t, k)$

Defining Negation

- › Usually, the duality of $\Box$ and $\Diamond$ is presented as $\Diamond \equiv \neg\Box\neg$
 - but we do not assume anything on the set of connectives
- › If we *do* want to define negation s.t. $(\Diamond\varphi, k) \leftrightarrow (\neg\Box\neg\varphi, k)$ and $(\Box\varphi, k) \leftrightarrow (\neg\Diamond\neg\varphi, k)$ are valid, there is exactly one way to do so in $mv - K$
 - Define $\neg(v_k)$ as v_{n-k+1} (“reflection”)

Finite Model Property and Strong Decidability

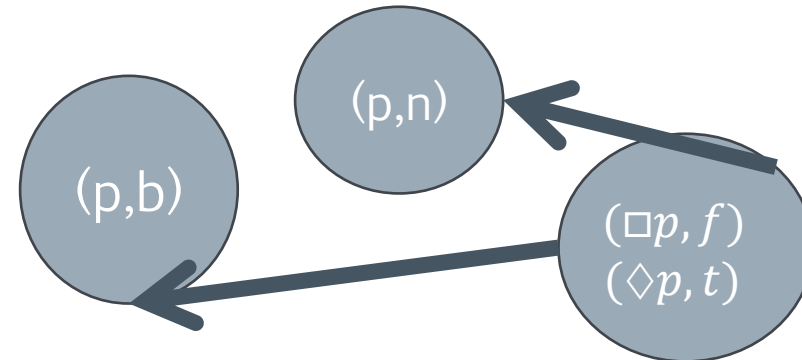
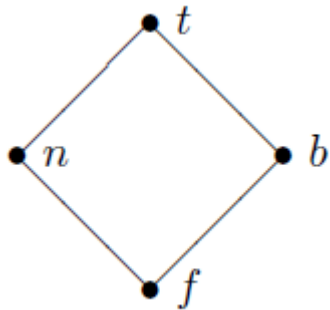
- › A modal logic has the **finite model property** if $\Sigma \not\models \Gamma \rightarrow \Delta$ (for a finite Σ) implies there is a *finite* model satisfying Σ but not $\Gamma \rightarrow \Delta$
 - $mv - K$ and all of its extensions have it
- › Main idea: filtrations
 - Given $M = (W, R, I)$, define its filtration
$$M^* = (W^*, R^*, I^*)$$
through a subformula-closed set of formulas Φ
 - › Equivalence relation on W : $u \approx v$ iff $I(u, \varphi) = I(v, \varphi)$ for all $\varphi \in \Phi$
 - › $W^* = W / \approx$
 - › $I^*([w], p) = I(w, p)$ for all $p \in \Phi$
 - › Definition of R^* depends on the specific logic, so that M^* is indeed an appropriate model and the following holds:
$$I^*([w], \varphi) = I(w, \varphi) \text{ for all } \varphi \in \Phi$$

Finite Model Property and Strong Decidability (2)

- e.g., for mv- K , $[u]R^*[v]$ iff there exist $u' \approx u, v' \approx v$ s.t. $u'Rv'$
- e.g., for mv- $S4$, $[u]R^*[v]$ iff $I(u, \Box\varphi) \leq I(v, \Box\varphi)$ for all $\Box\varphi \in \Phi$ and $I(u, \Diamond\varphi) \geq I(v, \Diamond\varphi)$ for all $\Diamond\varphi \in \Phi$
- › Now given $\Sigma \not\models \Gamma \rightarrow \Delta$, the filtration of a counterexample through the subformula-closure of the set of all formulas from Σ, Γ, Δ is a *finite* counterexample
- › Together with strong completeness, this implies *strong decidability*
 - a decision procedure for the problem “Does Σ satisfy $\Gamma \rightarrow \Delta$?” (for finite Σ)

Generalization: Modal Belnap's Logic

- › We can generalize all of our results (soundness, completeness, extensions, def of negation, finite model property and even embedding of the appropriate instance of *mvIL*) to a specific case of a partially-ordered set of truth values (Belnap-Dunn lattice)
- › The main difference is that now the value of $\Box\varphi$ (and $\Diamond\varphi$) does not have to be the value of φ in any successor



Generalization: Modal Belnap's Logic (2)

› The rules for the Belnap-counterpart of $mv\text{-}K$ are:

$$\frac{(\varphi, n) \rightarrow \Gamma^\times}{(\Box\varphi, n), \Gamma \rightarrow} \quad \frac{(\varphi, b) \rightarrow \Gamma^\times}{(\Box\varphi, b), \Gamma \rightarrow} \quad \frac{(\varphi, n) \rightarrow \Gamma^\times \quad (\varphi, f) \rightarrow \Gamma^\times}{(\Box\varphi, f), \Gamma \rightarrow}$$

$$\frac{(\varphi, b) \rightarrow \Gamma^\times \quad (\varphi, f) \rightarrow \Gamma^\times}{(\Box\varphi, f), \Gamma \rightarrow}$$

and dual rules for $\Diamond$

π

Thank you!