

Many-valued modal logics

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The author of this thesis states that the research, including the collection, processing and presentation of data, addressing and comparing to previous research, etc., was done entirely in an honest way, as expected from scientific research that is conducted according to the ethical standards of the academic world. Also, reporting the research and its results in this thesis was done in an honest and complete manner, according to the same standards.

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Abstract

We combine the concepts of modal logics and many-valued logics in a general and comprehensive way. Namely, given any finite linearly ordered set of truth values and any set of propositional connectives defined by truth tables, we define two kinds of many-valued modal logics.

One is the many-valued pure logic of necessitation, which is a counterpart of the two-valued subnormal modal logic of pure necessitation N. The other, that is an extension of the first one, is the many-valued minimal normal modal logic, which is a counterpart of the two-valued minimal normal modal logic K.

Both are presented as a Gentzen-like sequent calculus. We prove their soundness and strong completeness with respect to many-valued Kripke structures (for the first logic) or models (for the second logic).

Both logics treat necessitation and possibility independently, i.e., they are not defined by each other, so that the duality between them is reflected in the proof systems themselves.

We also prove the finite model property (that implies strong decidability) of these logics and consider some of their extensions, e.g., counterparts of the two-valued logics T, S4, and S5.

In addition, we show that there is exactly one way to define negation such that De Morgan's duality between necessitation and possibility holds in the normal logic.

We also prove that many-valued intuitionistic logic is a fragment of the S4 extension of our normal logic.

Finally, we consider the case where the set of truth values is a certain four-valued (non-linear) lattice, and generalize all of the above results to this case as well.

Notation and Abbreviations

B_4-B	the many-valued counterpart of B based on the Dunn-Belnap lattice of truth values
B_4-D	the many-valued counterpart of D based on the Dunn-Belnap lattice of truth values
B_4-IL	the many-valued counterpart of intuitionistic logic based on the Dunn-Belnap lattice of truth values
B_4-K	the many-valued counterpart of K based on the Dunn-Belnap lattice of truth values
B_4-K4	the many-valued counterpart of K4 based on the Dunn-Belnap lattice of truth values
B_4-N	the many-valued counterpart of N based on the Dunn-Belnap lattice of truth values
B_4-S5	the many-valued counterpart of S5 based on the Dunn-Belnap lattice of truth values
B_4-S4	the many-valued counterpart of S4 based on the Dunn-Belnap lattice of truth values
B_4-T	the many-valued counterpart of T based on the Dunn-Belnap lattice of truth values
$mv-B$	the many-valued counterpart of B
$mv-D$	the many-valued counterpart of D
$mv-IL$	the many-valued counterpart of intuitionistic logic
$mv-K$	the many-valued counterpart of K
$mv-K4$	the many-valued counterpart of K4
$mv-N$	the many-valued counterpart of N
$mv-S5$	the many-valued counterpart of S5
$mv-S4$	the many-valued counterpart of S4
$mv-T$	the many-valued counterpart of T

Chapter 1

Introduction

Modal logic is a kind of logic used to represent statements about necessity and possibility. It is used in computer science to reason about concepts such as uncertainty and time-dependent properties, for example in formal verification and in knowledge representation.

Many-valued logic is a logic in which there are more than two truth values. It is used, for example, to represent partial or inconsistent information in databases and in information retrieval systems.

The (two-valued) logic N of pure necessitation [FMT92] is a subnormal modal logic that extends classical propositional calculus with the modal connective $\Box$ and the rule of inference

$$\frac{\varphi}{\Box\varphi} \tag{1.1}$$

but contains no axioms for manipulating modalities. Semantically, N is characterized by a generalization of Kripke models (called Kripke structures), in which each formula has its own accessibility relation. The theory of normal modal logics, i.e., extensions of N containing the axiom scheme

$$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi) \tag{1.2}$$

is characterized by Kripke models [Kri59; Kri63] and can be developed from the theory of N .¹ The main motivation for dealing with N comes from the area of nonmonotonic reasoning, because the nonmonotonic logic of N seems to be particularly useful in investigations of knowledge sets based on partial information, see [MT93]. The minimal normal modal logic, i.e., the one that is obtained from N by adding (1.2), is denoted by K .

In our thesis we define many-valued counterparts $\mathbf{mv}\text{-}N$ and $\mathbf{mv}\text{-}K$ of N and K , respectively, in which the necessity connective is interpreted as the infimum of all relevant values and the possibility connective is interpreted as their supremum, and

¹ As we show in this thesis, many-valued normal modal logics can be developed from many-valued N .

nothing is assumed about the underlying propositional connectives.

Syntactically, our proof systems are extensions of that in [KF21] to the modal case. The possibility connective $\Diamond$ is treated explicitly. The reason for such a treatment is that, in $\mathbf{mv-K}$, $\Box$ and $\Diamond$ are not necessarily interdefinable. This is because our set of connectives does not necessarily contain negation, and even if it does, nothing is assumed about its truth table. Moreover, in $\mathbf{mv-N}$, they are not interdefinable even in the presence of any kind of negation, because they depend on the same accessibility relation (see Chapter 11).

We also show extensions of our logics, which are counterparts of some well-known extensions of K. We prove the finite model property of $\mathbf{mv-N}$, $\mathbf{mv-K}$ and their extensions. We then show the unique definition of negation such that De Morgan's duality between $\Box$ and $\Diamond$ holds in $\mathbf{mv-K}$. We also prove that many-valued intuitionistic logic, as defined in [Tak86], is a fragment of one of the extensions of $\mathbf{mv-K}$. Finally, we deal with a specific case where the set of truth values is not linearly ordered. We consider the case in which the partially ordered set is the Dunn-Belnap lattice [Dun76; Bel75; Bel77], and show that all of the above results hold in this case as well (apart from the uniqueness of the definition of negation).

A number of many-valued normal modal logics is known from the literature. In [Ost88], an n -valued modal logic is based on the Łukasiewicz classical n -valued connectives. The paper contains Hilbert-style calculi for the generalizations of the two-valued normal modal logics T, S4, and S5, and the author notes that other generalizations are also possible. This work seems to generalize [SJLM78], in which three-valued modal Łukasiewicz logics are considered. Three-valued modal logics with different connectives are considered in [Seg67].

In [Tho78] and [Mor79], general notions of many-valued modal logics are suggested, using designated values and a rather general interpretation of the modal connective.

In [Mor89], proof systems relying on matrices are presented for normal three-valued modal logics based on any arbitrary set of propositional connectives. Among other logics, the three-valued counterparts of the two-valued T, S4, and S5 are presented.

In [Fit91], the author presents a sequent calculus for modal logics based on any finite lattice of truth values. These logics, in addition to all propositional constants, have all classical propositional connectives. The semantics relies on a many-valued accessibility relation, that is further discussed in [Fit92]. This paper also addresses the possibility connective $\Diamond$ that is treated explicitly, because, in general, $\Diamond$ and $\Box$ are not interdefinable.² In addition, some extensions of the many-valued modal logics are mentioned at the end of [Fit92].

The most general approach (for our purposes) was, probably, taken in [Tak94], where a proof system, relying on matrices of labelled formulas, is presented for many-valued normal modal logics with an arbitrary set of propositional connectives. The

² As noted above, these connectives are not interdefinable in our thesis either, but for a different reason.

system is appropriate for any semantic interpretation of the necessity connective satisfying certain conditions – not only for its interpretation as the infimum. The system is weakly complete and possesses the subformula property (that implies weak decidability). However, the possibility connective is not addressed in [Tak94] at all, and extensions of the logic are not presented there.

Another general approach is taken in [FL98], where proof systems using tableaux are suggested for a variety of finite-valued modal logics with generalized modalities.

In [BEGR11], counterparts of K, using Hilbert-style proof systems, are presented for any finite residuated lattice of truth values, allowing many-valued accessibility relations. The logics address only the necessity operator (it is only mentioned that the possibility operator should, in general, be addressed separately and not as an abbreviation of $\neg\Box\neg$), and they are based on a fixed set of propositional connectives. The semantics use designated values to interpret validity of formulas.

In [MOG08, Chapter 9.1], many-valued modal logics are discussed, referring also to Gentzen systems, logic extensions and logic embeddings. Again, a fixed set of connectives is assumed and the semantic interpretation is algebraic.

Our research introduces a novel and comprehensive framework for many-valued modal logics that stands out by integrating several key features simultaneously: the use of an arbitrary “base logic”, the use of Gentzen-like sequents of labelled formulas, the independent treatment of both necessity and possibility modalities, the demonstration of strong completeness and strong decidability, and addressing all basic logic extensions. While each of these elements has been explored individually in previous studies, our work combines them into a single coherent system. This combination allows for a more robust and flexible logical framework that can handle a wider variety of logical scenarios and applications. By employing labeled formulas (discussed, e.g., in [BFSZ98] and [KF21]), we can address any truth value rather than being limited to designated ones, providing a significant advantage in terms of expressive power. The motivation behind this research lies in the importance of many-valued modal logics in contexts with inherent uncertainty or gradations of truth, such as fuzzy logic systems and multi-agent systems. Additionally, extending these logics to include features like transitive accessibility relations is crucial for modeling more complex systems. Our results include the finite model property, ensuring the logics’ strong decidability, and embedding many-valued intuitionistic logic within our framework, thus offering a comprehensive and robust tool for logical analysis. Moreover, we develop the theory of ***mv-K*** (the definitions and many of the results) from the theory of ***mv-N***. ***mv-N*** itself does not have counterparts in any previous paper.

The thesis is organized as follows. In Chapter 2 we introduce a many-valued logic ***mv-N*** of pure necessitation and present a sound and strongly complete³ proof system for it. Chapter 3 contains the proof of the strong completeness of ***mv-N***. In Chapter 4

³ That is, complete with respect to the consequence relation.

we consider some extensions of $\mathbf{mv-N}$ and prove their soundness and completeness with respect to certain classes of Kripke structures. In Chapter 5 we show that $\mathbf{mv-N}$ and its extensions from Chapter 4 possess the finite model property⁴. Then in Chapter 6 we generalize the results regarding $\mathbf{mv-N}$ to the case of a certain lattice of truth values.

In Chapter 7, we introduce a many-valued modal logic $\mathbf{mv-K}$ and present a sound and strongly complete proof system for it. Chapter 8 contains the proof of the strong completeness of $\mathbf{mv-K}$. Chapter 9 contains some extensions of $\mathbf{mv-K}$ and the proof of their soundness and completeness with respect to certain classes of Kripke models and, in Chapter 10, we show that $\mathbf{mv-K}$ and its extensions from Chapter 9 possess the finite model property. Then, in Chapter 11, we present the appropriate definition of negation so that $\Box$ and $\Diamond$ are interdefinable. In Chapter 12, we embed many-valued intuitionistic logic in our many-valued counterpart of S4. Finally, in Chapter 13 we generalize the results regarding $\mathbf{mv-K}$ to the case of a certain lattice of truth values.

⁴ Thus, they are strongly decidable.

Chapter 2

Many-valued logic of pure necessitation $mv-N$

In this chapter we define a many-valued logic, $mv-N$, assuming a linear order on the set of truth values.

In what follows, $V = \{v_1, \dots, v_n\}$, $n \geq 2$, is a set of truth values ordered by

$$v_1 < v_2 < \dots < v_n$$

Formulas of $mv-N$ are built from propositional variables by means of propositional connectives (of arbitrary arities) and the modal connectives $\Box$ and $\Diamond$. The set of all $mv-N$ formulas will be denoted by $\mathcal{F}$. The semantics of propositional connectives is given by truth tables, where, as usual, the truth table of an ℓ -ary propositional connective $*$ is a function $*$: $V^\ell \rightarrow V$ and the semantics of the modal connectives is given below.

A *labelled* formula is a pair (φ, k) , where φ is a formula and $k = 1, \dots, n$. The intended meaning of such a labelled formula is that v_k is the truth value associated with φ .

Define $(\varphi, k)^+ = \{\varphi\} \times \{k, \dots, n\}$ and $(\varphi, k)^- = \{\varphi\} \times \{1, \dots, k\}$.

Sequents are expressions of the form $\Gamma \rightarrow \Delta$, where Γ and Δ are finite (possibly empty) sets of labelled formulas and $\rightarrow$ is not a symbol of the underlying language.

The $mv-N$ semantics is as follows.

A *many-valued Kripke structure* (or *many-valued N -structure* or just Kripke structure) is a triple $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$, where

- W is a nonempty set (of possible worlds),
- $\{R_\varphi\}_{\varphi \in \mathcal{F}}$ is a family of binary (accessibility) relations on W , and
- $I : W \times \mathcal{P} \rightarrow V$, where $\mathcal{P}$ is the set of propositional variables, is a (valuation) function.

For a world $u \in W$ and a formula $\varphi \in \mathcal{F}$, we define the set of R_φ -successors of u ,

denoted by $S_\varphi(u)$, as

$$S_\varphi(u) = \{v \in W : uR_\varphi v\}$$

and extend I to $W \times \mathcal{F}$, recursively, as follows.

- $I(u, *(\varphi_1, \dots, \varphi_\ell)) = *(I(u, \varphi_1), \dots, I(u, \varphi_\ell))$,
- $I(u, \Box\varphi) = \inf(\{I(v, \varphi) : v \in S_\varphi(u)\})$, where $\inf(\emptyset)$ is $\mathbf{v}_n$, and
- $I(u, \Diamond\varphi) = \sup(\{I(v, \varphi) : v \in S_\varphi(u)\})$, where $\sup(\emptyset)$ is $\mathbf{v}_1$.

Note that, if $S_\varphi(u) \neq \emptyset$, then, since $\mathbf{V}$ is finite and linearly ordered, $\inf(\{I(v, \varphi) : v \in S_\varphi(u)\})$ and $\sup(\{I(v, \varphi) : v \in S_\varphi(u)\})$ are, actually, $\min(\{I(v, \varphi) : v \in S_\varphi(u)\})$ and $\max(\{I(v, \varphi) : v \in S_\varphi(u)\})$, respectively.

Remark 2.1. Note that satisfiability of a formula φ depends only on the set of relations $\{R_{\varphi'}\}_{\varphi' \text{ is a subformula of } \varphi}$.

We also write $M, u \models (\varphi, k)$, if $I(u, \varphi) = \mathbf{v}_k$.

The satisfiability relation $\models$ between worlds of W and sequents of labelled formulas is defined as follows.

A world u satisfies a sequent $\Gamma \rightarrow \Delta$, denoted $M, u \models \Gamma \rightarrow \Delta$, if the following holds.

- If for each $(\varphi, k) \in \Gamma$, $I(u, \varphi) = \mathbf{v}_k$, then for some $(\varphi, k) \in \Delta$, $I(u, \varphi) = \mathbf{v}_k$.¹

A Kripke structure M satisfies a sequent $\Gamma \rightarrow \Delta$, if each world in W satisfies $\Gamma \rightarrow \Delta$ and M satisfies a set of sequents Σ , if it satisfies each sequent in Σ . Finally, a set of sequents Σ *semantically entails* a sequent $\Gamma \rightarrow \Delta$, denoted $\Sigma \models \Gamma \rightarrow \Delta$, if each many-valued Kripke structure satisfying all sequents from Σ also satisfies $\Gamma \rightarrow \Delta$.

We define next the proof system of **mv-N**.

The axioms are:

$$(\varphi, k) \rightarrow (\varphi, k) \tag{2.1}$$

$$(\varphi_1, k_1), \dots, (\varphi_\ell, k_\ell) \rightarrow (*(\varphi_1, \dots, \varphi_\ell), k) \tag{2.2}$$

for each table entry $\mathbf{v}_{k_1}, \dots, \mathbf{v}_{k_\ell}$ such that $*(\mathbf{v}_{k_1}, \dots, \mathbf{v}_{k_\ell}) = \mathbf{v}_k$,

$$(\Box\varphi, k) \rightarrow (\Diamond\varphi, k)^+ \quad k \neq n \tag{2.3}$$

and

$$(\Box\varphi, n) \rightarrow (\Diamond\varphi, 1), (\Diamond\varphi, n) \tag{2.4}$$

and the logical rules are

¹ In other words, v satisfies a sequent $\Gamma \rightarrow \Delta$, if the metavalue of the classical metasequent $\{I(u, \varphi) = \mathbf{v}_k : (\varphi, k) \in \Gamma\} \rightarrow \{I(u, \varphi) = \mathbf{v}_k : (\varphi, k) \in \Delta\}$ is “true.”

$\Box$ -introduction:

$$\frac{(\varphi, k) \rightarrow}{(\Box \varphi, k) \rightarrow} \quad k \neq n \quad (2.5)$$

$\Diamond$ -introduction:

$$\frac{(\varphi, k) \rightarrow}{(\Diamond \varphi, k) \rightarrow} \quad k \neq 1 \quad (2.6)$$

and the structural rules below.

k -left-shift:

$$\frac{\Gamma, (\varphi, k) \rightarrow \Delta}{\Gamma \rightarrow \Delta, \{\varphi\} \times \{k\}} \quad (2.7)$$

k', k'' -right-shift:

$$\frac{\Gamma \rightarrow \Delta, (\varphi, k')}{\Gamma, (\varphi, k'') \rightarrow \Delta} \quad k' \neq k'' \quad (2.8)$$

k -left-weakening:

$$\frac{\Gamma \rightarrow \Delta}{\Gamma, (\varphi, k) \rightarrow \Delta} \quad (2.9)$$

k -right-weakening:

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, (\varphi, k)} \quad (2.10)$$

k -cut:

$$\frac{\Gamma' \rightarrow \Delta', (\varphi, k) \quad \Gamma'', (\varphi, k) \rightarrow \Delta''}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''} \quad (2.11)$$

k', k'' -resolution:

$$\frac{\Gamma' \rightarrow \Delta', (\varphi, k') \quad \Gamma'' \rightarrow \Delta'', (\varphi, k'')}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''} \quad k' \neq k'' \quad (2.12)$$

In fact, cut and resolution are derivable from each other, see [KF21, Proposition 3.3].

We shall also need the two following derivable rules. One is “multi-cut”

$$\frac{\{\Gamma_k, (\varphi, k) \rightarrow \Delta_k : k \in \{1, \dots, n\}\}}{\bigcup_{k \in \{1, \dots, n\}} \Gamma_k \rightarrow \bigcup_{k \in \{1, \dots, n\}} \Delta_k} \quad (2.13)$$

that is derivable by cuts from the sequent

$$\rightarrow \{\varphi\} \times \{1, 2, \dots, n\} \quad (2.14)$$

see [KF21, Proposition 3.4], and the other is “multi-shift”

$$\frac{\{\Gamma_k, (\varphi, k) \rightarrow \Delta_k : k \in K\}}{\bigcup_{k \in K} \Gamma_k \rightarrow \bigcup_{k \in K} \Delta_k, \{\varphi\} \times \overline{K}} \quad (2.15)$$

for $K \subset \{1, \dots, n\}$ and $\overline{K} = \{1, \dots, n\} \setminus K$, see [KF21, Remark 3.5].

Theorem 2.2. *Let Σ and $\Gamma \rightarrow \Delta$ be a set of sequents and a sequent, respectively. Then $\Sigma \vdash \Gamma \rightarrow \Delta$ if and only if $\Sigma \models \Gamma \rightarrow \Delta$.*

The proof of the “only if” part of theorem (soundness) is standard and is presented below. The proof of the “if” part of theorem (strong completeness) is rather involved and is presented in the next chapter.

Proof of the “only if” part of Theorem 2.2 The proof is by induction on the derivation length of $\Gamma \rightarrow \Delta$ from Σ . For the case of sequents from Σ , axioms (2.1) and (2.2), and the structural rules of inference we refer the reader to [KF21, Section 3.2].

For the case of axioms and rules of inference involving modal connectives, the induction step is as follows.

Let $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ be a Kripke structure satisfying Σ .

Let $\Gamma \rightarrow \Delta$ be an instance of (2.3) and assume $M, u \models (\Box\varphi, k)$. Since $k \neq n$, $S_\varphi(u) \neq \emptyset$, implying

$$v_k = \inf(\{I(v, \varphi) : v \in S_\varphi(u)\}) \leq \sup(\{I(v, \varphi) | v \in S_\varphi(u)\}) = I(u, \Diamond\varphi)$$

Thus, $M, u \models (\Diamond\varphi, k)^+$.

Let $\Gamma \rightarrow \Delta$ be an instance of (2.4) and assume $M, u \models (\Box\varphi, n)$. If $S_\varphi(u)$ is empty, $M, u \models (\Diamond\varphi, 1)$, implying $M, u \models \Gamma \rightarrow \Delta$. Otherwise,

$$\{I(v, \varphi) : v \in S_\varphi(u)\} = \{v_n\}$$

implying $M, u \models (\Diamond\varphi, n)$. Thus, $M, u \models \Gamma \rightarrow \Delta$.

Let $\Gamma \rightarrow \Delta$ be the conclusion of (2.5) and assume to the contrary that, for some $u \in W$, $M, u \models (\Box\varphi, k)$. Since $k \neq n$, $S_\varphi(u) \neq \emptyset$, implying $v_k \in \{I(v, \varphi) : v \in S_\varphi(u)\}$. In particular there exists a world $v \in W$ such that $M, v \models (\varphi, k)$, contradicting the induction hypothesis $M \models (\varphi, k) \rightarrow$.

The case of (2.6) is dual to that of (2.5). ■

We conclude this chapter with examples of **mv-N** derivability.

Example 2.3. The rule

$$\frac{\rightarrow \{\varphi\} \times K}{\rightarrow \{\Box\varphi\} \times K, (\Box\varphi, n)} \quad (2.16)$$

is derivable in **mv-N** as follows.

1. $\rightarrow \{\varphi\} \times K$ assumption
2. $\{(\varphi, k) \rightarrow : k \in \overline{K}\}$ follows from 1 by a number of (2.8)
3. $\{(\Box\varphi, k) \rightarrow : k \in \overline{K} \cup \{n\}\}$ follows from 2 by (2.5)
4. $\rightarrow \{\Box\varphi\} \times K, (\Box\varphi, n)$ follows from 3 by (2.15)

The dual rule

$$\frac{\rightarrow \{\varphi\} \times K}{\rightarrow \{\Diamond\varphi\} \times K, (\Diamond\varphi, 1)}$$

can be derived in a similar manner.

Example 2.4. For $n = 3$, the rule

$$\frac{\rightarrow (\varphi, 2), (\varphi, 3)}{(\Diamond \varphi, 2) \rightarrow (\Box \varphi, 2)} \quad (2.17)$$

is derivable in $\mathbf{mv-N}$ as follows.

1. $\rightarrow (\varphi, 2), (\varphi, 3)$ assumption
2. $\rightarrow (\Box \varphi, 2), (\Box \varphi, 3)$ follows from 1 by (2.16)
3. $(\Diamond \varphi, 2) \rightarrow (\Box \varphi, 1), (\Box \varphi, 2)$ (2.18)
4. $(\Diamond \varphi, 2) \rightarrow (\Box \varphi, 2)$ follows from 2 and 3 by (2.12).

