



Directional wavelet packets originating from polynomial splines

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Abstract

The paper presents a versatile library of quasi-analytic complex-valued wavelet packets (qWPs) which originate from polynomial splines of arbitrary orders. Discrete Fourier transforms (DFT) of qWPs are located in either positive or negative half-band of the frequency domain. Consequently, the DFTs of 2D qWPs, which are derived by the tensor products of 1D qWPs, occupy one of quadrants of 2D frequency domain. Such a structure of the DFT spectra of the 2D qWPs results in the directionality of their real parts. Due to the fact that the spectra of qWPs are well localized in the frequency domain, the shapes of real parts of the qWPs are close to windowed cosine waves oscillating in a variety of different directions with a variety of frequencies. For example, a set of the fourth-level qWPs comprises 314 different directions and 256 different frequencies. The above properties combined with a fast transform implementation make directional qWPs a strong tool for the application to a variety of image processing tasks such as restoration of degraded images and extraction of characteristic features from images to use them in deep learning. A few illustrations of successful application of the designed qWPs to image denoising, inpainting