Example 2.5. The sequent

$$(\Diamond \varphi, k) \rightarrow (\Box \varphi, k)^- \quad k \neq 1 \quad (2.18)$$

is $\mathbf{mv-N}$ derivable.

If $k = n$, then (2.18) follows from (2.14) by (2.9).

If $k = n - 1$, then

1. $(\Diamond \varphi, n - 1) \rightarrow (\Box \varphi, n - 1)^-, (\Box \varphi, n)$ follows from (2.14) by (2.9)
2. $(\Diamond \varphi, n - 1) \rightarrow (\Box \varphi, n - 1)^-, (\Diamond \varphi, 1), (\Diamond \varphi, n)$ follows from 1 and (2.4) by (2.11)
3. $(\Diamond \varphi, n - 1) \rightarrow (\Box \varphi, n - 1)^-$ follows from 2 by (2.8),
because $k \neq 1$

Otherwise,

1. $\{(\Box \varphi, j) \rightarrow (\Diamond \varphi, j)^+ : j > k, j \neq n\}$ axiom (2.3)
2. $\{(\Box \varphi, j), (\Diamond \varphi, k) \rightarrow : j > k, j \neq n\}$ follows from 1 by (2.8)
3. $(\Diamond \varphi, k) \rightarrow (\Box \varphi, k)^-, (\Box \varphi, n)$ follows from 2 by (2.15)
4. $(\Diamond \varphi, k) \rightarrow (\Box \varphi, k)^-, (\Diamond \varphi, 1), (\Diamond \varphi, n)$ follows from 3 and (2.4) by (2.11)
5. $(\Diamond \varphi, k) \rightarrow (\Box \varphi, k)^-$ follows from 4 by (2.8),
because $k \neq 1, n$

In a similar fashion, it can be shown that the sequent

$$(\Diamond \varphi, 1) \rightarrow (\Box \varphi, 1), (\Box \varphi, n) \quad (2.19)$$

is derivable as well and that each of the following modifications of the given proof system results in an equivalent system:²

- Replacing (2.3) by (2.18).
- Replacing (2.4) by (2.19).

² That is, the set of provable sequents from a given set of assumptions remains the same.

- Replacing (2.3) and (2.4) by (2.18) and (2.19), respectively.
- Replacing (2.4) by (2.18).

We omit the proofs.

Chapter 3

The proof of the “if” part of Theorem 2.2

For the proof of the strong completeness of $\mathbf{mv-N}$, we extend the definition of provability to infinite sets of labeled formulas.

For a set of sequents Σ , a (not necessarily finite) set of labeled formulas Γ , and a finite set of labeled formulas Δ , we write $\Sigma \vdash \Gamma \rightarrow \Delta$, if there exists a finite subset Γ' of Γ such that $\Sigma \vdash \Gamma' \rightarrow \Delta$.

A set of labeled formulas Γ is called Σ -consistent, if $\Sigma \not\vdash \Gamma \rightarrow$.

A set of sequents Σ is called *consistent*, if $\Sigma \not\vdash \rightarrow$.¹

Lemma 3.1. *If $\Sigma \not\vdash \Gamma \rightarrow \Delta$, then there exists a maximal (with respect to inclusion) Σ -consistent set Γ' including Γ such that $\Sigma \not\vdash \Gamma' \rightarrow \Delta$.*

The proof is straightforward, by Zorn’s lemma, and is omitted.

Lemma 3.2. ([KF21, Lemma 3.12 and the following observation]) *If Γ is a maximal set for which $\Sigma \not\vdash \Gamma \rightarrow \Delta$, then for every formula $\varphi \in \mathcal{F}$ there exists a unique $k \in \{1, \dots, n\}$ such that $(\varphi, k) \in \Gamma$.*

For a consistent set of sequents Σ , the Σ -canonical structure $M_\Sigma = \langle W_\Sigma, \{R_{\Sigma, \varphi}\}_{\varphi \in \mathcal{F}}, I_\Sigma \rangle$ is defined as follows.

- W_Σ is the set of all maximal Σ -consistent sets. Since Σ is consistent, by Lemma 3.1, W_Σ is nonempty.
- For a formula $\varphi \in \mathcal{F}$ and worlds $\Gamma', \Gamma'' \in W_\Sigma$, $\Gamma' R_{\Sigma, \varphi} \Gamma''$ if and only if for the unique i, j , and k such that $(\Box\varphi, i), (\Diamond\varphi, j) \in \Gamma'$ and $(\varphi, k) \in \Gamma''$, provided by Lemma 3.2, $i \leq k \leq j$.²
- For $u \in W_\Sigma$ and $p \in \mathcal{P}$, $I_\Sigma(u, p)$ is the unique value v_k such that $(p, k) \in u$.

¹ Equivalently, Σ is consistent, if there exists a Σ -consistent set of formulas Γ .

² If $i > j$, then $S_\varphi(\Gamma')$ is empty.

Theorem 3.3. *For all labelled formulas (φ, k) and all $u \in W_\Sigma$, $(\varphi, k) \in u$ if and only if $M_\Sigma, u \models (\varphi, k)$.*

Proof It is sufficient to show the “only if” part of the theorem, i.e., that $(\varphi, k) \in u$ implies $M_\Sigma, u \models (\varphi, k)$. This is because, if $(\varphi, k) \notin u$, then, by Lemma 3.2, $(\varphi, k') \in u$ for $k' \neq k$. Therefore by the “only if” part of the theorem, $M_\Sigma, u \models (\varphi, k')$, implying $M_\Sigma, u \not\models (\varphi, k)$.

The proof is by induction on the complexity of φ . For the cases of an atomic formula and a propositional principal connective, see [KF21, Proposition 3.13].

Let φ be of the form $\Box\varphi'$ and assume $(\Box\varphi', i), (\Diamond\varphi', j) \in u$ for some j (and $i = k$, of course). We distinguish between the cases of $i \neq n$ and $i = n$.

- Let $i \neq n$. By the induction hypothesis and the definition of $R_{\Sigma, \varphi'}$, for each world $v \in S_{\varphi'}(u)$, $\mathbf{v}_i \leq I_\Sigma(v, \varphi')$. Therefore, for the proof of

$$\mathbf{v}_i = \min(\{I_\Sigma(v, \varphi') : v \in S_{\varphi'}(u)\}) = I_\Sigma(u, \varphi)$$

it suffices to show that

$$\mathbf{v}_i \in \{I_\Sigma(v, \varphi') : v \in S_{\varphi'}(u)\} \quad (3.1)$$

Since $(\Box\varphi', i) \in u$, it follows from (2.3) that $\Sigma \vdash u \rightarrow (\Diamond\varphi', i)^+$. Thus, $i \leq j$, because, otherwise, by i, j -right-shifts (2.8), we obtain a contradiction.

Note that $\Sigma \not\vdash (\varphi', i) \rightarrow$. Were $\Sigma \vdash (\varphi', i) \rightarrow$, by (2.5), we would have $\Sigma \vdash (\Box\varphi', i) \rightarrow$, in contradiction with Σ -consistency of u .

Since $\Sigma \not\vdash (\varphi', i) \rightarrow$, by Lemma 3.1, there exists a world $w \in W_\Sigma$ containing (φ', i) , which, together with $i \leq i \leq j$ and the definition of $R_{\Sigma, \varphi'}$, implies $uR_{\Sigma, \varphi'}w$. By the induction hypothesis, $M_\Sigma, w \models (\varphi', i)$ and (3.1) follows.

- Let $i = n$. If $S_{\varphi'}(u) = \emptyset$, then, trivially, $M_\Sigma, u \models (\varphi, k)$. Otherwise, by the induction hypothesis and the definition of $R_{\Sigma, \varphi'}$, all worlds $v \in S_{\varphi'}(u)$ satisfy $\mathbf{v}_n \leq I_\Sigma(v, \varphi')$, implying $I_\Sigma(v, \varphi') = \mathbf{v}_n$. Thus,

$$\min(\{I_\Sigma(v, \varphi') : v \in S_{\varphi'}(u)\}) = \min(\{\mathbf{v}_n\}) = \mathbf{v}_n$$

and $M_\Sigma, u \models (\varphi, k)$ follows.

The case of $\Diamond$ is dual to that of $\Box$. We just replace $\Box$ with $\Diamond$, $\Diamond$ with $\Box$, min with max, n with 1, and $\leq$ with $\geq$. We omit the details. $\blacksquare$

Corollary 3.4. *We have $M_\Sigma \models \Sigma$.*

Proof Let $u \in W$ and let $\Gamma \rightarrow \Delta \in \Sigma$. Assume that u satisfies all labelled formulas in Γ and assume to the contrary that u satisfies no labelled formula in Δ . By Theorem 3.3, $\Gamma \subseteq u$, and for all $(\varphi, k_\varphi) \in \Delta$ there is $k'_\varphi \neq k_\varphi$ such that $(\varphi, k'_\varphi) \in u$. By definition, $\Sigma \vdash \Gamma \rightarrow \Delta$. Therefore, by k_φ, k'_φ -right-shifts (2.8),

$$\Sigma \vdash \Gamma, \{(\varphi, k'_\varphi) : (\varphi, k_\varphi) \in \Delta\} \rightarrow$$

implying $\Sigma \vdash u \rightarrow$, because $\Gamma, \{(\varphi, k'_\varphi) : (\varphi, k_\varphi) \in \Delta\} \subseteq u$. This, however, contradicts Σ -consistency of u . ■

Proof of the “if” part of Theorem 2.2 Assume $\Sigma \not\vdash \Gamma \rightarrow \Delta$. By Lemma 3.1, there exists a maximal set Γ' including Γ such that $\Sigma \not\vdash \Gamma' \rightarrow \Delta$. By the definition of M_Σ , $\Gamma' \in W_\Sigma$. We contend that $M_\Sigma \not\models \Gamma \rightarrow \Delta$. Namely, $M_\Sigma, \Gamma' \not\models \Gamma \rightarrow \Delta$.

Since $\Gamma \subseteq \Gamma'$, by Theorem 3.3, Γ' satisfies all labelled formulas in Γ . However, it satisfies no labelled formula in Δ , because, otherwise, by Theorem 3.3, such a formula would belong to Γ' , implying $\Sigma \vdash \Gamma' \rightarrow \Delta$, in contradiction with the definition of Γ' . Thus, $M_\Sigma \not\models \Gamma \rightarrow \Delta$, which completes the proof of our contention and, together with Corollary 3.4, completes the proof of the “if” part of the theorem. ■

Chapter 4

Two extensions of $mv-N$

In this chapter, L is an extension of $mv-N$ with additional axioms or rules. The axioms we shall discuss here are many-valued counterparts of axioms D , see [HC84, p. 29], and T , see [HC84, p. 10].

We write $\Sigma \vdash_L \Gamma \rightarrow \Delta$, if Σ proves $\Gamma \rightarrow \Delta$ in L (and we keep writing $\Sigma \vdash \Gamma \rightarrow \Delta$, if L is $mv-N$ itself). We generalize this notation to sequents $\Gamma \rightarrow \Delta$ with an infinite antecedent Γ , like in the previous chapter.

Clearly, the results of the previous chapter apply also to any extension L . Below, we just rewrite them with respect to L .

Definition 4.1. A set of labeled formulas Γ is called *L - Σ -consistent*, if $\Sigma \not\vdash_L \Gamma \rightarrow$.

Definition 4.2. A set of sequents Σ is called *L -consistent*, if $\Sigma \not\vdash_L \rightarrow$, or, equivalently, if there exists an L - Σ -consistent set, cf. footnote 1.

Lemma 4.3. (Cf. Lemma 3.1.) *If $\Sigma \not\vdash_L \Gamma \rightarrow \Delta$, then there exists a maximal L - Σ -consistent set Γ' including Γ such that $\Sigma \not\vdash_L \Gamma' \rightarrow \Delta$.*

Lemma 4.4. (Cf. Lemma 3.2.) *If Γ is a maximal set for which $\Sigma \not\vdash_L \Gamma \rightarrow \Delta$, then for every formula $\varphi \in \mathcal{F}$ there exists a unique $k \in \{1, \dots, n\}$ such that $(\varphi, k) \in \Gamma$.*

For an L -consistent set of sequents Σ , we define the *L - Σ -canonical structure* $M_{L,\Sigma} = \langle W_{L,\Sigma}, \{R_{L,\Sigma,\varphi}\}_{\varphi \in \mathcal{F}}, I_{L,\Sigma} \rangle$ just like the Σ -canonical structure M_Σ in Chapter 3, except that $W_{L,\Sigma}$ is the set of all maximal L - Σ -consistent sets. Note that $W_{L,\Sigma}$ is nonempty, because Σ is L -consistent.

Corollary 4.5. *For an L -consistent set of sequents Σ the following holds.*

- (i) (Cf. Theorem 3.3.) *For all labelled formulas (φ, k) and all $u \in W_{L,\Sigma}$, $(\varphi, k) \in u$ if and only if $M_{L,\Sigma}, u \models (\varphi, k)$.*
- (ii) (Cf. Corollary 3.4.) $M_{L,\Sigma} \models \Sigma$.
- (iii) (Cf. the “if” part of Theorem 2.2.) *If $\Sigma \not\vdash_L \Gamma \rightarrow \Delta$, then $M_{L,\Sigma} \not\models \Gamma \rightarrow \Delta$.*

We proceed with two extensions of $\mathbf{mv-N}$ which are sound and strongly complete for the many-valued Kripke structures defined below.

Definition 4.6. A many-valued Kripke structure $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ is called *serial* (or *with no dead-ends*), if for all formulas $\varphi \in \mathcal{F}$, R_φ is a serial relation, i.e., for all $u \in W$, $S_\varphi(u) \neq \emptyset$.

Theorem 4.7. *If $\mathbf{L}$ contains the axioms*

$$(\Box\varphi, n) \rightarrow (\Diamond\varphi, n) \quad (4.1)$$

then $M_{\mathbf{L}, \Sigma}$ is serial.

Proof By Remark 4.8 below, we may remove the restriction on n in axiom (2.3) and in rule (2.5). Then, we are, actually, in the argument of the case of $i \neq n$ in the proof of Theorem 3.3, where, in particular, we have shown that for each formula φ and each world u , $S_\varphi(u)$ is nonempty. ■

Remark 4.8. Axiom (4.1) is, actually, removing the requirement “ $k \neq n$ ” from (2.3). In addition, axiom (2.4) becomes redundant. Also, the requirement “ $k \neq n$ ” in rule (2.5) is redundant as well, which can be shown as follows.

1. $(\varphi, n) \rightarrow$ assumption
2. $(\Diamond\varphi, n) \rightarrow$ follows from 1 by (2.6)
3. $(\Box\varphi, n) \rightarrow (\Diamond\varphi, n)$ axiom (4.1)
4. $(\Box\varphi, n) \rightarrow$ follows from 2 and 3 by n -cut (2.11)

Dually, it can be shown that the requirement “ $k \neq 1$ ” in rule (2.6) is also redundant.

Definition 4.9. A many-valued Kripke structure $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ is called *reflexive*, if for all formulas $\varphi \in \mathcal{F}$, R_φ is a reflexive relation.

Theorem 4.10. *If $\mathbf{L}$ contains the axioms*

$$(\Box\varphi, k) \rightarrow (\varphi, k)^+ \quad (4.2)$$

and

$$(\varphi, k) \rightarrow (\Diamond\varphi, k)^+ \quad (4.3)$$

*then $M_{\mathbf{L}, \Sigma}$ is reflexive.*¹

Proof For a formula φ and a world $u \in W_{\mathbf{L}, \Sigma}$, we have to show that $uR_{\mathbf{L}, \varphi}u$. Let i, j , and k be such that $(\Box\varphi, i), (\Diamond\varphi, j), (\varphi, k) \in u$. By axiom (4.2), $\Sigma \vdash_{\mathbf{L}} u \rightarrow (\varphi, i)^+$,

¹ The classical axiom $\Box\varphi \supset \varphi$ is replaced with axioms (4.2) and (4.3), because, in $\mathbf{mv-N}$, $\Box$ and $\Diamond$ are not interdefinable.

implying $i \leq k$, because, otherwise, like in the proof of Theorem 3.3, by right-shifts we arrive at a contradiction. Similarly, by axiom (4.3), $k \leq j$. Thus, by definition of $R_{L,\varphi}$, we have $uR_{L,\varphi}u$. ■

Remark 4.11. Axioms (4.2) and (4.3), with k being n , imply (4.1), by n -cut (2.11), and also imply axioms (2.3) by a number of cuts. In addition, by Remark 4.8, these axioms also imply (2.4) and make redundant the restrictions on k in rules (2.5), and (2.6).

Theorem 4.12. *The extension of $\mathbf{mv-N}$ with (4.1) is sound and strongly complete with respect to serial Kripke structures.*

Theorem 4.13. *The extension of $\mathbf{mv-N}$ with (4.2) and (4.3) is sound and strongly complete with respect to reflexive Kripke structures.*

The completeness of each logic follows immediately from Corollary 4.5 (iii) and Theorems 4.7 and 4.10. It remains to show soundness. In view of the “only if” part of Theorem 2.2, we only have to check that every world $u \in W$ satisfies the respective axioms.

Proof of the soundness part of Theorem 4.12 If $u \models (\Box\varphi, n)$, then

$$v_n = \inf(\{I(v, \varphi) : v \in S_\varphi(u)\}) \leq \sup(\{I(v, \varphi) : v \in S_\varphi(u)\}) = I(u, \Diamond\varphi)$$

because $S_\varphi(u) \neq \emptyset$. Thus, $I(u, \Diamond\varphi) = v_n$ follows. ■

Proof of the soundness part of Theorem 4.13 Since $u \in S_\varphi(u)$,

$$I(u, \Box\varphi) \leq I(u, \varphi) \leq I(u, \Diamond\varphi)$$

both (4.2) and (4.3) are satisfied by u . ■

Remark 4.14. For a set of $\mathbf{mv-N}$ formulas Φ , the proofs in this chapter are easily modified to show that if we add to $\mathbf{mv-N}$ only the Φ -instances of (4.1), i.e., the set of sequents

$$\{(\Box\varphi, n) \rightarrow (\Diamond\varphi, n) : \varphi \in \Phi\}$$

then we obtain a sound and complete logic with respect to Kripke structures in which R_φ is serial for all $\varphi \in \Phi$.

Similarly, adding only the Φ -instances of (4.2) and (4.3) results in a sound and complete logic with respect to Kripke structures in which R_φ is reflexive for all $\varphi \in \Phi$.

Chapter 5

Decidability of $mv\text{-}N$ and its extensions

In what follows, L can be any of the logics $mv\text{-}N$, $mv\text{-}N+(4.1)$, or $mv\text{-}N+(4.2),(4.3)$ and C_L is the class of the respective Kripke structures.¹

We show that it possesses the finite model property. The proof is based on the filtration technique, cf. [Seg71, Chapter I, Section 7], where this technique is applied to some two-valued modal logics.

Let Φ be a subformula-closed set of formulas² and let $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ be a Kripke structure. The equivalence relation $\equiv_\Phi$ on W is defined as follows.

$$u \equiv_\Phi v \text{ if and only if } I(u, \varphi) = I(v, \varphi) \text{ for all } \varphi \in \Phi.$$

The L -filtration of M through Φ is the Kripke structure

$$M_{L,\Phi}^* = \langle W_{L,\Phi}^*, \{R_{L,\Phi,\varphi}^*\}_{\varphi \in \mathcal{F}}, I_{L,\Phi}^* \rangle$$

where³

- $W_{L,\Phi}^*$ is the set of all equivalence classes of $\equiv_\Phi$. That is, $W_{L,\Phi}^* = \{[u] : u \in W\}$ where $[u]$ is the $\equiv_\Phi$ equivalence class of u .
- $[u]R_{L,\Phi,\varphi}^*[v]$ if and only if there exist $u' \in [u]$ and $v' \in [v]$ such that $u'R_\varphi v'$.
- For $[u] \in W^*$ and a propositional variable $p \in \Phi$, $I_{L,\Phi}^*([u], p) = I(u, p)$. By the definition of $\equiv_\Phi$, $I_{L,\Phi}^*$ is well defined and the value of $I_{L,\Phi}^*$ for $p \notin \Phi$ does not matter for our purposes.

¹ That is, C_L consists of all Kripke structures, if L is $mv\text{-}N$, consists of all serial Kripke structures, if L is $mv\text{-}N+(4.1)$, and consists of all reflexive Kripke structures, if L is $mv\text{-}N+(4.2),(4.3)$.

² That is, if $\varphi \in \Phi$, then each subformula of φ also belongs to Φ .

³ For the three logics considered here, the filtration does not, actually, depend on L . However, in Chapter 10, the accessibility relation of the filtration is going to depend on the underlying logic, hence we indicate the logic in the subscript.

Theorem 5.1. For all $\varphi \in \Phi$ and $u \in W$,

$$I(u, \varphi) = I_{\mathbf{L}, \Phi}^*([u], \varphi) \quad (5.1)$$

Proof The proof is by induction on the complexity of φ .

If φ is a propositional variable, the equality immediately follows from the definition of $I_{\mathbf{L}, \Phi}^*$.

If φ is of the form $\ast(\varphi_1, \dots, \varphi_\ell)$, then $\varphi_1, \dots, \varphi_\ell \in \Phi$, because Φ is subformula closed. Therefore, by the induction hypothesis, for all $i = 1, 2, \dots, \ell$, $I(u, \varphi_i) = I_{\mathbf{L}, \Phi}^*([u], \varphi_i)$, implying

$$\begin{aligned} I(u, \varphi) &= \ast(I(u, \varphi_1), \dots, I(u, \varphi_\ell)) \\ &= \ast(I_{\mathbf{L}, \Phi}^*([u], \varphi_1), \dots, I_{\mathbf{L}, \Phi}^*([u], \varphi_\ell)) = I_{\mathbf{L}, \Phi}^*([u], \varphi) \end{aligned}$$

Let φ be of the form $\Box\varphi'$ and let $I(u, \varphi) = \mathbf{v}_i$. Let $[v] \in W_{\mathbf{L}, \Phi}^*$ be such that $[u]R_{\mathbf{L}, \Phi, \varphi'}^*[v]$. By the definition of $R_{\mathbf{L}, \Phi, \varphi'}^*$, there exist $u' \in [u]$ and $v' \in [v]$ such that $u'R_{\varphi'}v'$. Since $u' \equiv_\Phi u$ and $\varphi \in \Phi$, $I(u', \varphi) = \mathbf{v}_i$ and since $u'R_{\varphi'}v'$, $\mathbf{v}_i \leq I(v', \varphi')$. Thus, by the induction hypothesis (because $\varphi' \in \Phi$),

$$I(v', \varphi') = I_{\mathbf{L}, \Phi}^*([v'], \varphi') = I_{\mathbf{L}, \Phi}^*([v], \varphi')$$

implying $\mathbf{v}_i \leq I_{\mathbf{L}, \Phi}^*([v], \varphi')$, that, in turn, implies

$$\mathbf{v}_i \leq \inf(\{I_{\mathbf{L}, \Phi}^*([v], \varphi') : [u]R_{\mathbf{L}, \Phi, \varphi'}^*[v]\}) = I_{\mathbf{L}, \Phi}^*([u], \varphi) \quad (5.2)$$

If $i = n$, then we have (5.1).

Assume $i \neq n$. Since $I(u, \varphi) = \mathbf{v}_i$, there exists $v \in W$ such that $uR_{\varphi'}v$ and $I(v, \varphi') = \mathbf{v}_i$. By the induction hypothesis, $I_{\mathbf{L}, \Phi}^*([v], \varphi') = \mathbf{v}_i$ and, since $uR_{\varphi'}v$, $[u]R_{\mathbf{L}, \Phi, \varphi'}^*[v]$, by the definition of $R_{\mathbf{L}, \Phi, \varphi'}^*$. Therefore,

$$\mathbf{v}_i \in \{I_{\mathbf{L}, \Phi}^*([v], \varphi') : [u]R_{\mathbf{L}, \Phi, \varphi'}^*[v]\}$$

implying

$$\mathbf{v}_i \geq \inf(\{I_{\mathbf{L}, \Phi}^*([v], \varphi') : [u]R_{\mathbf{L}, \Phi, \varphi'}^*[v]\}) = I_{\mathbf{L}, \Phi}^*([u], \varphi)$$

which, together with (5.2), implies (5.1).

The case in which φ is of the form $\Diamond\varphi'$ is dual to the above and is omitted. ■

Proposition 5.2. If a Kripke structure M is serial, then $M_{\mathbf{L}, \Phi}^*$ is also serial.

Proof Let $[u] \in W_{\mathbf{L}, \Phi}^*$ and let $\varphi \in \mathcal{F}$. Since M is serial, there exists $v \in W$ such that $uR_\varphi v$. By the definition of $R_{\mathbf{L}, \Phi, \varphi}^*$, $[u]R_{\mathbf{L}, \Phi, \varphi}^*[v]$. ■

Proposition 5.3. If a Kripke structure M is reflexive, then $M_{\mathbf{L}, \Phi}^*$ is also reflexive.

Proof Let $[u] \in W_{L,\Phi}^*$ and let $\varphi \in \mathcal{F}$. Since M is reflexive, $uR_\varphi u$. By the definition of $R_{L,\Phi,\varphi}^*$, $[u]R_{L,\Phi,\varphi}^*[u]$. ■

Definition 5.4. We say that a logic L possesses the *finite model property*, if for each finite set of sequents Σ and each sequent $\Gamma \rightarrow \Delta$ such that $\Sigma \not\vdash_L \Gamma \rightarrow \Delta$, there exists a finite Kripke structure $M \in \mathbf{C}_L$ (i.e. the set of worlds of M is finite) such that $M \models \Sigma$, but $M \not\models \Gamma \rightarrow \Delta$.

Theorem 5.5. *Each of the logics $\mathbf{mv-N}$, $\mathbf{mv-N}+(4.1)$, or $\mathbf{mv-N}+(4.2)+(4.3)$ possesses the finite model property.*

Proof Let L be any of the logics $\mathbf{mv-N}$, $\mathbf{mv-N}+(4.1)$, or $\mathbf{mv-N}+(4.2)+(4.3)$, and let Σ and $\Gamma \rightarrow \Delta$ be a finite set of sequents and a sequent, respectively, such that $\Sigma \not\vdash_L \Gamma \rightarrow \Delta$. By Theorems 2.2, 4.12, and 4.13, there exist a respective (“general,” serial, or reflexive) Kripke structure $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ such that $M \models \Sigma$, but $M \not\models \Gamma \rightarrow \Delta$.

Let Φ be the subformula-closure of the formulas appearing in Σ and in $\Gamma \rightarrow \Delta$.⁴ Then Φ is finite because Σ is finite, there are finitely many formulas in a sequent, and there are finitely many subformulas of each given formula.

Let $M^* = \langle W_{L,\Phi}^*, \{R_\varphi^*\}_{\varphi \in \mathcal{F}}, I_{L,\Phi}^* \rangle$, where R_φ^* is $R_{L,\Phi,\varphi}^*$, if $\varphi \in \Phi$ and is total⁵, otherwise.

The set of worlds W^* is finite, because an equivalence class of $\equiv_\Phi$ is characterized by a choice of values from $\mathbf{V}$ to each element of Φ (that is finite) and there are finitely many such choices (at most $n^{|\Phi|}$).

It immediately follows from Theorem 5.1 that for a sequent Σ consisting only of formulas from Φ , $M \models \Sigma$ if and only if $M^* \models \Sigma$, implying $M^* \models \Sigma$ and $M^* \not\models \Gamma \rightarrow \Delta$. Since, by Propositions 5.2 and 5.3, R_φ^* is serial or reflexive, respectively, for $\varphi \in \Phi$, and, obviously, total relations are both serial and reflexive, the proof is complete. ■

Corollary 5.6. *Each of the logics $\mathbf{mv-N}$, $\mathbf{mv-N}+(4.1)$, and $\mathbf{mv-N}+(4.2)+(4.3)$ is strongly decidable. That is, if Σ and $\Gamma \rightarrow \Delta$ are a finite set of sequents and a sequent, respectively, then there is a procedure deciding whether $\Sigma \vdash_L \Gamma \rightarrow \Delta$.*

Proof The decision procedure is standard. We in parallel search for a proof of $\Gamma \rightarrow \Delta$ from Σ and for a finite Kripke structure provided by Theorem 5.5 that satisfies Σ , but does not satisfy $\Gamma \rightarrow \Delta$. Note that, by Remark 2.1, the above satisfiability and unsatisfiability involve only (finitely many) relations $R_{L,\Phi,\varphi}^*$, where $\varphi \in \Phi$. ■

⁴ That is, Φ is the minimal set containing all formulas appearing in Σ and in $\Gamma \rightarrow \Delta$ together with all of their subformulas.

⁵ That is, $R_\varphi^* = W_{L,\Phi}^* \times W_{L,\Phi}^*$.

Chapter 6

Dunn-Belnap logic of pure necessitation $B_4\text{-}N$

The four-valued logic B_4 [Dun76; Bel75; Bel77] was suggested as the logic that a question-answering machine should use. The logic consists of four truth values, f, n, b and t . Its truth tables constitute a lattice, which the author called “the logical lattice” and denoted it by $L4$, see Figure 6.1. Namely, the conjunction of two values is their infimum and the disjunction is their supremum.

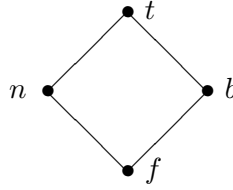


Figure 6.1: Dunn-Belnap’s logical lattice $L4$

In this chapter we briefly discuss a four-valued logic of pure necessitation, $B_4\text{-}N$, with the B_4 truth values. Again, the necessity connective is interpreted as the infimum of all relevant values and the possibility connective is interpreted as their supremum, and nothing is assumed about the underlying propositional connectives. That is, the B_4 propositional connectives are not assumed, and the lattice $L4$ is only used for interpreting the modal connectives $\Box$ and $\Diamond$.

In what follows, $V = \{f, n, b, t\}$ is a set of truth values ordered by

$$f < n, f < b, f < t, n < t, b < t$$

(see Figure 6.1). In particular, n and b are incomparable.

The set of formulas $\mathcal{F}$ is defined as in $mv\text{-}N$, but a (*Dunn-Belnap*) *labelled formula* is now a pair (φ, v) , where φ is a formula and $v \in V$. The intended meaning of such a

labelled formula is that $\mathbf{v}$ is the truth value associated with φ .

A **B₄ Kripke structure** (or B_4 *N-structure* or just Kripke structure) is defined exactly like **mv-N**-structures, with the same extension of I to $W \times \mathcal{F}$ (with $\inf(\emptyset) = \mathbf{t}$ and $\sup(\emptyset) = \mathbf{f}$), but note that now $\inf(\{I(v, \varphi) : v \in S_\varphi(u)\})$ and $\sup(\{I(v, \varphi) : v \in S_\varphi(u)\})$ might not be $\min(\{I(v, \varphi) : v \in S_\varphi(u)\})$ and $\max(\{I(v, \varphi) : v \in S_\varphi(u)\})$, even if $S_\varphi(u) \neq \emptyset$.

Naturally, we write $M, u \models (\varphi, \mathbf{v})$, if $I(u, \varphi) = \mathbf{v}$.

Let $\mathbf{u}, \mathbf{v} \in \mathbf{V}$. We denote the set of values between $\mathbf{u}$ and $\mathbf{v}$ by $[\mathbf{u}, \mathbf{v}]$:

$$[\mathbf{u}, \mathbf{w}] = \{\mathbf{v} \in \mathbf{V} : \mathbf{u} \leq \mathbf{v} \leq \mathbf{w}\}$$

That is, $[\mathbf{v}, \mathbf{v}] = \{\mathbf{v}\}$, for all $\mathbf{v} \in \mathbf{V}$, $[\mathbf{f}, \mathbf{n}] = \{\mathbf{f}, \mathbf{n}\}$, $[\mathbf{f}, \mathbf{b}] = \{\mathbf{f}, \mathbf{b}\}$, $[\mathbf{f}, \mathbf{t}] = \mathbf{V}$, $[\mathbf{n}, \mathbf{t}] = \{\mathbf{n}, \mathbf{t}\}$, $[\mathbf{b}, \mathbf{t}] = \{\mathbf{b}, \mathbf{t}\}$, and the other intervals $[\mathbf{n}, \mathbf{b}]$ and $[\mathbf{b}, \mathbf{n}]$ are empty.

As usual, $\overline{[\mathbf{u}, \mathbf{w}]}$ is the complement of $[\mathbf{u}, \mathbf{w}]$:

$$\overline{[\mathbf{u}, \mathbf{w}]} = \mathbf{V} \setminus [\mathbf{u}, \mathbf{w}]$$

We also define

$$(\varphi, \mathbf{v})^+ = \{\varphi\} \times [\mathbf{v}, \mathbf{t}]$$

and

$$(\varphi, \mathbf{v})^- = \{\varphi\} \times [\mathbf{f}, \mathbf{v}]$$

The proof system of **B₄-N** is presented below. It is similar to that of **mv-N**, the most important difference being the rules involving modal connectives.

The axioms are:

$$(\varphi, \mathbf{v}) \rightarrow (\varphi, \mathbf{v}) \tag{6.1}$$

$$(\varphi_1, \mathbf{v}_1), \dots, (\varphi_\ell, \mathbf{v}_\ell) \rightarrow (*(\varphi_1, \dots, \varphi_\ell), \mathbf{v}) \tag{6.2}$$

for each table entry $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ such that $*(\mathbf{v}_1, \dots, \mathbf{v}_\ell) = \mathbf{v}$,

$$(\Box\varphi, \mathbf{v}) \rightarrow (\Diamond\varphi, \mathbf{v})^+ \quad \mathbf{v} \neq \mathbf{t} \tag{6.3}$$

and

$$(\Box\varphi, \mathbf{t}) \rightarrow (\Diamond\varphi, \mathbf{t}), (\Diamond\varphi, \mathbf{f}) \tag{6.4}$$

and the logical rules are

$$\frac{(\varphi, \mathbf{v}) \rightarrow}{(\Box\varphi, \mathbf{v}) \rightarrow} \quad \mathbf{v} \in \{\mathbf{n}, \mathbf{b}\} \tag{6.5}$$

$$\frac{(\varphi, \mathbf{v}) \rightarrow}{(\Diamond\varphi, \mathbf{v}) \rightarrow} \quad \mathbf{v} \in \{\mathbf{n}, \mathbf{b}\} \tag{6.6}$$

$$\frac{(\varphi, \mathbf{f}) \rightarrow \quad (\varphi, \mathbf{v}) \rightarrow}{(\Box \varphi, \mathbf{f}) \rightarrow} \quad \mathbf{v} \in \{\mathbf{n}, \mathbf{b}\} \quad (6.7)$$

$$\frac{(\varphi, \mathbf{t}) \rightarrow \quad (\varphi, \mathbf{v}) \rightarrow}{(\Diamond \varphi, \mathbf{t}) \rightarrow} \quad \mathbf{v} \in \{\mathbf{n}, \mathbf{b}\} \quad (6.8)$$

$$\frac{(\varphi, \mathbf{f}) \rightarrow}{(\Box \varphi, \mathbf{f}) \rightarrow (\Diamond \varphi, \mathbf{t})} \quad (6.9)$$

$$\frac{(\varphi, \mathbf{t}) \rightarrow}{(\Diamond \varphi, \mathbf{t}) \rightarrow (\Box \varphi, \mathbf{f})} \quad (6.10)$$

and the structural rules below.

$\mathbf{v}$ -left-shift:

$$\frac{\Gamma, (\varphi, \mathbf{v}) \rightarrow \Delta}{\Gamma \rightarrow \Delta, \{\varphi\} \times \{\mathbf{v}\}} \quad (6.11)$$

$\mathbf{v}', \mathbf{v}''$ -right-shift:

$$\frac{\Gamma \rightarrow \Delta, (\varphi, \mathbf{v}')}{\Gamma, (\varphi, \mathbf{v}'') \rightarrow \Delta} \quad \mathbf{v}' \neq \mathbf{v}'' \quad (6.12)$$

$\mathbf{v}$ -left-weakening:

$$\frac{\Gamma \rightarrow \Delta}{\Gamma, (\varphi, \mathbf{v}) \rightarrow \Delta} \quad (6.13)$$

$\mathbf{v}$ -right-weakening:

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, (\varphi, \mathbf{v})} \quad (6.14)$$

$\mathbf{v}$ -cut:

$$\frac{\Gamma' \rightarrow \Delta', (\varphi, \mathbf{v}) \quad \Gamma'', (\varphi, \mathbf{v}) \rightarrow \Delta''}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''} \quad (6.15)$$

$\mathbf{v}', \mathbf{v}''$ -resolution:

$$\frac{\Gamma' \rightarrow \Delta', (\varphi, \mathbf{v}') \quad \Gamma'' \rightarrow \Delta'', (\varphi, \mathbf{v}'')}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''} \quad \mathbf{v}' \neq \mathbf{v}'' \quad (6.16)$$

Again, cut and resolution are derivable from each other, see [KF21, Proposition 3.3].

We shall also need the two following derivable rules. One is “multi-cut”

$$\frac{\{\Gamma_{\mathbf{v}}, (\varphi, \mathbf{v}) \rightarrow \Delta_{\mathbf{v}} : \mathbf{v} \in \mathbf{V}\}}{\bigcup_{\mathbf{v} \in \mathbf{V}} \Gamma_{\mathbf{v}} \rightarrow \bigcup_{\mathbf{v} \in \mathbf{V}} \Delta_{\mathbf{v}}} \quad (6.17)$$

that is derivable by cuts from the derivable sequent

$$\rightarrow \{\varphi\} \times \mathbf{V} \quad (6.18)$$

see [KF21, Proposition 3.4], and the other is “multi-shift”

$$\frac{\{\Gamma_{\mathbf{v}}, (\varphi, \mathbf{v}) \rightarrow \Delta_{\mathbf{v}} : \mathbf{v} \in \mathbf{W}\}}{\bigcup_{\mathbf{v} \in \mathbf{W}} \Gamma_{\mathbf{v}} \rightarrow \bigcup_{\mathbf{v} \in \mathbf{W}} \Delta_{\mathbf{v}}, \{\varphi\} \times \overline{\mathbf{W}}} \quad (6.19)$$

for $\mathbf{W} \subset \mathbf{V}$ and $\overline{\mathbf{W}} = \mathbf{V} \setminus \mathbf{W}$, see [KF21, Remark 3.5].

In the current chapter, we write $\vdash$ for a proof in $\mathbf{B}_4\text{-}\mathbf{N}$.

Theorem 6.1. *Let Σ and $\Gamma \rightarrow \Delta$ be a set of sequents and a sequent, respectively. Then $\Sigma \vdash \Gamma \rightarrow \Delta$ if and only if $\Sigma \models \Gamma \rightarrow \Delta$.*

6.1 Soundness of $\mathbf{B}_4\text{-}\mathbf{N}$

The proof of soundness is very similar to the “only if” part of Theorem 2.2, and we shall only show the most “interesting” parts of the proof.

Let $\Gamma \rightarrow \Delta$ be the conclusion of (6.7) and assume to the contrary that, for some $u \in W$, $M, u \models (\Box\varphi, \mathbf{f})$.

If $\mathbf{f} \in \{I(v, \varphi) \mid v \in S_\varphi(u)\}$, there exists a world $v \in W$ such that $M, v \models (\varphi, \mathbf{f})$, contradicting the induction hypothesis $M \models (\varphi, \mathbf{f}) \rightarrow$.

Therefore, $M, u \models (\Box\varphi, \mathbf{f})$ implies $\mathbf{n}, \mathbf{b} \in \{I(v, \varphi) : v \in S_\varphi(u)\}$. In particular, there exist worlds $v, w \in W$ such that $M, v \models (\varphi, \mathbf{n})$ and $M, w \models (\varphi, \mathbf{b})$. One of these worlds contradicts the induction hypothesis $M \models (\varphi, \mathbf{v}) \rightarrow$.

Now let $\Gamma \rightarrow \Delta$ be the conclusion of (6.9). Assume a world u satisfies $(\Box\varphi, \mathbf{f})$. If $\mathbf{f} \in \{I(v, \varphi) : v \in S_\varphi(u)\}$, then there exists $v \in W$ satisfying $(\varphi, \mathbf{f})$ contradicting $M \models (\varphi, \mathbf{f}) \rightarrow$. Therefore, $M, u \models (\Box\varphi, \mathbf{f})$ implies $\mathbf{b}, \mathbf{n} \in \{I(v, \varphi) : v \in S_\varphi(u)\}$, that, in turn, implies $M, u \models (\Diamond\varphi, \mathbf{t})$, as desired.

6.2 Completeness of $\mathbf{B}_4\text{-}\mathbf{N}$

Consistency and Σ -consistency are defined exactly as in Chapter 3. Lemma 3.1 and Lemma 3.2 (with obvious syntactic modifications, such as replacing $k \in \{1, \dots, n\}$ by $\mathbf{v} \in \mathbf{V}$, that we shall ignore in the future) are proved in the same way.

A $\mathbf{B}_4\text{-}\mathbf{N}$ - Σ -canonical structure is defined the same as well.

The proof of Theorem 3.3 for $\mathbf{B}_4\text{-}\mathbf{N}$ is very similar. Again, we shall focus on a specific case that is new to $\mathbf{B}_4\text{-}\mathbf{N}$. Namely, let the labelled formula be $(\Box\varphi', \mathbf{f}) \in u$.

If $\Sigma \not\vdash (\varphi', \mathbf{f}) \rightarrow$, then, by Lemma 3.1, there exists $v \in W_\Sigma$ such that $(\varphi', \mathbf{f}) \in v$. The label of $\Diamond\varphi'$ in u is clearly at least $\mathbf{f}$, so, by definition of $R_{\Sigma, \varphi'}$, $uR_{\Sigma, \varphi'}v$. By the induction hypothesis, $M_\Sigma, v \models (\varphi', \mathbf{f})$. Thus $M_\Sigma, u \models (\Box\varphi', \mathbf{f})$ as desired.

Now assume $\Sigma \vdash (\varphi', \mathbf{f}) \rightarrow$.

If there exists $\mathbf{v} \in \{\mathbf{n}, \mathbf{b}\}$ such that $\Sigma \vdash (\varphi', \mathbf{v}) \rightarrow$, then, by (6.7), $\Sigma \vdash (\Box\varphi', \mathbf{f}) \rightarrow$, contradicting the Σ -consistency of u .

Therefore, we have both $\Sigma \not\vdash (\varphi', \mathbf{n}) \rightarrow$ and $\Sigma \not\vdash (\varphi', \mathbf{b}) \rightarrow$. By Lemma 3.1, there exist $v, w \in W_\Sigma$ such that $(\varphi', \mathbf{n}) \in v$ and $(\varphi', \mathbf{b}) \in w$.

Since we assumed $\Sigma \vdash (\varphi', \mathbf{f}) \rightarrow$, by (6.9), $\Sigma \vdash (\Box\varphi', \mathbf{f}) \rightarrow (\Diamond\varphi', \mathbf{t})$, implying $\Sigma \vdash u \rightarrow (\Diamond\varphi', \mathbf{t})$. By maximal Σ -consistency of u , $(\Diamond\varphi', \mathbf{t}) \in u$. Therefore, since

$\mathbf{n}, \mathbf{b} \in [\mathbf{f}, \mathbf{t}]$, we have both $uR_{\Sigma, \varphi'} v$ and $uR_{\Sigma, \varphi'} w$. By the induction hypothesis, $M_{\Sigma}, v \models (\varphi', \mathbf{n})$ and $M_{\Sigma}, w \models (\varphi', \mathbf{b})$. Thus $M_{\Sigma}, u \models (\Box \varphi', \mathbf{f})$ as desired.

The rest of the proof of completeness – that is, Corollary 3.4 and the proof of the “if” part of Theorem 2.2 – is identical.

6.3 Two extensions of $B_4\text{-}N$

All of the results and proofs in Chapter 4 are applicable for $B_4\text{-}N$ in a straightforward manner (taking $\mathbf{t}$ for $\mathbf{n}$ in axiom (4.1), of course). The only proof that is slightly modified is the one of Remark 4.8:

1. $(\varphi, \mathbf{t}) \rightarrow$ assumption
2. $(\Diamond \varphi, \mathbf{t}) \rightarrow (\Box \varphi, \mathbf{f})$ follows from 1 by (6.10)
3. $(\Box \varphi, \mathbf{t}) \rightarrow (\Diamond \varphi, \mathbf{t})$ axiom (4.1)
4. $(\Box \varphi, \mathbf{t}) \rightarrow (\Box \varphi, \mathbf{f})$ follows from 2 and 3 by $\mathbf{t}$ -cut (6.15)
5. $(\Box \varphi, \mathbf{t}) \rightarrow$ follows from 4 by right-shift (6.12)

So, $B_4\text{-}N+(4.1)$ and $B_4\text{-}N+(4.2),(4.3)$ are sound and strongly complete with respect to serial and reflexive B_4 -structures, respectively.

6.4 Decidability of $B_4\text{-}N$ and its extensions

Filtrations are defined in the same manner. Again, we shall focus on one case in the proof of Theorem 5.1: φ is of the form $\Box \varphi'$ and $I(u, \varphi) = \mathbf{f}$.

If there exists $v \in W$ such that $uR_{\varphi'} v$ and $I(v, \varphi') = \mathbf{f}$, then, by the induction hypothesis, $I_{L, \Phi}^*([v], \varphi') = \mathbf{f}$ and, since $uR_{\varphi'} v$, $[u]R_{L, \Phi, \varphi'}^*[v]$, by the definition of $R_{L, \Phi, \varphi'}^*$. Therefore $I_{L, \Phi}^*([u], \varphi) = \mathbf{f}$ as desired.

Otherwise, there exist $v, w \in W$ such that $uR_{\varphi'} v$, $uR_{\varphi'} w$, $I(v, \varphi') = \mathbf{n}$ and $I(w, \varphi') = \mathbf{b}$. The rest is similar to the previous argument.

We conclude, like in Chapter 5, that $B_4\text{-}N$ and its extensions possess the finite model property and thus strongly decidable

Chapter 7

Many-valued normal modal logic *mv-K*

We shall develop ***mv-K***, the many valued counterpart of the normal modal logic K, taking a similar approach to that presented in [MT93], where K is developed from N.

Let $i, j \in \{1, \dots, n\}$. We denote the set of integers between i and j by $[i, j]$. That is

$$[i, j] = \{k \in \{1, \dots, n\} : i \leq k \leq j\}$$

By definition,

$$\overline{[i, j]} = \{1, \dots, n\} \setminus [i, j] = [1, i-1] \cup [j+1, n]$$

In particular, we define

$$(\varphi, k)^+ = \{\varphi\} \times [k, n]$$

and

$$(\varphi, k)^- = \{\varphi\} \times [1, k]$$

Definition 7.1. For a set of labelled formulas Γ , the set of labelled formulas $\Gamma^\times$ is defined as follows.

$$\Gamma^\times = \bigcup \{ \{\psi\} \times \overline{[i_\psi, j_\psi]} : (\Box\psi, i_\psi), (\Diamond\psi, j_\psi) \in \Gamma \}$$

That is, for all ψ such that $(\Box\psi, i_\psi), (\Diamond\psi, j_\psi) \in \Gamma$, $\Gamma^\times$ includes the set $\{\psi\} \times \overline{[i_\psi, j_\psi]}$ and nothing else.

Remark 7.2. In the above definition, it may be safely assumed that for all $\psi \in \mathcal{F}$, there exist at most one i_ψ and one j_ψ such that $(\Box\psi, i_\psi), (\Diamond\psi, j_\psi) \in \Gamma$. In other cases, the definition of $\Gamma^\times$ does not matter for our purposes.

The idea lying behind the definition of $\Gamma^\times$ is, that in a Kripke *model* (as will soon be defined), if uRv and u satisfies *every* element of Γ , then v satisfies *no* element of $\Gamma^\times$, see Lemma 7.5.

Many-Valued K, or **mv-K**, is the extension of **mv-N** (whose proof system was presented above) with the following rules:

$$\frac{(\varphi, k) \rightarrow \Gamma^\times}{(\Box\varphi, k), \Gamma \rightarrow} \quad k \neq n \quad (7.1)$$

$$\frac{(\varphi, k) \rightarrow \Gamma^\times}{(\Diamond\varphi, k), \Gamma \rightarrow} \quad k \neq 1 \quad (7.2)$$

A *many-valued normal modal logic* is any extension of **mv-K**.

In the case of normal modal logics, simpler interpretations than the ones presented in Chapter 2 can be used to semantically characterize the provability relation. We used the term *Kripke structure* to refer to interpretations with separate accessibility relations for different formulas. From now on, we will use the term *Kripke model* when referring to interpretations with one accessibility relation for all formulas. The terminology is adapted from [MT93; FMT92].

Definition 7.3. A many-valued Kripke structure $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ is called a *many-valued Kripke model* (or *many-valued K-model* or just *Kripke model*) if there exists a relation R on W such that $R_\varphi = R$ for all $\varphi \in \mathcal{F}$. In this case we denote it by $M = \langle W, R, I \rangle$.

Kripke models are a particular case of Kripke structures. In particular, the semantics given in Chapter 2 applies to models as well, but note that $S_\varphi(u)$ no longer depends on φ . Therefore, for models, it will be simply denoted by $S(u)$. That is, $S(u) = \{v \in W \mid uRv\}$.

Theorem 7.4. **mv-K** is sound and strongly complete with respect to Kripke models.

The proof of the soundness is standard and is presented below. The proof of the strong completeness is rather involved and is presented in the next chapter.

Lemma 7.5. Let $M = \langle W, R, I \rangle$ be a Kripke model, let $u, v \in W$ and let Γ be a set of labelled formulas. If uRv and u satisfies every element of Γ , then v satisfies no element of $\Gamma^\times$.

Proof Assume to the contrary that v satisfies an element of $\Gamma^\times$. That is, v satisfies a labelled formula in $\{\psi\} \times \overline{[i_\psi, j_\psi]}$, for some ψ such that

$$(\Box\psi, i_\psi), (\Diamond\psi, j_\psi) \in \Gamma$$

However, u satisfies both $(\Box\psi, i_\psi)$ and $(\Diamond\psi, j_\psi)$ which, together with uRv , implies $v_{i_\psi} \leq I(v, \psi) \leq v_{j_\psi}$. Thus, v cannot satisfy an element of $\{\psi\} \times \overline{[i_\psi, j_\psi]}$, and we have arrived at a contradiction. ■

Proof of the soundness part of Theorem 7.4 A Kripke model is a Kripke structure. Thus, following the proof of soundness for **mv-N**, it is sufficient to consider the cases of (7.1),(7.2) for a Kripke model M .

Let $M = \langle W, R, I \rangle$ be a Kripke model satisfying Σ .

Let $(\Box\varphi, k), \Gamma \rightarrow$ be the conclusion of (7.1) and assume to the contrary that there exists a world u satisfying $(\Box\varphi, k)$ and every labelled formula in Γ . Since $k \neq n$, $S(u) \neq \emptyset$. Thus, there exists $v \in S(u)$ satisfying (φ, k) .¹

By Lemma 7.5, v satisfies no element of $\Gamma^\times$. Therefore (and by the definition of the satisfiability relation $\models$, of course), $M, v \not\models (\varphi, k) \rightarrow \Gamma^\times$, contradicting the induction hypothesis $M \models (\varphi, k) \rightarrow \Gamma^\times$.

The case of (7.2) is dual to that of (7.1). ■

We conclude this chapter with examples of **mv-K** derivability, some of which will be used in the proof of the completeness theorem.

Example 7.6. Apply (7.1) with $\Gamma = \emptyset$ to obtain (2.5). Apply (7.2) with $\Gamma = \emptyset$ to obtain (2.6). Thus rules (2.5), (2.6) are redundant in **mv-K**.

Example 7.7. The sequents

$$(\Box\varphi, k) \rightarrow (\Diamond\varphi, k)^+ \quad k \neq n \tag{7.3}$$

and

$$(\Diamond\varphi, k) \rightarrow (\Box\varphi, k)^- \quad k \neq 1 \tag{7.4}$$

are **mv-K**-derivable. In particular, axiom (2.3) is redundant in **mv-K**.

The derivation of (7.3) is as follows, where, in steps 2_j and 3_j , $j < k$.

1. $(\varphi, k) \rightarrow (\varphi, k)$ axiom (2.1)
2. $(\varphi, k) \rightarrow \{\varphi\} \times [1, n]$ follows from 1 by $n - 1$ right weakenings (2.10)
- 2_j . $(\varphi, k) \rightarrow \{\varphi\} \times \overline{[k, j]}$ follows from 2, because, for $j < k$, $[k, j] = \emptyset$
- 3_j . $(\Box\varphi, k), (\Diamond\varphi, j) \rightarrow$ follows from 2_j by (7.1) with Γ being $\{(\Box\varphi, k), (\Diamond\varphi, j)\}$
4. $(\Box\varphi, k) \rightarrow (\Diamond\varphi, k)^+$ follows from all 3_j , $j < k$, by multi-shift (2.15)

The derivation of (7.4) is dual to that of (7.3) and is omitted.

Now, for $k \neq 1, n$, by k n, j -right-shifts (2.8), $j \leq k$, on (7.4), we obtain

$$(\Diamond\varphi, k), (\Box\varphi, n) \rightarrow \quad k = 2, 3, \dots, n - 1$$

from which, by multi-shift (2.15), we obtain

$$(\Box\varphi, n) \rightarrow (\Diamond\varphi, 1), (\Diamond\varphi, n) \tag{7.5}$$

¹ Actually, the existence of such a world v is all we need from the satisfiability of $(\Box\varphi, k)$.

That is, axiom (2.4) is redundant as well.

Dually, one can derive

$$(\Diamond\varphi, 1) \rightarrow (\Box\varphi, 1), (\Box\varphi, n) \quad (7.6)$$

Example 7.8. The sequent

$$(\Box\varphi, n), (\Diamond\varphi, 1) \rightarrow (\Box\psi, n) \quad (7.7)$$

is **mv-K**-derivable.² The derivation is as follows, where, in steps 2_i and 3_i , $i \neq n$.

1. $\rightarrow \{\varphi\} \times [1, n]$ derivable sequent (2.14)
- 2_i. $(\psi, i) \rightarrow \{\varphi\} \times [1, n]$ follows from 1 by i -left-weakening (2.9)
- 3_i. $(\Box\psi, i), (\Box\varphi, n), (\Diamond\varphi, 1) \rightarrow$ follows from 2_i by (7.1), with Γ being
 $\{(\Box\varphi, n), (\Diamond\varphi, 1)\}$, because $\overline{[n, 1]} = [1, n]$
4. $(\Box\varphi, n), (\Diamond\varphi, 1) \rightarrow (\Box\psi, n)$ follows from all 3_i, $i \neq n$, by multi-shift (2.15)

Dually, one can derive

$$(\Box\varphi, n), (\Diamond\varphi, 1) \rightarrow (\Diamond\psi, 1) \quad (7.8)$$

Example 7.9. The rule

$$\frac{\{(\varphi, \ell) \rightarrow (\psi, k_{\ell-i+1}) : \ell = i, i+1, \dots, j\}}{(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Box\psi, k_1), (\Box\psi, k_2), \dots, (\Box\psi, k_{j-i+1})} \quad i \leq j \quad (7.9)$$

is **mv-K**-derivable.³ The derivation is as follows.

We contend that sequent

$$(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Box\psi, k_1), (\Box\psi, k_2), \dots, (\Box\psi, k_{j-i+1}), (\Box\psi, n) \quad (7.10)$$

is derivable from the premises of this rule.

If $\{k_1, \dots, k_{j-i+1}, n\} = [1, n]$, then it is derivable from (2.14), with φ being $\Box\psi$, by two left-weakenings (2.9).

Otherwise, let $k \neq k_1, k_2, \dots, k_{j-i+1}, n$.

Applying right-shift (2.8) on each assumption, we obtain the sequents

$$(\varphi, \ell), (\psi, k) \rightarrow \quad \ell = i, i+1, \dots, j$$

² Semantically, this sequent is valid, because, if a world u satisfies both $(\Box\varphi, n)$ and $(\Diamond\varphi, 1)$, then $S(u) = \emptyset$. Therefore u satisfies $(\Box\psi, n)$ for every ψ .

³ Semantically, this rule is valid, because, if a world u satisfies $(\Box\varphi, i)$ and $(\Diamond\varphi, j)$, for $i \leq j$, then for all $v \in S(u)$, $\mathbf{v}_i \leq I(\mathbf{v}, \varphi) \leq \mathbf{v}_j$. Therefore, by the premises of the rule, $I(v, \psi) \in \{\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \dots, \mathbf{v}_{k_{j-i+1}}\}$, implying $I(u, \Box\psi) = \min\{I(v, \psi) : v \in S(u)\} \in \{\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \dots, \mathbf{v}_{k_{j-i+1}}\}$.

from which, by multi-shift (2.15) (or multi-cut (2.13) if $i = 1$ and $j = n$), we obtain

$$(\psi, k) \rightarrow \{\varphi\} \times \overline{[i, j]}$$

Since $k \neq n$, we may apply rule (7.1) to this sequent, which results in

$$(\Box\psi, k), (\Box\varphi, i), (\Diamond\varphi, j) \rightarrow$$

Since the latter is derivable for all $k \neq k_1, k_1, \dots, k_{j-i+1}, n$, by multi-shift (2.15), we obtain (7.10), which proves our contention.

If $n \in \{k_1, k_2, \dots, k_{j-i+1}\}$ we are done. Therefore, we assume that $n \neq k_1, k_2, \dots, k_{j-i+1}$.

From (7.5), and (7.10), by cut, we derive

$$(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Box\psi, k_1), \dots, (\Box\psi, k_{j-i+1}), (\Diamond\psi, n), (\Diamond\psi, 1) \quad (7.11)$$

To proceed, we observe that, dually, to the derivation of (7.10), from the assumptions of (7.9), one can derive

$$(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Diamond\psi, k_1), \dots, (\Diamond\psi, k_{j-i+1}), (\Diamond\psi, 1)$$

from which, since $n \neq k_1, k_2, \dots, k_{j-i+1}$, by right-shifts (2.8), we obtain

$$(\Diamond\psi, n), (\Box\varphi, i), (\Diamond\varphi, j) \rightarrow \quad (7.12)$$

From (7.11) and (7.12), by cut, we derive

$$(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Box\psi, k_1), \dots, (\Box\psi, k_{j-i+1}), (\Diamond\psi, 1) \quad (7.13)$$

Since $i \leq j$, either $i \neq n$ or $j \neq 1$.

Assume $i \neq n$. Interchanging φ and ψ in (7.7), we obtain

$$(\Box\psi, n), (\Diamond\psi, 1) \rightarrow (\Box\varphi, n)$$

from which, by i -right-shift (2.8), we derive

$$(\Box\psi, n), (\Diamond\psi, 1), (\Box\varphi, i) \rightarrow \quad (7.14)$$

Similarly, if $j \neq 1$, then, interchanging φ and ψ in (7.8), we obtain

$$(\Box\psi, n), (\Diamond\psi, 1) \rightarrow (\Diamond\varphi, 1)$$

from which, by j -right-shift (2.8), we derive

$$(\Box\psi, n), (\Diamond\psi, 1), (\Diamond\varphi, j) \rightarrow \quad (7.15)$$

From either (7.14), if $i \neq n$, or (7.15), if $j \neq 1$, and (7.10), by cut, we have

$$(\Box\varphi, i), (\Diamond\varphi, j), (\Diamond\psi, 1) \rightarrow (\Box\psi, k_1), \dots, (\Box\psi, k_{j-i+1})$$

from which and (7.13), by cut, we obtain the desired conclusion

$$(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Box\psi, k_1), \dots, (\Box\psi, k_{j-i+1})$$

of (7.9).

The dual rule of (7.16)

$$\frac{\{(\varphi, \ell) \rightarrow (\psi, k_{\ell-i+1}) : \ell = i, i+1, \dots, j\}}{(\Box\varphi, i), (\Diamond\varphi, j) \rightarrow (\Diamond\psi, k_1), (\Diamond\psi, k_2), \dots, (\Diamond\psi, k_{j-i+1})} \quad i \leq j \quad (7.16)$$

is derivable as well.

Example 7.10. The rule

$$\frac{(\varphi, n) \rightarrow (\psi, n)}{(\Box\varphi, n) \rightarrow (\Box\psi, n)} \quad (7.17)$$

is **mv-K**-derivable.

This rule can be thought of as a generalization of the rule

$$\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

from [OM57].

The derivation of (7.17) is as follows.

1. $(\varphi, n) \rightarrow (\psi, n)$ assumption
2. $(\Box\varphi, n), (\Diamond\varphi, n) \rightarrow (\Box\psi, n)$ follows from 1, by (7.9) with $i = j = n = k_1$
3. $(\Box\varphi, n) \rightarrow (\Diamond\varphi, 1), (\Diamond\varphi, n)$ (7.5)
4. $(\Box\varphi, n) \rightarrow (\Box\psi, n), (\Diamond\varphi, 1)$ follows from 2 and 3 by n -cut (2.11)
5. $(\Box\varphi, n), (\Diamond\varphi, 1) \rightarrow (\Box\psi, n)$ (7.7)
6. $(\Box\varphi, n) \rightarrow (\Box\psi, n)$ follows from 4 and 5 by 1-cut (2.11)

We shall also need the following generalization of multi-shift (2.15).

Proposition 7.11. *The following generalization of multi-shift (2.15)*

$$\frac{\{\Gamma_{k_1, \dots, k_\ell}, (\varphi_1, k_1), \dots, (\varphi_\ell, k_\ell) \rightarrow \Delta_{k_1, \dots, k_\ell} : k_1 \in K_1, \dots, k_\ell \in K_\ell\}}{\bigcup_{k_1 \in K_1, \dots, k_\ell \in K_\ell} \Gamma_{k_1, \dots, k_\ell} \rightarrow \bigcup_{k_1 \in K_1, \dots, k_\ell \in K_\ell} \Delta_{k_1, \dots, k_\ell}, \{\varphi_1\} \times \overline{K_1}, \dots, \{\varphi_\ell\} \times \overline{K_\ell}} \quad (7.18)$$

is **mv-K**-derivable.

Proof Derivability of (7.18) can be shown as follows. Fix $k_2 \in K_2, \dots, k_\ell \in K_\ell$ and apply multi-shift (2.15), (or multi-cut (2.13), if $K_1 = [1, n]$) to the set of sequents

$$\{\Gamma_{k_1, \dots, k_\ell}, (\varphi_1, k_1), \dots, (\varphi_\ell, k_\ell) \rightarrow \Delta_{k_1, \dots, k_\ell} : k_1 \in K_1\}$$

to obtain

$$\bigcup_{k_1 \in K_1} \Gamma_{k_1, \dots, k_\ell, (\varphi_2, k_2), \dots, (\varphi_\ell, k_\ell)} \rightarrow \bigcup_{k_1 \in K_1} \Delta_{k_1, \dots, k_\ell, \{\varphi_1\} \times \overline{K_1}}$$

Then, fix $k_3 \in K_3, \dots, k_\ell \in K_\ell$ and apply multi-shift (2.15), (or multi-cut (2.13), if $K_2 = [1, n]$) to the set of sequents

$$\left\{ \bigcup_{k_1 \in K_1} \Gamma_{k_1, \dots, k_\ell, (\varphi_2, k_2), \dots, (\varphi_\ell, k_\ell)} \rightarrow \bigcup_{k_1 \in K_1} \Delta_{k_1, \dots, k_\ell, \{\varphi_1\} \times \overline{K_1}} : k_2 \in K_2 \right\}$$

to obtain

$$\bigcup_{k_1 \in K_1, k_2 \in K_2} \Gamma_{k_1, \dots, k_\ell, (\varphi_3, k_3), \dots, (\varphi_\ell, k_\ell)} \rightarrow \bigcup_{k_1 \in K_1, k_2 \in K_2} \Delta_{k_1, \dots, k_\ell, \{\varphi_1\} \times \overline{K_1}, \{\varphi_2\} \times \overline{K_2}}$$

and so on (ℓ times in total).⁴ ■

⁴ The proof is, actually, by induction on ℓ that is omitted.

Chapter 8

The canonical model theorem and the proof of the completeness part of Theorem 7.4

Like in [MT93], to every structure we assign a model. Namely, for a structure $M = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ we define $M^\cap = \langle W, \bigcap_{\varphi \in \mathcal{F}} R_\varphi, I \rangle$. Recall that I is originally defined only on $W \times \mathcal{P}$; its extensions to $W \times \mathcal{F}$ on M and on $M^\cap$ may, in general, differ from each other. We denote these extensions by $I_M, I_{M^\cap}$ respectively.

Let Σ be $\mathbf{mv}\text{-}\mathbf{K}$ -consistent and let $M_{\mathbf{mv}\text{-}\mathbf{K}, \Sigma}$ be the $\mathbf{mv}\text{-}\mathbf{K}\text{-}\Sigma$ -canonical structure (recall the discussion at Chapter 4, with $\mathbf{mv}\text{-}\mathbf{K}$ as $\mathbf{L}$). We call $M_{\mathbf{mv}\text{-}\mathbf{K}, \Sigma}^\cap$ the Σ -canonical model.

From now on, we enumerate $\mathcal{F}$ as $\psi_1, \psi_2, \dots$

Remark 8.1. The accessibility relation $R = \bigcap_{\varphi \in \mathcal{F}} R_\varphi$ of the canonical model can be explicitly defined as follows.

For worlds $\Gamma', \Gamma'' \in W_{\mathbf{mv}\text{-}\mathbf{K}, \Sigma}$, $\Gamma' R \Gamma''$ if and only if for the unique $i_1, i_2, \dots, j_1, j_2, \dots$, and $k_1, k_2, \dots$ such that

$$(\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \dots \in \Gamma'$$

and

$$(\psi_1, k_1), (\psi_2, k_2), \dots \in \Gamma''$$

provided by Lemma 4.4, $i_m \leq k_m \leq j_m$ for all $m = 1, 2, \dots$ ¹

Lemma 8.2. *Let Γ be a Σ -consistent set of formulas and let*

$$(\Box\varphi', k)(\Box\psi, i), (\Diamond\psi, j) \in \Gamma \tag{8.1}$$

where $k \neq n$. Then $i \leq j$.

¹ Equivalently, $\Gamma' R \Gamma''$ if and only if $(\Gamma')^\times \cap \Gamma'' = \emptyset$.

Proof Assume to the contrary that $i > j$. We distinguish among the cases of, $i \neq n$, $j \neq 1$, and $i = n$ and $j = 1$.

If $i \neq n$, then

1. $\Gamma \rightarrow (\Box\psi, i)$ follows from axiom (2.1), with φ being $\Box\psi$ and k being i , and (8.1)
2. $\Gamma \rightarrow (\Diamond\psi, j)$ follows from axiom (2.1), with φ being $\Diamond\psi$ and k being j , and (8.1)
3. $(\Box\psi, i) \rightarrow (\Diamond\psi, i)^+$ (7.3) with φ being ψ and k being i
4. $\Gamma \rightarrow (\Diamond\psi, i)^+$ follows from 1 and 3 by cut
5. $\Gamma \rightarrow$ follows from 2 and 4 by $n - i$ resolutions (2.12)

However, $\Sigma \vdash \Gamma \rightarrow$ contradicts the Σ -consistency of Γ .

The case of $j \neq 1$ is dual to that of $i \neq n$ and is omitted.

Let $i = n$ and $j = 1$. Then

1. $\Gamma \rightarrow (\Box\varphi', k)$ follows from axiom (2.1), with φ being $\Box\varphi'$
2. $\Gamma \rightarrow (\Box\psi, n)$ follows from axiom (2.1), with φ being $\Box\psi$ and k being n , and (8.1)
3. $\Gamma \rightarrow (\Diamond\psi, 1)$ follows from axiom (2.1), with φ being $\Diamond\psi$ and k being 1 , and (8.1)
4. $(\Box\psi, n), (\Diamond\psi, 1) \rightarrow (\Box\varphi', n)$ (7.7) with φ being ψ and ψ being φ'
5. $\Gamma \rightarrow (\Box\varphi', n)$ follows from 2, 3, and 4 by two cuts
6. $\Gamma \rightarrow$ follows from 1 and 5 by resolution (2.12)

Again, $\Sigma \vdash \Gamma \rightarrow$ contradicts the Σ -consistency of Γ . ■

Lemma 8.3. *Let (φ', k) be a labelled formula and let $i_1, j_1, i_2, j_2, \dots \in \{1, \dots, n\}$ such that $i_m \leq j_m$, for all $m = 1, 2, \dots$. If for all $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots$, the set $\{(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots\}$ is $\mathbf{L}$ - Σ -inconsistent, then there exists a non-negative integer L such that*

$$\Sigma \vdash_{\mathbf{L}} (\varphi', k) \rightarrow \{\psi_1\} \times \overline{[i_1, j_1]}, \{\psi_2\} \times \overline{[i_2, j_2]}, \dots, \{\psi_L\} \times \overline{[i_L, j_L]}$$

Proof We contend that there exists a non-negative integer L such that

$$\Sigma \vdash_{\mathbf{L}} (\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots, (\psi_L, k_L) \rightarrow \quad (8.2)$$

$$k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots, k_L \in [i_L, j_L]$$

For the proof of our contention, we consider a tree T whose nodes are sets of labelled formulas of the form

$$\{(\psi_1, k_1), (\psi_2, k_2), \dots, (\psi_m, k_m)\} \quad (8.3)$$

$$k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots, k_m \in [i_m, j_m],$$

$m = 0, 1, \dots$, such that each node (8.3) is $\mathbf{L}\text{-}\Sigma$ -consistent, when (φ', k) is added to it as an element; and the successors of a node (8.3) are nodes of the form

$$\{(\psi_1, k_1), (\psi_2, k_2), \dots, (\psi_m, k_m), (\psi_{m+1}, k_{m+1})\}$$

where $k_{m+1} \in [i_{m+1}, j_{m+1}]$.

Thus, nodes (8.3) are of height m . In particular, the root of T is $\emptyset$, if $\{(\varphi', k)\}$ is $\mathbf{L}\text{-}\Sigma$ -consistent. Otherwise, T is empty.

This tree T is of a finite branching degree, because a node of height m has at most $j_{m+1} - i_{m+1} + 1$ successors. Also, T has no infinite paths. Indeed, an infinite path would correspond to a choice of $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots$. However, the set of labelled formulas $\{(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots\}$ is $\mathbf{L}\text{-}\Sigma$ -inconsistent. Thus, it includes a node that becomes $\mathbf{L}\text{-}\Sigma$ -inconsistent, when (φ', k) is added to it, which contradicts the definition of T . Therefore, by the contraposition of the König infinite lemma [Kön26], T is finite.

Let H be the height of T .² Then, for $L = H + 1$, we have (8.2), which proves our contention.

Now, from (8.2), by (7.18) we obtain

$$\Sigma \vdash_{\mathbf{L}} (\varphi', k) \rightarrow \{\psi_1\} \times \overline{[i_1, j_1]}, \{\psi_2\} \times \overline{[i_2, j_2]}, \dots, \{\psi_L\} \times \overline{[i_L, j_L]}$$

as desired. ■

Theorem 8.4. *Let Σ be $\mathbf{mv}\text{-}\mathbf{K}$ -consistent and let $M = M_{\mathbf{mv}\text{-}\mathbf{K}, \Sigma} = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ be the $\mathbf{mv}\text{-}\mathbf{K}\text{-}\Sigma$ -canonical structure. Then for all $\varphi \in \mathcal{F}$ and $u \in W$, $I_{M^\cap}(u, \varphi) = I_M(u, \varphi)$.*

Proof The proof is by induction on the complexity of φ .

For $\varphi \in \mathcal{P}$, this is clear since $I_{M^\cap}(u, \varphi) = I(u, \varphi) = I_M(u, \varphi)$.

If $\varphi = *(\varphi_1, \dots, \varphi_\ell)$, then by the induction hypothesis,

$$I_{M^\cap}(u, \varphi_1) = I_M(u, \varphi_1), \dots, I_{M^\cap}(u, \varphi_\ell) = I_M(u, \varphi_\ell)$$

Therefore,

$$I_{M^\cap}(u, \varphi) = *(I_{M^\cap}(u, \varphi_1), \dots, I_{M^\cap}(u, \varphi_\ell)) = *(I_M(u, \varphi_1), \dots, I_M(u, \varphi_\ell)) = I_M(u, \varphi)$$

Now assume $\varphi = \Box\varphi'$. Since $R = \bigcap_{\varphi \in \mathcal{F}} R_\varphi \subseteq R_{\varphi'}$, we know $S(u) \subseteq S_{\varphi'}(u)$ (where $S(u) = \{v \in W \mid uRv\}$). Thus,

$$I_{M^\cap}(u, \varphi) = \inf\{I_{M^\cap}(v, \varphi') \mid v \in S(u)\} \geq \inf\{I_{M^\cap}(v, \varphi') \mid v \in S_{\varphi'}(u)\}$$

² We put $H = -1$, if T is empty.

$$= \inf\{I_M(v, \varphi') \mid v \in S_{\varphi'}(u)\} = I_M(u, \varphi)$$

(we used the induction hypothesis to see $I_{M^\cap}(v, \varphi') = I_M(v, \varphi')$). If $I_M(u, \varphi) = \mathbf{v}_n$ then $I_{M^\cap}(u, \varphi) \geq \mathbf{v}_n$ implying $I_{M^\cap}(u, \varphi) = \mathbf{v}_n$ and we are done. Therefore assume $I_M(u, \varphi) \neq \mathbf{v}_n$ and denote it by $\mathbf{v}_k$. It is sufficient to prove $I_{M^\cap}(u, \varphi) \leq \mathbf{v}_k$. For this, it is sufficient to show that there exists $v \in S(u)$ such that $I_{M^\cap}(v, \varphi') = \mathbf{v}_k$.

Assume $(\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \dots \in u$. We claim that it is sufficient to show that there exist $v \in W$ and $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots$ such that $(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots \in v$. Indeed, uRv by definition of R , and, by part (i) of Corollary 4.5, $I_M(v, \varphi') = \mathbf{v}_k$, so by the induction hypothesis $I_{M^\cap}(v, \varphi') = \mathbf{v}_k$ as desired.

By Lemma 4.3, for existence of such $k_1, k_2, \dots$ and v , it suffices to show that there exist $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots$ such that the set of labelled formulas $\{(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots\}$ is **mv-K-Σ**-consistent.

For the proof, assume to the contrary that for all $k_1 \in [i_1, j_1], k_2 \in [i_2, j_2], \dots$, the set $\{(\varphi', k), (\psi_1, k_1), (\psi_2, k_2), \dots\}$ is **mv-K-Σ**-inconsistent.

Note that, by Lemma 8.2, $i_m \leq j_m$, for all $m = 1, 2, \dots$

By Lemma 8.3, there exists a non-negative integer L such that

$$\Sigma \vdash_{\mathbf{mv-K}} (\varphi', k) \rightarrow \{\psi_1\} \times \overline{[i_1, j_1]}, \{\psi_2\} \times \overline{[i_2, j_2]}, \dots, \{\psi_L\} \times \overline{[i_L, j_L]}$$

from which, by (7.1), we obtain

$$\Sigma \vdash_{\mathbf{mv-K}} (\Box\varphi', k), (\Box\psi_1, i_1), (\Diamond\psi_1, j_1), (\Box\psi_2, i_2), (\Diamond\psi_2, j_2), \dots, (\Box\psi_L, i_L), (\Diamond\psi_L, j_L) \rightarrow$$

that contradicts the **mv-K-Σ**-consistency of u (note that indeed $(\Box\varphi', k) \in u$, since $I_M(u, \Box\varphi') = \mathbf{v}_k$ and by part (i) of Corollary 4.5).

The case of $\varphi = \Diamond\varphi'$ is dual, using (7.2), and is omitted. ■

An immediate consequence of Theorem 8.4 is that for every sequent $\Gamma \rightarrow \Delta$, $M \models \Gamma \rightarrow \Delta$ if and only if $M^\cap \models \Gamma \rightarrow \Delta$. Thus, we also see that for every set of sequents Σ , $M \models \Sigma$ if and only if $M^\cap \models \Sigma$.

Proof of the completeness part of Theorem 7.4 Assume $\Sigma \not\models_{\mathbf{mv-K}} \Gamma \rightarrow \Delta$ and let M be the **mv-K-Σ**-canonical structure. By parts (ii) and (iii) of Corollary 4.5, $M \models \Sigma$ but $M \not\models \Gamma \rightarrow \Delta$. Thus, by Theorem 8.4, $M^\cap \models \Sigma$ but $M^\cap \not\models \Gamma \rightarrow \Delta$. Since $M^\cap$ is a Kripke model, the proof of completeness of **mv-K** is complete. ■

Chapter 9

Extensions of $mv\text{-}\mathbf{K}$

Like in Chapter 4, we observe that the proof of Theorem 8.4 and its consequences apply for any extension $\mathbf{L}$ of $mv\text{-}\mathbf{K}$ over the same language. We state it formally.

Corollary 9.1. *Let $\mathbf{L}$ be an extension of $mv\text{-}\mathbf{K}$, let Σ be $\mathbf{L}$ -consistent and let $M = M_{\mathbf{L},\Sigma} = \langle W, \{R_\varphi\}_{\varphi \in \mathcal{F}}, I \rangle$ be the $\mathbf{L}$ - Σ -canonical structure, defined in Chapter 4. (We call $M^\cap = M_{\mathbf{L},\Sigma}^\cap$ the $\mathbf{L}$ - Σ -canonical model.)*

- (i) (Cf. Theorem 8.4.) *For all $\psi \in \mathcal{F}$ and $u \in W$, $I_{M^\cap}(u, \psi) = I_M(u, \psi)$.*
- (ii) (Cf. Corollary 3.4.) $M^\cap \models \Sigma$.
- (iii) (Cf. the completeness part of Theorem 7.4.) *If $\Sigma \not\vdash_{\mathbf{L}} \Gamma \rightarrow \Delta$, then $M^\cap \not\models \Gamma \rightarrow \Delta$.*

We proceed with some extensions of $mv\text{-}\mathbf{K}$ which are sound and strongly complete for the many-valued Kripke models defined below.

Definition 9.2. A binary relation $R \subseteq W \times W$ is called *Euclidian*, if for all $u, v, w \in W$, uRv and uRw imply vRw .

Definition 9.3. A many-valued Kripke model $M = \langle W, R, I \rangle$ is called *serial/ reflexive/ transitive/ symmetric/ Euclidean*, if the accessibility relation R is serial/ reflexive/ transitive/ symmetric/ Euclidean, respectively.

In this chapter, the many-valued modal logics, we shall deal with, result from $mv\text{-}\mathbf{K}$ by adding some subsets of the following axioms, as well as axioms (4.1), (4.2) and (4.3).

$$(\Box\varphi, k) \rightarrow (\Box\Box\varphi, k)^+ \tag{9.1}$$

$$(\Diamond\Diamond\varphi, k) \rightarrow (\Diamond\varphi, k)^+ \tag{9.2}$$

$$(\varphi, k) \rightarrow (\Box\Diamond\varphi, k)^+ \tag{9.3}$$

$$(\Diamond \Box \varphi, k) \rightarrow (\varphi, k)^+ \quad (9.4)$$

$$(\Diamond \varphi, k) \rightarrow (\Box \Diamond \varphi, k)^+ \quad (9.5)$$

$$(\Diamond \Box \varphi, k) \rightarrow (\Box \varphi, k)^+ \quad (9.6)$$

Theorem 9.4. *Let $\mathbf{L}$ be an extension of $\mathbf{mv-K}$, Σ be an $\mathbf{L}$ -consistent set of sequents and $M_{\mathbf{L}, \Sigma}$ be the $\mathbf{L}$ - Σ -canonical structure.*

- (i) *If (4.1) is an axiom of $\mathbf{L}$, then $M_{\mathbf{L}, \Sigma}^\cap$ is serial.*
- (ii) *If (4.2) and (4.3) are axioms of $\mathbf{L}$, then $M_{\mathbf{L}, \Sigma}^\cap$ is reflexive.*
- (iii) *If (9.1) and (9.2) are axioms of $\mathbf{L}$, then $M_{\mathbf{L}, \Sigma}^\cap$ is transitive.*
- (iv) *If (9.3) and (9.4) are axioms of $\mathbf{L}$, then $M_{\mathbf{L}, \Sigma}^\cap$ is symmetric.*
- (v) *If (9.5) and (9.6) are axioms of $\mathbf{L}$, then $M_{\mathbf{L}, \Sigma}^\cap$ is Euclidean.*

Proof Denote the relation of the canonical model $M_{\mathbf{L}, \Sigma}^\cap$ by $R_{\mathbf{L}, \Sigma}$.

- (i) Note that the restriction $k \neq n$ of rule (7.1) may be removed:

1. $(\varphi, n) \rightarrow \Gamma^\times$ assumption
2. $(\Diamond \varphi, n), \Gamma \rightarrow$ follows from 1 by (7.2)
3. $(\Box \varphi, n) \rightarrow (\Diamond \varphi, n)$ axiom (4.1)
4. $(\Box \varphi, n), \Gamma \rightarrow$ follows from 2 and 3 by cut

Also, in the proof of Lemma 8.2, it cannot be the case that $i = n$ and $j = 1$, because $i = n$ implies $j = n$. Therefore, in the proof of Theorem 8.4, in the case where φ is of the form $\Box \varphi'$ and $k \neq n$, the argument applies for $k = n$ as well. Thus, $S(u)$ is nonempty.

- (ii) By Theorem 4.10, $M_{\mathbf{L}, \Sigma}$ is reflexive (recall that it means each $R_{\mathbf{L}, \Sigma, \varphi}$, $\varphi \in \mathcal{F}$, is reflexive). Since an intersection of reflexive relations is reflexive, $M_{\mathbf{L}, \Sigma}^\cap$ is reflexive.
- (iii) Assume $uR_{\mathbf{L}, \Sigma}v$ and $vR_{\mathbf{L}, \Sigma}w$ and let

$$(\Box \psi, i), (\Diamond \psi, j), (\Box \Box \psi, i_u), (\Diamond \Diamond \psi, j_u) \in u$$

$$(\Box \psi, i_v), (\Diamond \psi, j_v) \in v$$

and

$$(\psi, k) \in w$$

for some $i, j, i_u, j_u, i_v, j_v, k$.

We contend that $i \leq i_u$. For the proof, assume to the contrary that $i > i_u$. Then

1. $(\Box\psi, i) \rightarrow (\Box\Box\psi, i)^+$ axiom (9.1)
2. $(\Box\psi, i), (\Box\Box\psi, i_u) \rightarrow$ follows from 1 by i', i_u -right-shifts (2.8), $i' \geq i$
3. $u \rightarrow$ follows from 2, because $(\Box\psi, i), (\Box\Box\psi, i_u) \in u$

Thus, the assumption $i > i_u$ contradicts the $\mathbf{L}$ - Σ -consistency of u .

Dually, $j_u \leq j$.

By the definition of $R_{\mathbf{L}, \Sigma}$, $uR_{\mathbf{L}, \Sigma}v$ implies $i_u \leq i_v$ and $j_u \geq j_v$, and $vR_{\mathbf{L}, \Sigma}w$ implies $i_v \leq k \leq j_v$, which, together with $i \leq i_u$ and $j_u \leq j$, in turn, imply

$$i \leq i_u \leq i_v \leq k \leq j_v \leq j_u \leq j$$

Thus, $k \in [i, j]$. Since this holds for every formula $\psi \in \mathcal{F}$, by the definition of $R_{\mathbf{L}, \Sigma}$, $uR_{\mathbf{L}, \Sigma}w$.

(iv) Assume $uR_{\mathbf{L}, \Sigma}v$ and let

$$(\psi, k), (\Box\Diamond\psi, i_u), (\Diamond\Box\psi, j_u) \in u$$

and

$$(\Box\psi, i), (\Diamond\psi, j) \in v$$

for some i, j, k, i_u, j_u .

By the definition of $R_{\mathbf{L}, \Sigma}$, $uR_{\mathbf{L}, \Sigma}v$ implies both $i_u \leq j$ and $j_u \geq i$.

We contend that $k \leq i_u$. For the proof, assume to the contrary that $k > i_u$. Then

1. $(\psi, k) \rightarrow (\Box\Diamond\psi, k)^+$ axiom (9.3)
2. $(\psi, k), (\Box\Diamond\psi, i_u) \rightarrow$ follows from 1 by i', i_u -right-shifts (2.8), $i' \geq k$
3. $u \rightarrow$ follows from 2, because $(\psi, k), (\Box\Diamond\psi, i_u) \in u$

Thus, the assumption $k > i_u$ contradicts the $\mathbf{L}$ - Σ -consistency of u .

Dually, $j_u \leq k$, implying

$$i \leq j_u \leq k \leq i_u \leq j$$

Thus, $k \in [i, j]$. Since this holds for every formula $\psi \in \mathcal{F}$, by the definition of $R_{\mathbf{L}, \Sigma}$, $vR_{\mathbf{L}, \Sigma}u$.

(v) Assume $uR_{\mathbf{L}, \Sigma}v$ and $uR_{\mathbf{L}, \Sigma}w$ and let, for some $i, j, k, i_u, j_u, i'_u, j'_u$,

$$(\Box\psi, i_u), (\Diamond\psi, j_u), (\Box\Diamond\psi, i'_u), (\Diamond\Box\psi, j'_u) \in u$$

$$(\Box\psi, i), (\Diamond\psi, j) \in v$$

and

$$(\psi, k) \in w$$

We contend that $j_u \leq i'_u$. For the proof, assume to the contrary that $j_u > i'_u$. Then

1. $(\Diamond\psi, j_u) \rightarrow (\Box\Diamond\psi, j_u)^+$ axiom (9.5)
2. $(\Diamond\psi, j_u), (\Box\Diamond\psi, i'_u) \rightarrow$ follows from 1, by i', i'_u -right-shifts (2.8), $i' \geq j_u$
3. $u \rightarrow$ follows from 2, because $(\Diamond\psi, j_u), (\Box\Diamond\psi, i'_u) \in u$

Thus, the assumption $i'_u < j_u$ contradicts the $\mathbf{L}\text{-}\Sigma$ -consistency of u .

Dually, $j'_u \leq i_u$.

Since $uR_{\mathbf{L},\Sigma}w$, $i_u \leq k \leq j_u$ and, since $uR_{\mathbf{L},\Sigma}v$, $i'_u \leq j$ and $j'_u \geq i$, which, together with $j_u \leq i'_u$ and $j'_u \leq i_u$, imply

$$i \leq j'_u \leq i_u \leq k \leq j_u \leq i'_u \leq j$$

Thus, $k \in [i, j]$. Since this holds for every formula $\psi \in \mathcal{F}$, by the definition of $R_{\mathbf{L},\Sigma}$, $vR_{\mathbf{L},\Sigma}w$. ■

Next we define the many-valued counterparts of the two-valued modal logics D, T, K4, S4, B, and S5.

Definition 9.5.

- The many-valued modal logic $\mathbf{mv}\text{-}\mathbf{D}$ is obtained from $\mathbf{mv}\text{-}\mathbf{K}$ by adding to it (4.1).
- The many-valued modal logic $\mathbf{mv}\text{-}\mathbf{T}$ is obtained from $\mathbf{mv}\text{-}\mathbf{K}$ by adding to it (4.2) and (4.3).
- The many-valued modal logic $\mathbf{mv}\text{-}\mathbf{K4}$ is obtained from $\mathbf{mv}\text{-}\mathbf{K}$ by adding to it (9.1) and (9.2).
- The many-valued modal logic $\mathbf{mv}\text{-}\mathbf{S4}$ is obtained from $\mathbf{mv}\text{-}\mathbf{T}$ by adding to it (9.1) and (9.2).
- The many-valued modal logic $\mathbf{mv}\text{-}\mathbf{B}$ is obtained from $\mathbf{mv}\text{-}\mathbf{K}$ by adding to it (9.3) and (9.4).
- The many-valued modal logic $\mathbf{mv}\text{-}\mathbf{S5}$ is obtained from $\mathbf{mv}\text{-}\mathbf{T}$ by adding to it (9.5) and (9.6).

The above many-valued logics, but $\mathbf{mv}\text{-}\mathbf{D}$ are defined by pairs of axioms - the many valued counterpart of the two-valued one and its dual, because the logics under consideration do not necessarily have negation. Thus, unlike in the two-valued case, $\square$

and $\diamond$ are not interdefinable. We address the extension of these logics with negation in Chapter 11.

The above axioms are many-valued counterparts of axioms **D**, see [HC84, p. 29], **T**, **4**, **B**, see [HC84, p. 10], and **E**, see [HC84, p. 11].

Theorem 9.6.

- (i) **mv-D** is sound and (strongly) complete with respect to serial Kripke models.
- (ii) **mv-T** is sound and (strongly) complete with respect to reflexive Kripke models.
- (iii) **mv-K4** is sound and (strongly) complete with respect to transitive Kripke models.
- (iv) **mv-S4** is sound and (strongly) complete with respect to reflexive and transitive (preordered) Kripke models.
- (v) **mv-B** is sound and (strongly) complete with respect to symmetric Kripke models.
- (vi) **mv-S5** is sound and (strongly) complete with respect to reflexive and Euclidean Kripke models.¹

Proof The completeness of each logic follows immediately from parts (ii) and (iii) of Corollary 9.1 and Theorem 9.4. It remains to show soundness.

- (i) We just need to show that serial models $M = \langle W, R, I \rangle$ satisfy (4.1), but since a serial model is a serial structure, this follows immediately from Theorem 4.12.
- (ii) We just need to show that reflexive models $M = \langle W, R, I \rangle$ satisfy (4.2) and (4.3), but since a reflexive model is a reflexive structure, this follows immediately from Theorem 4.13.
- (iii) We just need to show that transitive models satisfy (9.1) and (9.2). Let $M = \langle W, R, I \rangle$ be a transitive model and let $u \in W$. For every $v \in S(u)$, by transitivity, $S(v) \subseteq S(u)$. Therefore, $I(v, \Box\varphi) \geq I(u, \Box\varphi)$, implying

$$\inf\{I(v, \Box\varphi) : v \in S(u)\} \geq I(u, \Box\varphi)$$

that, in turn, implies $I(u, \Box\varphi) \leq I(u, \Box\Box\varphi)$. Thus (9.1) is satisfied. The case of (9.2) is dual.

- (iv) Follows from (ii) and (iii).
- (v) We just need to show that symmetric models satisfy (9.3) and (9.4). Let $M = \langle W, R, I \rangle$ be a symmetric model and let $u \in W$. For every $v \in S(u)$, by symmetry, $u \in S(v)$. Therefore, $I(u, \varphi) \leq I(v, \Diamond\varphi)$, implying

$$I(u, \varphi) \leq \inf\{I(v, \Diamond\varphi) : v \in S(u)\}$$

¹ This is the class of all Kripke models whose accessibility relation is an equivalence relation.

that, in turn, implies $I(u, \varphi) \leq I(u, \Box \Diamond \varphi)$. Thus (9.3) is satisfied. The case of (9.4) is dual.

(vi) It is sufficient to show that Euclidean models satisfy (9.5) and (9.6). Let $M = \langle W, R, I \rangle$ be an Euclidean model and let $u \in W$. For every $v \in S(u)$, by the Euclidean property, $S(u) \subseteq S(v)$ (if uRw then vRw , since uRv). Therefore, $I(u, \Diamond \varphi) \leq I(v, \Diamond \varphi)$, implying

$$I(u, \Diamond \varphi) \leq \inf \{ I(v, \Diamond \varphi) : v \in S(u) \}$$

that, in turn, implies $I(u, \Diamond \varphi) \leq I(u, \Box \Diamond \varphi)$. Thus (9.5) is satisfied. The case of (9.6) is dual. ■

Chapter 10

Decidability of $mv\text{-}K$ and its extensions

Recall the definition of an L -filtration of a structure M through Φ , given in Chapter 5 for L that is one of $mv\text{-}N, mv\text{-}N+(4.1)$ or $mv\text{-}N+(4.2), (4.3)$.

We extend this definition to the new logics we defined. That is, L can be any of the logics $mv\text{-}K, mv\text{-}D, mv\text{-}T, mv\text{-}K4, mv\text{-}S4, mv\text{-}B$ and $mv\text{-}S5$, and C_L is the class of the respective Kripke models, see Theorem 9.6. $M = \langle W, R, I \rangle$ is now assumed to be a model, and so will be its filtration. The definition of $W_{L,\Phi}^*$ and $I_{L,\Phi}^*$ is the same for all logics (same as in Chapter 5). The definition of the accessibility relation $R_{L,\Phi}^*$ depends on L .

- For $mv\text{-}K, mv\text{-}D$ and $mv\text{-}T$, $[u]R_{L,\Phi}^*[v]$ if and only if there exist $u' \in [u]$ and $v' \in [v]$ such that $u'Rv'$.
- For $mv\text{-}K4$, $[u]R_{L,\Phi}^*[v]$ if and only if
 - for all $\Box\varphi' \in \Phi$, $I(u, \Box\varphi') \leq I(v, \Box\varphi')$ and $I(u, \Box\varphi') \leq I(v, \varphi')$; and
 - for all $\Diamond\varphi' \in \Phi$, $I(u, \Diamond\varphi') \geq I(v, \Diamond\varphi')$ and $I(u, \Diamond\varphi') \geq I(v, \varphi')$.
- For $mv\text{-}S4$, $[u]R_{L,\Phi}^*[v]$ if and only if
 - for all $\Box\varphi' \in \Phi$, $I(u, \Box\varphi') \leq I(v, \Box\varphi')$; and
 - for all $\Diamond\varphi' \in \Phi$, $I(u, \Diamond\varphi') \geq I(v, \Diamond\varphi')$.
- For $mv\text{-}B$, $[u]R_{L,\Phi}^*[v]$ if and only if
 - for all $\Box\varphi' \in \Phi$, $I(u, \Box\varphi') \leq I(v, \varphi')$ and $I(v, \Box\varphi') \leq I(u, \varphi')$; and
 - for all $\Diamond\varphi' \in \Phi$, $I(u, \Diamond\varphi') \geq I(v, \varphi')$ and $I(v, \Diamond\varphi') \geq I(u, \varphi')$.
- For $mv\text{-}S5$, $[u]R_{L,\Phi}^*[v]$ if and only if
 - for all $\Box\varphi' \in \Phi$, $I(u, \Box\varphi') = I(v, \Box\varphi')$ and

– for all $\Diamond\varphi' \in \Phi$, $I(u, \Diamond\varphi') = I(v, \Diamond\varphi')$.

Lemma 10.1. *Let M be in $\mathbf{C}_L$ and let $M_{L,\Phi}^*$ be its L -filtration through Φ . If uRv , then*

$$[u]R_{L,\Phi}^*[v] \quad (10.1)$$

Proof For L being **mv-K**, **mv-D** or **mv-T**, since uRv , (10.1) follows from the definition of $R_{L,\Phi}^*$.

For L being **mv-K4** or **mv-S4**, since uRv , $I(u, \Box\varphi') \leq I(v, \varphi')$ and $I(u, \Diamond\varphi') \geq I(v, \varphi')$. Moreover, since M is transitive, $S(v) \subseteq S(u)$, implying $I(u, \Box\varphi') \leq I(v, \Box\varphi')$ and $I(u, \Diamond\varphi') \geq I(v, \Diamond\varphi')$. Since this holds for all formulas in the forms $\Box\varphi'$ or $\Diamond\varphi'$ in Φ , (10.1) follows from the definition of $R_{L,\Phi}^*$.

For L being **mv-B**, since uRv , $I(u, \Box\varphi') \leq I(v, \varphi')$ and $I(u, \Diamond\varphi') \geq I(v, \varphi')$. Moreover, since M is symmetric, $I(v, \Box\varphi') \leq I(u, \varphi')$ and $I(v, \Diamond\varphi') \geq I(u, \varphi')$. Since this holds for all formulas in the forms $\Box\varphi'$ or $\Diamond\varphi'$ in Φ , (10.1) follows from the definition of $R_{L,\Phi}^*$.

Finally, for L being **mv-S5**, since uRv and R is an equivalence relation, $I(u, \Box\varphi') = I(v, \Box\varphi')$ and $I(u, \Diamond\varphi') = I(v, \Diamond\varphi')$. Since this holds for all formulas in the forms $\Box\varphi'$ or $\Diamond\varphi'$ in Φ , (10.1) follows from the definition of $R_{L,\Phi}^*$.

This completes the proof of (10.1). ■

Theorem 10.2. *Let M be in $\mathbf{C}_L$ and let $M_{L,\Phi}^*$ be its L -filtration through Φ . Then*

- For all $\varphi \in \Phi$ and $u \in W$, $I(u, \varphi) = I_{L,\Phi}^*([u], \varphi)$ and
- $M_{L,\Phi}^*$ is in $\mathbf{C}_L$.

Proof We do not need to consider the cases of L being **mv-K**, **mv-D**, or **mv-T**, since they follow from Theorem 5.1. Indeed, the L -filtration (for any of the above three logics) of a model M is the same as its **mv-N**-filtration. Also, a model is serial/ reflexive as a model if and only if it is serial/ reflexive as a structure, respectively.

For the other logics, the proof of the first part of the theorem is by induction on the complexity of φ .

The cases where $\varphi \in \mathcal{P}$ or $\varphi = *(\varphi_1, \dots, \varphi_\ell)$ are handled like in the proof of Theorem 5.1.

Let φ be of the form $\Box\varphi'$ and let $I(u, \varphi) = v_k$. Let $[v] \in W_{L,\Phi}^*$ be such that $[u]R_{L,\Phi}^*[v]$. We contend that

$$v_k \leq \inf(\{I_{L,\Phi}^*([v], \varphi') : [u]R_{L,\Phi}^*[v]\}) = I_{L,\Phi}^*([u], \varphi) \quad (10.2)$$

For L being **mv-K4** or **mv-B**, by the definition of $R_{L,\Phi}^*$, $I(u, \Box\varphi') \leq I(v, \varphi')$. By the induction hypothesis, $I(v, \varphi') = I_{L,\Phi}^*([v], \varphi')$, implying $v_k \leq I_{L,\Phi}^*([v], \varphi')$, which, in turn, implies (10.2).

For L being **mv-S4** or **mv-S5**, by the definition of $R_{L,\Phi}^*$, $I(u, \Box\varphi') \leq I(v, \Box\varphi')$. In addition, by reflexivity of M , $I(v, \Box\varphi') \leq I(v, \varphi')$, implying $I(u, \Box\varphi') \leq I(v, \varphi')$. From this point, the proof of $v_k \leq I_{L,\Phi}^*([v], \varphi')$ is the same as in the previous case.

This proves our contention (10.2). To complete the proof of the first part of the theorem, we have to show that

$$v_k = \inf(\{I_{L,\Phi}^*([v], \varphi') : [u]R_{L,\Phi}^*[v]\}) = I_{L,\Phi}^*([u], \varphi) \quad (10.3)$$

If $k = n$, (10.3) follows from (10.2).

Assume $k \neq n$. Since $I(u, \varphi) = v_k$, there exists $v \in W$ such that uRv and $I(v, \varphi') = v_k$. By Lemma 10.1, $[u]R_{L,\Phi}^*[v]$.

By the induction hypothesis, $I_{L,\Phi}^*([v], \varphi') = v_k$, implying

$$v_k \in \{I_{L,\Phi}^*([v], \varphi') : [u]R_{L,\Phi}^*[v]\}$$

implying

$$v_k \geq \inf(\{I_{L,\Phi}^*([v], \varphi') : [u]R_{L,\Phi}^*[v]\}) = I_{L,\Phi}^*([u], \varphi)$$

which, together with (10.2), provides the desired equality (10.3).

The case in which φ is of the form $\Diamond\varphi'$ is dual to the above and is omitted.

It remains to show that $M_{L,\Phi}^*$ is in $\mathbf{CL}$.

For L being **mv-K4**, the transitivity of $M_{L,\Phi}^*$ can be shown as follows. Let $[u]R_{L,\Phi}^*[v]$ and $[v]R_{L,\Phi}^*[w]$ and let $\Box\varphi' \in \Phi$. Then, by the definition of $R_{L,\Phi}^*$, we have

$$I(u, \Box\varphi') \leq I(v, \Box\varphi') \leq I(w, \Box\varphi')$$

implying $I(u, \Box\varphi') \leq I(w, \Box\varphi')$. Also,

$$I(u, \Box\varphi') \leq I(v, \Box\varphi') \leq I(w, \varphi')$$

implying $I(u, \Box\varphi') \leq I(w, \varphi')$. Dually, one can show $I(u, \Diamond\varphi') \geq I(w, \Diamond\varphi')$ and $I(u, \Diamond\varphi') \geq I(w, \varphi')$. Then, by the definition of $R_{L,\Phi}^*$, $[u]R_{L,\Phi}^*[w]$ as desired.

For L being **mv-S4**, reflexivity and transitivity of $R_{L,\Phi}^*$ follow from reflexivity and transitivity of $\leq$, respectively.

For L being **mv-B**, $R_{L,\Phi}^*$ is symmetric, because it is defined by a symmetric relation.

For L being **mv-S5**, $R_{L,\Phi}^*$ is an equivalence relation, because so is equality =. ■

Now, the finite model property of each of the logics, and their strong decidability, follow immediately by the same arguments as in Chapter 5 (Theorem 5.5 and Corollary 5.6).

Theorem 10.3. *Each of the logics considered above possesses the finite model property.*

Corollary 10.4. *Each of the logics considered above is strongly decidable.*

Chapter 11

Duality of $\Box$ and $\Diamond$ via negation

In [MT93], the logic contains negation $\neg$ ($\neg t = f$, $\neg f = t$), and the modal operator $\Diamond$ is defined as the De Morgan dual $\neg\Box\neg$ of $\Box$. Consequently, the satisfiability of formulas of the form $\Diamond\varphi$ in the logic N depends merely on the relation $R_{\neg\varphi}$, which, in general, has no connection to the relation R_φ and, thus, to the satisfiability of $\Box\varphi$ (it is only related to the satisfiability of $\Box\neg\varphi$).

On the other hand, in $mv-N$ and $mv-K$, we do not assume the existence of any specific connective, and we define $\Diamond$ independently (not as an abbreviation), both semantically and syntactically via the proof system. In $mv-N$, the satisfiability of $\Diamond\varphi$ depends on the relation R_φ , just like the satisfiability of $\Box\varphi$, and there is no relation to $R_{\neg\varphi}$. This is a main difference between N and $mv-N$.

In particular, this means that there is no hope to find a definition for the unary connective $\neg$ such that the sequent $(\neg\Box\neg\varphi, k) \rightarrow (\Diamond\varphi, k)$ is derivable in $mv-N$. Indeed, take any structure M with $W = \{u\}$, $R_p = \{(u, u)\}$, $R_{\neg p} = \emptyset$ and $I(u, p) = v_k$ for some k .

Then $M, u \models (\Box\neg p, n)$, implying $M, u \models (\neg\Box\neg p, \neg n)$ ¹. Thus (by the assumed sequent), $M, u \models (\Diamond p, \neg n)$, so $M, u \models (p, \neg n)$ ². So $v_k = \neg(v_n)$. However, we chose k arbitrarily. That is, $\neg(v_n)$ cannot be uniquely defined.

In $mv-K$, however, we *can* define the truth table for negation $\neg$ in such a way that $\Box$ and $\Diamond$ become the De Morgan dual. That is, the sequents

$$(\Diamond\varphi, k) \rightarrow (\neg\Box\neg\varphi, k) \tag{11.1}$$

and

$$(\Box\varphi, k) \rightarrow (\neg\Diamond\neg\varphi, k) \tag{11.2}$$

are provable in $mv-K$.³ We shall show that this is the only appropriate definition of

¹We write $\neg k$ for the index of $\neg(v_k)$.

²The last step is the one that is not valid in N , i.e., if we defined $\Diamond$ as an abbreviation. (As noted above, the value of $\Diamond p$ in u would have no connection to the value of p in the elements of $S_p(u)$, only of $S_{\neg p}(u)$.)

³In particular, in the three-valued logics of Łukasiewicz [Łuk20] and Kleene [Kle38], these connec-

negation, for which (11.1) and (11.2) are derivable in $\mathbf{mv}\text{-}\mathbf{K}$.

The truth table of $\neg$ is

$$\neg(\mathbf{v}_k) = \mathbf{v}_{n-k+1} \quad k = 1, 2, \dots, n \quad (11.3)$$

That is,

$$\neg(\mathbf{v}_1) = \mathbf{v}_n, \neg(\mathbf{v}_2) = \mathbf{v}_{n-1}, \dots, \neg(\mathbf{v}_{n-1}) = \mathbf{v}_2, \text{ and } \neg(\mathbf{v}_n) = \mathbf{v}_1$$

Therefore, axioms (2.2) for $\neg$ are

$$(\varphi, k) \rightarrow (\neg\varphi, n - k + 1)$$

Example 11.1. Sequents

$$(\neg\varphi, n - k + 1) \rightarrow (\varphi, k) \quad (11.4)$$

are $\mathbf{mv}\text{-}\mathbf{N}$ derivable (with negation defined as in (11.3)).

The derivation is as follows.

- 1 _{$j \neq k$} . $(\varphi, j) \rightarrow (\neg\varphi, n - j + 1), \quad j \neq k$ axiom (2.2)
- 2 _{$j \neq k$} . $(\varphi, j), (\neg\varphi, n - k + 1) \rightarrow$ follows from 1 _{j} by
 $n - j + 1, n - k + 1$ -right-shift (2.8)
3. $(\neg\varphi, n - k + 1) \rightarrow (\varphi, k)$ follows from 2 _{$j \neq k$} by multi-shift (2.15)

Remark 11.2. Sequents (11.1) and (11.2) immediately imply their reversals. For (11.1), since each sequent in the set

$$\{(\Diamond\varphi, k') \rightarrow (\neg\Box\neg\varphi, k') : k' \neq k\}$$

is derivable, by right shifts, we derive

$$\{(\Diamond\varphi, k'), (\neg\Box\neg\varphi, k) \rightarrow : k' \neq k\}$$

from which, by multi-shift, we obtain

$$(\neg\Box\neg\varphi, k) \rightarrow (\Diamond\varphi, k)$$

and, dually, for (11.2).

Theorem 11.3. *Let $\neg$ be a unary connective. Then, sequents (11.1) and (11.2) are derivable in $\mathbf{mv}\text{-}\mathbf{K}$ if and only if, for all $k = 1, 2, \dots, n$, $\neg(\mathbf{v}_k) = \mathbf{v}_{n-k+1}$.*

tives are interdefinable.

Proof For the “only if” part of the theorem, we show first that, for all $k = 1, 2, \dots, n$,

$$\neg(\neg(\mathbf{v}_k)) = \mathbf{v}_k \quad (11.5)$$

Indeed, let $M = \langle W, R, I \rangle$ be a Kripke model such that, for some $u \in W$, $S(u) = \{u\}$ and $I(u, p) = \mathbf{v}_k$. Then $I(u, \Diamond p) = \mathbf{v}_k$ and, by (11.1),

$$I(u, \neg\Box\neg p) = \mathbf{v}_k \quad (11.6)$$

Since $I(u, \neg p) = \neg(\mathbf{v}_k)$ and $S(u) = \{u\}$, $I(u, \Box\neg p) = \neg(\mathbf{v}_k)$, implying $I(u, \neg\Box\neg p) = \neg(\neg(\mathbf{v}_k))$, which, together with (11.6), in turn, implies (11.5).

In particular, $\neg : \mathbf{V} \rightarrow \mathbf{V}$ is an invertible function.

Next, we shall prove that $\neg$ is an order-reversing function. For the proof, assume to the contrary that $\mathbf{v}_i < \mathbf{v}_j$, but $\neg(\mathbf{v}_i) < \neg(\mathbf{v}_j)$. Let $M = \langle W, R, I \rangle$ be a Kripke model such that, for some $u, v, w \in W$, $S(u) = \{v, w\}$, $I(v, p) = \mathbf{v}_i$, and $I(w, p) = \mathbf{v}_j$. Then, $I(u, \Diamond p) = \mathbf{v}_j$, implying, by (11.1), $I(u, \neg\Box\neg p) = \mathbf{v}_j$. However, $I(u, \Box\neg p) = \neg(\mathbf{v}_i)$, implying

$$I(u, \neg\Box\neg p) = \neg(\neg(\mathbf{v}_i)) = \mathbf{v}_i$$

Thus, $\mathbf{v}_i = \mathbf{v}_j$, contradicting our assumption $\mathbf{v}_i < \mathbf{v}_j$.

Clearly, the only order-reversing invertible function from $\mathbf{V}$ to $\mathbf{V}$ is (11.3).

It remains to prove the “if” part of the theorem. We prove it semantically.

Let $M = \langle W, R, I \rangle$ be a Kripke model, $u \in W$, and let $\varphi \in \mathcal{F}$. Then, it follows from (11.3) that

$$\sup\{I(v, \varphi) : v \in S(u)\} = \neg(\inf\{I(v, \neg\varphi) : v \in S(u)\})$$

implying the validity of (11.1).

The proof of (11.2) is dual. ■

Remark 11.4. If we define negation as above, then rule (7.2) becomes redundant, which can be shown as follows.

- | | |
|---|--|
| 1. $(\varphi, k) \rightarrow \Gamma^\times, \quad k \neq 1$ | assumption of (7.2) |
| 2. $(\neg\varphi, n - k + 1) \rightarrow (\varphi, k)$ | (11.4) |
| 3. $(\neg\varphi, n - k + 1) \rightarrow \Gamma^\times, \quad n - k + 1 \neq n$ | follows from 1 and 2 by cut |
| 4. $(\Box\neg\varphi, n - k + 1), \Gamma \rightarrow, \quad n - k + 1 \neq n$ | follows from 3 by (7.1) |
| 5. $(\neg\Box\neg\varphi, k) \rightarrow (\Box\neg\varphi, n - k + 1)$ | (11.4) |
| 6. $(\neg\Box\neg\varphi, k), \Gamma \rightarrow, \quad k \neq 1$ | follows from 4 and 5 by cut |
| 7. (7.2) | because, by (11.1), $\neg\Box\neg$ is $\Diamond$ |

Also, it can be shown that (4.3), (9.2), (9.4), and (9.6) follow from (4.2), (9.1), (9.3), and (9.5), respectively, and vice-versa.

We conclude this chapter by showing what is obtained by adding (11.1) and (11.2) (which, as noted above, are not theorems of $\mathbf{mv-N}$) to $\mathbf{mv-N}$ (with negation defined as in (11.3)).

Theorem 11.5. *Adding (11.1),(11.2) as axioms to $\mathbf{mv-N}$ results in a logic $(\mathbf{mv-N} + (11.1),(11.2))$ that is sound and (strongly) complete with respect to Kripke structures satisfying $R_\varphi = R_{\neg\varphi}$ for all $\varphi \in \mathcal{F}$. (In particular, $R_\varphi = R_{\neg\varphi} = R_{\neg\neg\varphi} = R_{\neg\neg\neg\varphi} = \dots$)*

Proof For soundness. observe that the semantic proofs of (11.1),(11.2) above use only the fact that $S_\varphi(u) = S_{\neg\varphi}(u)$.

For completeness, consider the $\mathbf{mv-N} + (11.1),(11.2)$ - Σ -canonical structure. Assume $uR_\varphi v$, $(\Box\varphi, i), (\Diamond\varphi, j) \in u, (\varphi, k) \in v$.

By (11.1), $\Sigma \vdash_{\mathbf{mv-N} + (11.1),(11.2)} u \rightarrow (\neg\Box\neg\varphi, j)$, thus, by (11.4), $\Sigma \vdash_{\mathbf{mv-N} + (11.1),(11.2)} u \rightarrow (\Box\neg\varphi, \neg j)$.

Similarly, by (11.2), $\Sigma \vdash_{\mathbf{mv-N} + (11.1),(11.2)} u \rightarrow (\Diamond\neg\varphi, \neg i)$.

Also, $\Sigma \vdash_{\mathbf{mv-N} + (11.1),(11.2)} v \rightarrow (\neg\varphi, \neg k)$. Therefore (since u and v are maximal consistent) $(\Box\neg\varphi, \neg j), (\Diamond\neg\varphi, \neg i) \in u, (\neg\varphi, \neg k) \in v$.

Since $uR_\varphi v$, $i \leq k \leq j$, implying $\neg j \leq \neg k \leq \neg i$, thus $uR_{\neg\varphi} v$. We thus proved $R_\varphi \subseteq R_{\neg\varphi}$. Similarly, using the reversals of (11.1),(11.2), we can prove $R_\varphi \supseteq R_{\neg\varphi}$. So, the canonical structure satisfies $R_\varphi = R_{\neg\varphi}$ and the proof is complete. ■

We thereby obtained a logic strictly between $\mathbf{mv-N}$ and $\mathbf{mv-K}$, and it is the weakest one satisfying the equivalences $\Diamond \leftrightarrow \neg\Box\neg$ and $\Box \leftrightarrow \neg\Diamond\neg^4$.

⁴Note, however, that it does *not* mean we can replace $\Diamond$ by $\neg\Box\neg$ inside any formula in this logic. For example, the value of $\Box\Diamond\varphi$ in a world might differ from the value of $\Box\neg\Box\neg\varphi$ in the same world, since $R_{\Diamond\varphi}$ may differ from $R_{\neg\Box\neg\varphi} = R_{\Box\neg\varphi}$. (In $\mathbf{mv-K}$, however, it *is* possible to replace subformulas by equivalent subformulas.)

Chapter 12

Embedding many-valued intuitionistic logic into $mv\text{-}S4$

In [Tak86], following [Rou70], Takano defined a quite general notion of many-valued intuitionistic logic, that we shall denote by $mv\text{-}IL$. We focus on the semantics, because we embed $mv\text{-}IL$ into $mv\text{-}S4$ semantically. Also, we restrict ourselves to the case of linearly ordered set of truth values V in which $mv\text{-}IL$ -interpretations may be defined, recursively, as follows.

The language of $mv\text{-}IL$ is that of many-valued propositional logic, i.e., it does not contain the modal connectives $\Box$ or $\Diamond$.

An $mv\text{-}IL$ -interpretation $M = \langle W, R, I \rangle$ is a preordered (reflexive and transitive) many-valued Kripke model satisfying the (monotonic valuation) requirement below.

For all propositional variables $p \in \mathcal{P}$ and for all $u, v \in W$ such that uRv ,

$$I(u, p) \leq I(v, p)$$

The definition of I extends to formulas of the form $*(\varphi_1, \dots, \varphi_\ell)$ as

$$I(u, *(\varphi_1, \dots, \varphi_\ell)) = \inf\{*(I(v, \varphi_1), \dots, I(v, \varphi_\ell)) : v \in S(u)\} \quad (12.1)$$

A straightforward induction on the formula complexity shows that I is monotonic not only on $W \times \mathcal{P}$, but on the whole $W \times \mathcal{F}$.

We write $M, u \models_{mv\text{-}IL} (\varphi, k)$, if $I(u, \varphi) = v_k$. For a sequent $\Gamma \rightarrow \Delta$ and a set of sequents Σ , we define the relations $M, u \models_{mv\text{-}IL} \Gamma \rightarrow \Delta$, $M \models_{mv\text{-}IL} \Gamma \rightarrow \Delta$, $M \models_{mv\text{-}IL} \Sigma$, and $\Sigma \models_{mv\text{-}IL} \Gamma \rightarrow \Delta$ like in the beginning of Chapter 2.

Our translation of $mv\text{-}IL$ to $mv\text{-}S4$, is a generalization of the two-valued case (first suggested in [Göd33]).

Definition 12.1. Let φ be a formula in the language of $mv\text{-}IL$. The translation φ^t of an $mv\text{-}IL$ formula φ is obtained from φ by inserting $\Box$ before each of its subformulas. That is, φ^t is defined recursively as follows.

- For a propositional variable p , p^t is $\Box p$, and
- if φ is of the form $\ast(\varphi_1, \dots, \varphi_\ell)$, then φ^t is $\Box \ast(\varphi_1^t, \dots, \varphi_\ell^t)$.

Lemma 12.2. *Let $M = \langle W, R, I \rangle$ be a preordered Kripke model and let $\widehat{M} = \langle W, R, \widehat{I} \rangle$ be such that, for all $u \in W$ and all $p \in \mathcal{P}$, $\widehat{I}(u, p) = I(u, \Box p)$. Then $\widehat{M}$ is an **mv-IL**-interpretation, and, for all $u \in W$ and all formulas φ in the language of **mv-IL**,*

$$\widehat{I}(u, \varphi) = I(u, \varphi^t) \quad (12.2)$$

Proof To show $\widehat{M}$ is an **mv-IL**-interpretation, we need to show that, for all $u, v \in W$ such that uRv and for all $p \in \mathcal{P}$, $\widehat{I}(u, p) \leq \widehat{I}(v, p)$, i.e., by the definition of $\widehat{I}$, we need to show $I(u, \Box p) \leq I(v, \Box p)$, which is clear, because M is transitive.

The proof of (12.2) is by induction on the complexity of φ (extending $\widehat{I}$ to an intuitionistic valuation).

The basis, i.e., the case of φ being a propositional variable, is by the definition of $\widehat{I}$, and, for the induction step, if φ is of the form $\ast(\varphi_1, \dots, \varphi_\ell)$, then

$$\begin{aligned} \widehat{I}(u, \varphi) &= \inf\{\ast(\widehat{I}(v, \varphi_1), \dots, \widehat{I}(v, \varphi_\ell)) : v \in S(u)\} \\ &= \inf\{\ast(I(v, \varphi_1^t), \dots, I(v, \varphi_\ell^t)) : v \in S(u)\} \\ &= \inf\{I(v, \ast(\varphi_1^t, \dots, \varphi_\ell^t)) : v \in S(u)\} \\ &= I(u, \Box \ast(\varphi_1^t, \dots, \varphi_\ell^t)) \\ &= I(u, \varphi^t) \end{aligned}$$

where the first equality is by (12.1), the second equality is by the induction hypothesis, the third and the fourth equalities are by the definition of the extension of I onto $W \times \mathcal{F}$, and the last equality is by the definition of translation t . $\blacksquare$

It follows from (12.2) that $\widehat{M} \models_{\mathbf{mv-IL}} \Gamma \rightarrow \Delta$ if and only if $M \models \Gamma^t \rightarrow \Delta^t$, where Γ^t and Δ^t are obtained from Γ and Δ , respectively, by translating every formula appearing in them. Similarly, $\widehat{M} \models_{\mathbf{mv-IL}} \Sigma$ if and only if $M \models \Sigma^t$ where Σ^t is obtained from Σ by translating every sequent appearing in it.

Theorem 12.3. *$\Sigma \models_{\mathbf{mv-IL}} \Gamma \rightarrow \Delta$ if and only if $\Sigma^t \models_{\mathbf{C}} \Gamma^t \rightarrow \Delta^t$, where $\mathbf{C}$ is the class of preordered Kripke models.*

Proof If $\Sigma^t \not\models_{\mathbf{C}} \Gamma^t \rightarrow \Delta^t$, there exists a preordered Kripke model M such that $M \models \Sigma^t$, but $M \not\models \Gamma^t \rightarrow \Delta^t$. By Lemma 12.2, $\widehat{M} \models_{\mathbf{mv-IL}} \Sigma$, but $\widehat{M} \not\models_{\mathbf{mv-IL}} \Gamma \rightarrow \Delta$. Thus, $\Sigma \not\models_{\mathbf{mv-IL}} \Gamma \rightarrow \Delta$.

Conversely, if $\Sigma \not\models_{\mathbf{mv-IL}} \Gamma \rightarrow \Delta$, there exists an **mv-IL** interpretation M such that $M \models \Sigma$, but $M \not\models \Gamma \rightarrow \Delta$. By definition, M is also a preordered Kripke model and $\widehat{M}$ defined in Lemma 12.2 is M itself, because by the definition of an intuitionistic valuation, the value of a propositional variable p in a world u is already the minimum

of the values of p in $S(u)$. Therefore M , as an ***mv-S4*** model, satisfies Σ^t but not $\Gamma^t \rightarrow \Delta^t$. ■

It follows that strong decidability (and completeness) of ***mv-S4*** implies strong decidability of ***mv-IL***.

Remark 12.4. If the principal connective $*$ of a formula is monotonic,¹ then there is no need to insert $\Box$ before $*$ in the translation. This is because $\hat{I}$ is “local” on this connective, like in modal logic.

¹ That is, if $\mathbf{v}_{k_1} \leq \mathbf{v}_{k'_1}, \dots, \mathbf{v}_{k_\ell} \leq \mathbf{v}_{k'_\ell}$, then $*(\mathbf{v}_{k_1}, \dots, \mathbf{v}_{k_\ell}) \leq *(\mathbf{v}_{k'_1}, \dots, \mathbf{v}_{k'_\ell})$. For example, in the three-valued logics of Łukasiewicz [Łuk20] and Kleene [Kle38], disjunction $\vee$ and conjunction $\wedge$ are monotonic.

Chapter 13

Dunn-Belnap normal modal logic $B_4\text{-}K$

In this chapter we briefly discuss a four-valued counterpart of K , $B_4\text{-}K$, assuming the set of truth values is ordered as in Figure 6.1.

$B_4\text{-}K$ can be thought of as an extension of $B_4\text{-}N$ (see Chapter 6), just like $mv\text{-}K$ is an extension of $mv\text{-}N$.

A **B_4 Kripke model** (or B_4 *K-model* or just Kripke model) is a B_4 -Kripke structure whose all relations are the same.

The following is a straightforward analogue of Definition 7.1.

Definition 13.1. For a set of labelled formulas Γ , the set of labelled formulas $\Gamma^\times$ is defined as follows.

$$\Gamma^\times = \bigcup \{ \{ \psi \} \times \overline{[u_\psi, w_\psi]} : (\Box\psi, u_\psi), (\Diamond\psi, w_\psi) \in \Gamma \}$$

That is, for all ψ such that $(\Box\psi, u_\psi), (\Diamond\psi, w_\psi) \in \Gamma$, $\Gamma^\times$ includes the set $\{ \psi \} \times \overline{[u_\psi, w_\psi]}$ and nothing else.

The proof system of $B_4\text{-}K$ is obtained from the one of $B_4\text{-}N$ by adding the following rules:

$$\frac{(\varphi, v) \rightarrow \Gamma^\times}{(\Box\varphi, v), \Gamma \rightarrow} \quad v \in \{n, b\} \quad (13.1)$$

$$\frac{(\varphi, v) \rightarrow \Gamma^\times}{(\Diamond\varphi, v), \Gamma \rightarrow} \quad v \in \{n, b\} \quad (13.2)$$

$$\frac{(\varphi, f) \rightarrow \Gamma^\times \quad (\varphi, v) \rightarrow \Gamma^\times}{(\Box\varphi, f), \Gamma \rightarrow} \quad v \in \{n, b\} \quad (13.3)$$

$$\frac{(\varphi, t) \rightarrow \Gamma^\times \quad (\varphi, v) \rightarrow \Gamma^\times}{(\Diamond\varphi, t), \Gamma \rightarrow} \quad v \in \{n, b\} \quad (13.4)$$

Theorem 13.2. $B_4\text{-}K$ is sound and strongly complete with respect to B_4 -Kripke models.

Like in Chapter 6, we shall not get into details that are similar to what we showed for $mv\text{-}K$.

13.1 Soundness of $B_4\text{-}K$

Lemma 7.5 has exactly the same proof in $B_4\text{-}K$.

Thus, the proof of soundness is similar to that of the "only if" part of Theorem 7.4. We treat, for example, the case of rule (13.3). let M satisfy both its premises $(\varphi, \mathbf{f}) \rightarrow \Gamma^\times$ and $(\varphi, \mathbf{v}) \rightarrow \Gamma^\times$ and assume to the contrary that there exists a world $u \in W$ not satisfying the conclusion $(\Box\varphi, \mathbf{f}), \Gamma \rightarrow$ of this rule. That is, u satisfies $(\Box\varphi, \mathbf{f})$ and every labelled formula in Γ . Since

$$\inf\{I(v, \varphi) : v \in S(u)\} = \mathbf{f}$$

either there exists $v \in S(u)$ satisfying $(\varphi, \mathbf{f})$, or there exist $v, w \in S(u)$ satisfying $(\varphi, \mathbf{n})$ and $(\varphi, \mathbf{b})$, respectively.

In the former case, by Lemma 7.5, $M, v \not\models (\varphi, \mathbf{f}) \rightarrow \Gamma^\times$, contradicting satisfiability of premise $(\varphi, \mathbf{f}) \rightarrow \Gamma^\times$ of (13.3).

In the latter case, by Lemma 7.5, $M, v \not\models (\varphi, \mathbf{n}) \rightarrow \Gamma^\times$ and $M, w \not\models (\varphi, \mathbf{b}) \rightarrow \Gamma^\times$. Since $\mathbf{v} \in \{\mathbf{n}, \mathbf{b}\}$, one of these conclusions contradicts satisfiability of premise $(\varphi, \mathbf{v}) \rightarrow \Gamma^\times$ of (13.3).

13.2 Derivability in $B_4\text{-}K$

Example 13.3. (6.5),(6.6),(6.7) and (6.8) follow immediately from (13.1),(13.2),(13.3) and (13.4). Thus they are redundant in $B_4\text{-}K$.

Example 13.4. The sequents

$$(\Box\varphi, \mathbf{v}) \rightarrow (\Diamond\varphi, \mathbf{v})^+ \quad \mathbf{v} \neq \mathbf{t} \tag{13.5}$$

and

$$(\Diamond\varphi, \mathbf{v}) \rightarrow (\Box\varphi, \mathbf{v})^- \quad \mathbf{v} \neq \mathbf{f} \tag{13.6}$$

are $B_4\text{-}K$ -derivable. In particular, (6.3) is redundant in $B_4\text{-}K$.

If $\mathbf{v} = \mathbf{f}$, (13.5) follows from (6.18) by left-weakening (6.13).

Otherwise, the derivation of (13.5) is as follows, where, in steps 2_w and 3_w , $\mathbf{w} \not\leq \mathbf{v}$

(i.e., if $v = n$, $w \in \{f, b\}$, and, if $v = b$, $w \in \{f, n\}$).

1. $(\varphi, v) \rightarrow (\varphi, v)$ axiom (6.1)
2. $(\varphi, v) \rightarrow \{\varphi\} \times V$ follows from 1 by three right-weakenings (6.14)
- 2_w . $(\varphi, v) \rightarrow \{\varphi\} \times \overline{[v, w]}$ follows from 2, because, for $w \not\geq v$, $[v, w] = \emptyset$
- 3_w . $(\Box\varphi, v), (\Diamond\varphi, w) \rightarrow$ follows from 2_w by (13.1) with Γ being $\{(\Box\varphi, v), (\Diamond\varphi, w)\}$
4. $(\Box\varphi, v) \rightarrow (\Diamond\varphi, v)^+$ follows from all 3_w , $w \not\geq v$, by multi-shift (6.19)

The derivation of (13.6) is dual to that of (13.5) and is omitted.

Now, for $v \in \{n, b\}$, by two w, t -right-shifts (6.12), $w \leq v$, on (13.6), we obtain

$$(\Diamond\varphi, v), (\Box\varphi, t) \rightarrow v \in \{n, b\}$$

from which, by multi-shift (6.19), we obtain

$$(\Box\varphi, t) \rightarrow (\Diamond\varphi, f), (\Diamond\varphi, t) \quad (13.7)$$

So, (6.4) is redundant as well. Dually, one can derive

$$(\Diamond\varphi, f) \rightarrow (\Box\varphi, f), (\Box\varphi, t) \quad (13.8)$$

Example 13.5. The sequent

$$(\Box\varphi, t), (\Diamond\varphi, f) \rightarrow (\Box\psi, t) \quad (13.9)$$

is B_4 - K -derivable.

The derivation of (13.9) is as follows, where, in step 2_u , $u \neq t$, and, in step 3_w , $w \in \{n, b\}$.

1. $\rightarrow \{\varphi\} \times V$ derivable sequent (6.18)
- 2_u . $(\psi, u) \rightarrow \{\varphi\} \times V$ follows from 1 by u -left-weakening (6.13)
- 3_w . $(\Box\psi, w), (\Box\varphi, t), (\Diamond\varphi, f) \rightarrow$ follows from 2_w by (13.1), with Γ being $\{(\Box\varphi, t), (\Diamond\varphi, f)\}$, because $\overline{[t, f]} = V$
4. $(\Box\psi, f), (\Box\varphi, t), (\Diamond\varphi, f) \rightarrow$ follows from 2_f and 2_n , by (13.3), with Γ being $\{(\Box\varphi, t), (\Diamond\varphi, f)\}$, because $\overline{[t, f]} = V$
5. $(\Box\varphi, t), (\Diamond\varphi, f) \rightarrow (\Box\psi, t)$ follows from all 3_w , $w \in \{n, b\}$, and 4, by multi-shift (6.19)

Dually, one can derive

$$(\Box\varphi, t), (\Diamond\varphi, f) \rightarrow (\Diamond\psi, f) \quad (13.10)$$

Example 13.6. The rule

$$\frac{(\varphi, \mathbf{t}) \rightarrow \Gamma^\times}{(\Diamond\varphi, \mathbf{t}), \Gamma \rightarrow (\Box\varphi, \mathbf{f})} \quad (13.11)$$

is $\mathbf{B}_4\text{-}\mathbf{K}$ -derivable. In particular, (6.10) is redundant in $\mathbf{B}_4\text{-}\mathbf{K}$. This rule is valid, which can be shown as follows. Assume $M \models (\varphi, \mathbf{t}) \rightarrow \Gamma^\times$ and let a world u satisfy $(\Diamond\varphi, \mathbf{t})$ and every element of Γ . If $\mathbf{t} \in \{I(v, \varphi) : v \in S(u)\}$, then there exists $v \in S(u)$ satisfying $(\varphi, \mathbf{t})$. However, by Lemma 7.5, v satisfies no element of $\Gamma^\times$, in contradiction with $M \models (\varphi, \mathbf{t}) \rightarrow \Gamma^\times$. Therefore, $M, u \models (\Diamond\varphi, \mathbf{t})$ implies $\mathbf{b}, \mathbf{n} \in \{I(v, \varphi) : v \in S(u)\}$, that, in turn, implies $M, u \models (\Box\varphi, \mathbf{f})$, as desired.

The derivation of the conclusion of (13.11) from its premise is presented below, where, in step 2_u , $\mathbf{u} \neq \mathbf{f}$, and, in step 3_w , $\mathbf{w} \in \{\mathbf{n}, \mathbf{b}\}$.

1. $(\varphi, \mathbf{t}) \rightarrow \Gamma^\times$ assumption
- 2_u . $(\varphi, \mathbf{t}) \rightarrow \Gamma^\times, \{\varphi\} \times \overline{[\mathbf{u}, \mathbf{t}]}$ follows from 1 by right-weakenings (6.14)
- 3_w . $(\varphi, \mathbf{w}) \rightarrow (\varphi, \mathbf{w})$ axiom (6.1)
4. $(\varphi, \mathbf{n}) \rightarrow \Gamma^\times, \{\varphi\} \times \overline{[\mathbf{b}, \mathbf{t}]}$ follows from 3_n by right-weakening (6.14)
5. $(\varphi, \mathbf{b}) \rightarrow \Gamma^\times, \{\varphi\} \times \overline{[\mathbf{n}, \mathbf{t}]}$ follows from 3_b by right-weakening (6.14)
6. $(\varphi, \mathbf{n}) \rightarrow \Gamma^\times, \{\varphi\} \times \overline{[\mathbf{t}, \mathbf{t}]}$ follows from 3_n by right-weakenings (6.14)
7. $(\Box\varphi, \mathbf{n}), (\Diamond\varphi, \mathbf{t}), \Gamma \rightarrow$ follows from 2_n and 5 by (13.4), with Γ being $\Gamma \cup \{(\Box\varphi, \mathbf{n}), (\Diamond\varphi, \mathbf{t})\}$ and $\mathbf{v}$ being $\mathbf{b}$
8. $(\Box\varphi, \mathbf{b}), (\Diamond\varphi, \mathbf{t}), \Gamma \rightarrow$ follows from 2_b and 4 by (13.4), with Γ being $\Gamma \cup \{(\Box\varphi, \mathbf{b}), (\Diamond\varphi, \mathbf{t})\}$ and $\mathbf{v}$ being $\mathbf{n}$
9. $(\Box\varphi, \mathbf{t}), (\Diamond\varphi, \mathbf{t}), \Gamma \rightarrow$ follows from 2_t and 6 by (13.4), with Γ being $\Gamma \cup \{(\Box\varphi, \mathbf{t}), (\Diamond\varphi, \mathbf{t})\}$ and $\mathbf{v}$ being $\mathbf{n}$
10. $(\Diamond\varphi, \mathbf{t}), \Gamma \rightarrow (\Box\varphi, \mathbf{f})$ follows from 7, 8 and 9 by multi-shift (6.19)

Dually, one can derive

$$\frac{(\varphi, \mathbf{f}) \rightarrow \Gamma^\times}{(\Box\varphi, \mathbf{f}), \Gamma \rightarrow (\Diamond\varphi, \mathbf{t})} \quad (13.12)$$

So, (6.9) is redundant as well.

All in all, we see $\mathbf{B}_4\text{-}\mathbf{K}$ can be built by replacing (7.1) and (7.2) by (13.1), (13.2), (13.3) and (13.4) in the proof system of $\mathbf{mv}\text{-}\mathbf{K}$; instead of as an extension of $\mathbf{B}_4\text{-}\mathbf{N}$.

13.3 Completeness of $\mathbf{B}_4\text{-}\mathbf{K}$

A $\mathbf{B}_4\text{-}\mathbf{K}\text{-}\Sigma$ -canonical model is defined the same as in Chapter 8. Lemmas 8.2 and 8.3 are proved the same way in $\mathbf{B}_4\text{-}\mathbf{K}$ (again, we ignore syntactic modifications such as using $\mathbf{v} \in \mathbf{V}$ instead of $k \in \{1, \dots, n\}$ and $\mathbf{t}$ instead of n).

Thus, the proof of Theorem 8.4 for $\mathbf{B}_4\text{-}\mathbf{K}$ is very similar. Again, we shall focus on a specific case that is new to $\mathbf{B}_4\text{-}\mathbf{K}$. Namely, let the formula φ be $\Box\varphi'$ and let $I_M(u, \Box\varphi') = \mathbf{f}$.

Assume that for some $\mathbf{u}_1, \mathbf{u}_2, \dots$ and $\mathbf{w}_1, \mathbf{w}_2, \dots$,

$$(\Box\psi_1, \mathbf{u}_1), (\Diamond\psi_1, \mathbf{w}_1), (\Box\psi_2, \mathbf{u}_2), (\Diamond\psi_2, \mathbf{w}_2), \dots \in u$$

For the proof of

$$\mathbf{f} = \inf\{I_{M^\cap}(v, \varphi') : v \in S(u)\} = I_{M^\cap}(u, \varphi)$$

it suffices to prove that we have one of the following containments:

$$\mathbf{f} \in \{I_{M^\cap}(v, \varphi') : v \in S(u)\} \quad (13.13)$$

or

$$\mathbf{n}, \mathbf{b} \in \{I_{M^\cap}(v, \varphi') : v \in S(u)\} \quad (13.14)$$

Containment (13.13) means that there exist a world $v \in W_{\mathbf{B}_4\text{-}\mathbf{K}, \Sigma}$ and $\mathbf{v}_1 \in [\mathbf{u}_1, \mathbf{w}_1], \mathbf{v}_2 \in [\mathbf{u}_2, \mathbf{w}_2], \dots$ such that

$$(\varphi', \mathbf{f}), (\psi_1, \mathbf{v}_1), (\psi_2, \mathbf{v}_2), \dots \in v$$

This is because, by the definition of R_Σ (the accessibility relation of $M^\cap$), $uR_\Sigma v$ and, by the induction hypothesis (and part (i) of Corollary 4.5), $I_{M^\cap}(v, \varphi') = \mathbf{f}$.

Thus, by Lemma 4.3, for the proof of (13.13), it suffices to show that

- (*) there exist $\mathbf{v}_1 \in [\mathbf{u}_1, \mathbf{w}_1], \mathbf{v}_2 \in [\mathbf{u}_2, \mathbf{w}_2], \dots$ such that the set of labelled formulas $\{(\varphi', \mathbf{f}), (\psi_1, \mathbf{v}_1), (\psi_2, \mathbf{v}_2), \dots\}$ is $\mathbf{B}_4\text{-}\mathbf{K}\text{-}\Sigma$ -consistent.

Similarly, containment (13.14) means that

- (**) there exist $\mathbf{v}'_1 \in [\mathbf{u}_1, \mathbf{w}_1], \mathbf{v}'_2 \in [\mathbf{u}_2, \mathbf{w}_2], \dots$ and $\mathbf{v}''_1 \in [\mathbf{u}_1, \mathbf{w}_1], \mathbf{v}''_2 \in [\mathbf{u}_2, \mathbf{w}_2], \dots$ such that the sets of labelled formulas $\{(\varphi', \mathbf{n}), (\psi_1, \mathbf{v}'_1), (\psi_2, \mathbf{v}'_2), \dots\}$ and $\{(\varphi', \mathbf{b}), (\psi_1, \mathbf{v}''_1), (\psi_2, \mathbf{v}''_2), \dots\}$ are $\mathbf{B}_4\text{-}\mathbf{K}\text{-}\Sigma$ -consistent.

So, the proof of (13.13) or (13.14), assume to the contrary that neither of (*) or (**) hold. That is,

- for all $\mathbf{v}_1 \in [\mathbf{u}_1, \mathbf{w}_1], \mathbf{v}_2 \in [\mathbf{u}_2, \mathbf{w}_2], \dots$, the set $\{(\varphi', \mathbf{f}), (\psi_1, \mathbf{v}_1), (\psi_2, \mathbf{v}_2), \dots\}$ is $\mathbf{B}_4\text{-}\mathbf{K}\text{-}\Sigma$ -inconsistent, and
- there exists $\mathbf{v}' \in \{\mathbf{n}, \mathbf{b}\}$ such that for all $\mathbf{v}_1 \in [\mathbf{u}_1, \mathbf{w}_1], \mathbf{v}_2 \in [\mathbf{u}_2, \mathbf{w}_2], \dots$, the set $\{(\varphi', \mathbf{v}'), (\psi_1, \mathbf{v}_1), (\psi_2, \mathbf{v}_2), \dots\}$ is $\mathbf{B}_4\text{-}\mathbf{K}\text{-}\Sigma$ -inconsistent.

Note that, by Lemma 8.2, $\mathbf{u}_m \leq \mathbf{w}_m$, for all $m = 1, 2, \dots$

By Lemma 8.3, there exist non-negative integers L' and L'' such that

$$\Sigma \vdash_{\mathbf{B}_4\text{-}\mathbf{K}} (\varphi', \mathbf{f}) \rightarrow \{\psi_1\} \times \overline{[\mathbf{u}_1, \mathbf{w}_1]}, \{\psi_2\} \times \overline{[\mathbf{u}_2, \mathbf{w}_2]}, \dots, \{\psi_{L'}\} \times \overline{[\mathbf{u}_{L'}, \mathbf{w}_{L'}]}$$

and

$$\Sigma \vdash_{B_4-K} (\varphi', \mathbf{v}') \rightarrow \{\psi_1\} \times \overline{[\mathbf{u}_1, \mathbf{w}_1]}, \{\psi_2\} \times \overline{[\mathbf{u}_2, \mathbf{w}_2]}, \dots, \{\psi_{L''}\} \times \overline{[\mathbf{u}_{L''}, \mathbf{w}_{L''}]}$$

Let $L = \max(\{L', L''\})$. By right-weakenings,

$$\Sigma \vdash_{B_4-K} (\varphi', \mathbf{f}) \rightarrow \{\psi_1\} \times \overline{[\mathbf{u}_1, \mathbf{w}_1]}, \{\psi_2\} \times \overline{[\mathbf{u}_2, \mathbf{w}_2]}, \dots, \{\psi_L\} \times \overline{[\mathbf{u}_L, \mathbf{w}_L]}$$

and

$$\Sigma \vdash_{B_4-K} (\varphi', \mathbf{v}') \rightarrow \{\psi_1\} \times \overline{[\mathbf{u}_1, \mathbf{w}_1]}, \{\psi_2\} \times \overline{[\mathbf{u}_2, \mathbf{w}_2]}, \dots, \{\psi_L\} \times \overline{[\mathbf{u}_L, \mathbf{w}_L]}$$

from which, by (13.3), we obtain

$$\Sigma \vdash_{B_4-K} (\Box \varphi', \mathbf{f}), (\Box \psi_1, \mathbf{u}_1), (\Diamond \psi_1, \mathbf{w}_1), (\Box \psi_2, \mathbf{u}_2), (\Diamond \psi_2, \mathbf{w}_2), \dots, (\Box \psi_L, \mathbf{u}_L), (\Diamond \psi_L, \mathbf{w}_L) \rightarrow$$

that contradicts the $B_4-K-\Sigma$ -consistency of u .

The rest of the proof of completeness is the same.

13.4 Extensions of B_4-K

The results of Chapter 9 pass almost verbatim to B_4-K (the proof of part (i) of Theorem 9.4 is very similar to the one we showed for the analogue of Remark 4.8 in Chapter 6, using (13.11)). We state the main result (again, we ignore syntactic modifications such as using $\mathbf{v} \in \mathbf{V}$ instead of $k \in \{1, \dots, n\}$ and $\mathbf{t}$ instead of n).

Definition 13.7.

- The many-valued modal logic B_4-D is obtained from B_4-K by adding to it (4.1).
- The many-valued modal logic B_4-T is obtained from B_4-K by adding to it (4.2) and (4.3).
- The many-valued modal logic B_4-K4 is obtained from B_4-K by adding to it (9.1) and (9.2).
- The many-valued modal logic B_4-S4 is obtained from B_4-T by adding to it (9.1) and (9.2).
- The many-valued modal logic B_4-B is obtained from B_4-K by adding to it (9.3) and (9.4).
- The many-valued modal logic B_4-S5 is obtained from B_4-T by adding to it (9.5) and (9.6).

Theorem 13.8.

- (i) B_4-D is sound and (strongly) complete with respect to serial Kripke models.
- (ii) B_4-T is sound and (strongly) complete with respect to reflexive Kripke models.
- (iii) B_4-K4 is sound and (strongly) complete with respect to transitive Kripke models.
- (iv) B_4-S4 is sound and (strongly) complete with respect to reflexive and transitive (preordered) Kripke models.
- (v) B_4-B is sound and (strongly) complete with respect to symmetric Kripke models.
- (vi) B_4-S5 is sound and (strongly) complete with respect to reflexive and Euclidean Kripke models.¹

13.5 Decidability of B_4-K and its extensions

Filtrations are defined in the same manner. Lemma 10.1 is proved in the same way, so the adaption of the proof of Theorem 10.2 to B_4-K is clear and similar to arguments we have already shown, therefore we omit it.

Thus, B_4-K and all of its extensions possess the finite model property and are thus strongly decidable.

13.6 Duality of $\Box$ and $\Diamond$ via negation in B_4-K

Unlike in Chapter 11, where the truth table of negation was determined uniquely by De Morgan duality, in B_4-K we may define the truth table for negation $\neg$ in two different ways, so that $\Box$ and $\Diamond$ become the De Morgan dual, that is, the sequents

$$(\Diamond\varphi, v) \rightarrow (\neg\Box\neg\varphi, v) \quad (13.15)$$

and

$$(\Box\varphi, v) \rightarrow (\neg\Diamond\neg\varphi, v) \quad (13.16)$$

are provable in B_4-K .² We shall show that these are the only appropriate definitions of negation, for which (13.15) and (13.16) are derivable in B_4-K .

The truth table of $\neg$ is either

$\neg$	
f	t
n	n
b	b
t	f

(13.17)

¹ This is the class of all Kripke models whose accessibility relation is an equivalence relation.

² In particular, in B_4 (with Belnap's original definition of negation, see [Bel75; Bel77]), these modal connectives are interdefinable.

or

$$\begin{array}{c|c}
 \neg & \\
 \hline
 \mathbf{f} & \mathbf{t} \\
 \mathbf{n} & \mathbf{b} \\
 \mathbf{b} & \mathbf{n} \\
 \mathbf{t} & \mathbf{f}
 \end{array} \tag{13.18}$$

Example 13.9. (Cf. Example 11.1.) Sequents

$$(\neg\varphi, \neg(\mathbf{v})) \rightarrow (\varphi, \mathbf{v})$$

are $\mathbf{B}_4\text{-}\mathbf{K}$ derivable.

The derivation is as follows.

- 1 _{$w \neq v$} . $(\varphi, \mathbf{w}) \rightarrow (\neg\varphi, \neg(\mathbf{w})), \quad \mathbf{w} \neq \mathbf{v}$ axiom (6.2)
- 2 _{$w \neq v$} . $(\varphi, \mathbf{w}), (\neg\varphi, \neg(\mathbf{v})) \rightarrow$ follows from 1 _{w} by $\neg(\mathbf{w}), \neg(\mathbf{v})$ -right-shift (6.12)
3. $(\neg\varphi, \neg(\mathbf{v})) \rightarrow (\varphi, \mathbf{v})$ follows from 2 _{$w \neq v$} by multi-shift (6.19)

Theorem 13.10. *Let $\neg$ be a unary connective. Then, sequents (13.15) and (13.16) are derivable in $\mathbf{B}_4\text{-}\mathbf{K}$ if and only if the truth table is either (13.17) or (13.18).*

Proof For the “only if” part of the theorem, we show first that, for all $\mathbf{v} \in \mathbf{V}$,

$$\neg(\neg(\mathbf{v})) = \mathbf{v} \tag{13.19}$$

Indeed, let $M = \langle W, R, I \rangle$ be a Kripke model such that, for some $u \in W$, $S(u) = \{u\}$ and $I(u, p) = \mathbf{v}$. Then $I(u, \Diamond p) = \mathbf{v}$ and, by (13.15),

$$I(u, \neg\Box\neg p) = \mathbf{v} \tag{13.20}$$

Since $I(u, \neg p) = \neg(\mathbf{v})$ and $S(u) = \{u\}$, $I(u, \Box\neg p) = \neg(\mathbf{v})$, implying $I(u, \neg\Box\neg p) = \neg(\neg(\mathbf{v}))$, which, together with (13.20), in turn, implies (13.19).

In particular, $\neg : \mathbf{V} \rightarrow \mathbf{V}$ is an invertible function.

Next, we shall prove that $\neg(\mathbf{t}) = \mathbf{f}$. For the proof, let $M = \langle W, R, I \rangle$ be a Kripke model such that, for some $u \in W$, $S(u) = \emptyset$. Then, $I(u, \Diamond p) = \mathbf{f}$, implying, by (13.15), $I(u, \neg\Box\neg p) = \mathbf{f}$. However, $I(u, \Box\neg p) = \mathbf{t}$, implying

$$I(u, \neg\Box\neg p) = \neg(\mathbf{t})$$

Thus, $\neg(\mathbf{t}) = \mathbf{f}$. We also have $\neg(\mathbf{f}) = \neg(\neg(\mathbf{t})) = \mathbf{t}$.

Clearly, the only invertible functions from $\mathbf{V}$ to $\mathbf{V}$ satisfying $\neg(\mathbf{t}) = \mathbf{f}$ and $\neg(\mathbf{f}) = \mathbf{t}$ are (13.17) and (13.18).

It remains to prove the “if” part of the theorem. We prove it semantically.

Let $M = \langle W, R, I \rangle$ be a Kripke model, $u \in W$, and let $\varphi \in \mathcal{F}$. Then, it follows from

either (13.17) or (13.18) that

$$\sup\{I(v, \varphi) : v \in S(u)\} = \neg \inf\{I(v, \neg\varphi) : v \in S(u)\}$$

implying the validity of (13.15).

The proof of (13.16) is dual. ■

Remark 13.11. (Cf. Remark 11.4.) If we define negation as either (13.17) or (13.18), then rule (13.2) becomes redundant, which can be shown as follows.

- | | |
|---|---|
| 1. $(\varphi, \mathbf{v}) \rightarrow \Gamma^\times, \mathbf{v} \in \{\mathbf{n}, \mathbf{b}\}$ | assumption of (13.2) |
| 2. $(\neg\varphi, \neg(\mathbf{v})) \rightarrow (\varphi, \mathbf{v})$ | Example 13.9 |
| 3. $(\neg\varphi, \neg(\mathbf{v})) \rightarrow \Gamma^\times, \neg(\mathbf{v}) \in \{\mathbf{n}, \mathbf{b}\}$ | follows from 1 and 2 by cut |
| 4. $(\Box\neg\varphi, \neg(\mathbf{v})), \Gamma \rightarrow, \neg(\mathbf{v}) \in \{\mathbf{n}, \mathbf{b}\}$ | follows from 3 by (13.1) |
| 5. $(\neg\Box\neg\varphi, \mathbf{v}) \rightarrow (\Box\neg\varphi, \neg(\mathbf{v}))$ | Example 13.9 |
| 6. $(\neg\Box\neg\varphi, \mathbf{v}), \Gamma \rightarrow, \mathbf{v} \in \{\mathbf{n}, \mathbf{b}\}$ | follows from 4 and 5 by cut |
| 7. (13.2) | because, by (13.15), $\neg\Box\neg$ is $\Diamond$ |

It can be shown that rule (13.4) becomes redundant as well. Also, it can be shown that (4.3), (9.2), (9.4), and (9.6) follow from (4.2), (9.1), (9.3), and (9.5), respectively, and vice-versa.

13.7 Embedding B_4 intuitionistic logic into $B_4\text{-}S4$

We consider an instance of the many-valued intuitionistic logic presented in [Tak86], that we shall denote by $B_4\text{-}IL$, and is based on the Dunn-Belnap lattice (Figure 6.1) of truth values. Its semantic definition is identical to the one given in Chapter 12. The results and their proofs are also identical. We only state the main results.

Definition 13.12. Let φ be a formula in the language of $B_4\text{-}IL$. The translation φ^t of a $B_4\text{-}IL$ formula φ is obtained from φ by inserting $\Box$ before each of its subformulas. That is, φ^t is defined recursively as follows.

- For a propositional variable p , p^t is $\Box p$, and
- if φ is of the form $\ast(\varphi_1, \dots, \varphi_\ell)$, then φ^t is $\Box \ast(\varphi_1^t, \dots, \varphi_\ell^t)$.

Theorem 13.13. $\Sigma \models_{B_4\text{-}IL} \Gamma \rightarrow \Delta$ if and only if $\Sigma^t \models_{\mathbf{C}} \Gamma^t \rightarrow \Delta^t$, where $\mathbf{C}$ is the class of preordered Kripke models.

It follows that strong decidability (and completeness) of $B_4\text{-}S4$ implies strong decidability of $B_4\text{-}IL$.

Remark 13.14. (Cf. Remark 12.4.) If the principal connective $*$ of a formula is monotonic,³ then there is no need to insert $\Box$ before $*$ in the translation. This is because $\hat{I}$ is “local” on this connective, like in modal logic.

³ That is, if $\mathbf{v}_1 \leq \mathbf{v}'_1, \dots, \mathbf{v}_\ell \leq \mathbf{v}'_\ell$, then $*(\mathbf{v}_1, \dots, \mathbf{v}_\ell) \leq *(\mathbf{v}'_1, \dots, \mathbf{v}'_\ell)$. For example, the standard connectives $\wedge$ and $\vee$ of B_4 (see [Bel75; Bel77]) are monotonic.

Chapter 14

Conclusion and open questions

In this thesis we considered two main families of many-valued modal logics, $mv-N$ and $mv-K$, assuming a finite linearly ordered set of truth values and an arbitrary set of propositional connectives defined by truth tables.

We defined appropriate semantics for these logics and constructed sound and strongly complete proof systems for them. We explored some of their extensions, finding counterparts for several well-known two-valued modal logics. We proved that all of our logics are strongly decidable by proving their finite model property. We also discussed the connection between many-valued intuitionistic logics and our many-valued modal logics.

In fact, we presented $mv-K$ as an extension of $mv-N$, so that the theory of $mv-N$ underlies the theories of all other logics presented in this thesis.

Throughout the thesis we emphasized our separate treatment of the two modal connectives, "necessary" and "possible". They are not interdefinable, so they are treated independently and their duality is reflected in the proof systems, as well as in the pairs of axioms needed for most of the extensions. If one wants to add negation in order to make the modal connectives interdefinable (so that only one of them is needed for the definition of the logic), then, as we have shown, there is only one way to do so for our normal modal logics.

In addition, we treated a specific case of a non-linear lattice of truth values, and studied the corresponding results in this case.

Some questions, for further research, naturally arise from our thesis.

- How can one use $mv-N$ in the area of many-valued nonmonotonic reasoning? It seems that, similarly to the logic N, $mv-N$ can be useful in investigations of knowledge sets based on partial information in the many-valued case.
- How can one generalize both $mv-N$ and $mv-K$, and all of the results regarding them, to the case of a general finite lattice of truth values? It seems reasonable that the step we took from linear sets to the Dunn-Belnap lattice L4 can be taken further and be applied to the more general case.

- How can one generalize $mv-N$ and $mv-K$ to the case of non-deterministic valuations, as discussed in [AL05]?
- How can one develop hypersequent calculi for these logics and establish the results of [Lah13] for many-valued modal logics?
- Do our proof systems (or any equivalent formulations of them) possess the cut-elimination property? One can try to establish it either syntactically, like in [KF21], or semantically, like in [LA13].

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שיחס הנגישות שלהם הוא יחס שקילות.

אנו מוכיחים גם ששתי הלוגיקות שלנו וכל הרחבותיהן מקיימות את תכונת המודל הסופי, ומכאן מסיקים שכל אחת מהלוגיקות היא כריעה במובן החזק.

נוסף על כך, אנו מוכיחים שקיימת דרך יחידה להגדיר את קשר השלילה כך שתתקיים דואליות דה-מורגן בין האופרטורים המודליים בלוגיקה הנורמלית שלנו. אנו מראים שבלוגיקה התת-נורמלית הדבר אינו אפשרי, ובוחנים את לוגיקת הביניים החלשה ביותר שבה הדואליות מתקיימת.

כמו כן, כידוע, לוגיקה אינטואיציוניסטית דו-ערכית ניתנת לשיכון בלוגיקה המודלית הדו-ערכית S4 ע"י תרגום של הנוסחאות. אנו מוכיחים תוצאה דומה עבור לוגיקה אינטואיציוניסטית רב-ערכית.

לסיום, אנו מתבוננים במקרה מסוים שבו קבוצת ערכי האמת אינה סדורה לינארית, אלא סדורה חלקית כסריג ארבע-ערכי. אנו מוצאים מערכות הוכחה מתאימות ומראים שכל תוצאותינו הקודמות תקפות גם במקרה הזה, עד כדי יחידות ההגדרה של קשר השלילה, שבמקרה זה יש שתי דרכים שונות להגדירו.

תקציר

בחיבור זה אנו משלבים בין התחומים "לוגיקה מודלית" ו"לוגיקה רב-ערכית" בצורה כללית ומקיפה.

לוגיקה מודלית היא הרחבה של לוגיקה קלאסית ע"י הוספת שני אופרטורים מודליים, דואליים זה לזה, המייצגים "הכרחי" ו"אפשרי". הלוגיקה המודלית N מתקבלת מלוגיקה פסוקית קלאסית ע"י הוספת כלל היסק אחד, ללא כל אקסיומות. לוגיקה זו ניתנת לאפיון סמנטי ע"י "מבני קריפקה". מבנה קריפקה הוא קבוצת עולמות אפשריים, יחד עם אינסוף יחסי נגישות בין העולמות (לכל נוסחה בשפה קיים יחס משלו על קבוצת העולמות), וכן פונקציית הערכה אשר מצמידה לכל פסוק יסודי בכל עולם ערך אמת ("אמת" או "שקר", במקרה הדו-ערכי). ללוגיקה זו יש שימוש בתחום הלוגיקות הלא מונוטוניות.

הלוגיקה K היא הרחבה של N המתקבלת ממנה ע"י הוספת תבנית אקסיומה אחת. הלוגיקה K מכונה לוגיקה נורמלית, וכך גם כל הרחבה שלה. N היא לוגיקה תת-נורמלית.

הסמנטיקה המקובלת של לוגיקה מודלית נורמלית פסוקית מוגדרת באמצעות "מודלי קריפקה", שהם מקרה פרטי ופשוט יותר של מבני קריפקה, שבו כל יחסי הנגישות זהים זה לזה. כלומר, ישנו יחס נגישות אחד.

לוגיקה רב-ערכית היא כל לוגיקה שבה גודל קבוצת ערכי האמת הוא יותר מ-2. בתזה שלנו, אנו מתמקדים בלוגיקות בעלות מספר סופי של ערכי אמת.

ספציפית, בהינתן קבוצה סופית וסדורה לינארית של ערכי אמת, וכן קבוצה שרירותית של קשרים לוגיים המוגדרים ע"י טבלאות אמת, אנו מגדירים שתי לוגיקות מודליות. האחת היא מקבילה רב-ערכית של N והשנייה היא מקבילה רב-ערכית של K . אנו מגדירים את הסמנטיקה של כל אחת מהן, וכן מוצאים לכל אחת מהן מערכת הוכחה נאותה ושלמה במובן החזק. מערכות ההוכחה רשומות בשפה של סיקוונטים (במקרה שלנו, אלו הם ביטויים מטא-לוגיים המייצגים גרירה) של נוסחאות מתוגות. נדגיש כי אנו מפתחים את התאוריה (ההגדרות והתוצאות) של הלוגיקה הנורמלית מתוך התאוריה של הלוגיקה התת-נורמלית.

שתי הלוגיקות הנ"ל מפרשות את האופרטורים המודליים "הכרחי" ו"אפשרי" ע"י אינפימום וסופרימום של ערכי האמת הרלוונטיים, בהתאמה. יתרה מכך, כל אחת מהלוגיקות מתייחסת ל"הכרחי" ול"אפשרי" באופן בלתי-תלוי, כלומר, אף אחד מהם לא מוגדר ע"י השני, כך שהדואליות ביניהם באה לידי ביטוי במערכות ההוכחה עצמן. זאת בניגוד לגרסה המקובלת של לוגיקה מודלית דו-ערכית, שבה מניחים את קיומו של קשר השלילה ומשתמשים בו כדי להגדיר את אחד האופרטורים באמצעות השני.

בנוסף, אנו מגדירים מספר הרחבות של הלוגיקות הנ"ל ומוכיחים שהן נאותות ושלמות עבור מחלקות מסוימות של מבני קריפקה. לדוגמה, אנו מגדירים מקבילה רב-ערכית ללוגיקה S_4 , אשר מאופיינת ע"י מודלי קריפקה שיחס הנגישות שלהם רפלקסיבי וטרנזיטיבי, וכן מקבילה רב-ערכית ללוגיקה S_5 , אשר מאופיינת ע"י מודלי קריפקה

המחקר בוצע בהנחייתו של פרופסור מיכאל קמינסקי, בפקולטה למתמטיקה.

מחבר חיבור זה מצהיר כי המחקר, כולל איסוף הנתונים, עיבודם והצגתם, התייחסות והשוואה למחקרים קודמים וכו', נעשה כולו בצורה ישרה, כמצופה ממחקר מדעי המבוצע לפי אמות המידה האתיות של העולם האקדמי. כמו כן, הדיווח על המחקר ותוצאותיו בחיבור זה נעשה בצורה ישרה ומלאה, לפי אותן אמות מידה.

הכרת תודה מסורה לטכניון על מימון מחקר זה.

לוגיקות מודליות רב-ערכיות

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
מגיסטר למדעים במתמטיקה

עמיר קרניאל

לוגיקות מודליות רב-ערכיות

עמיר קרניאל

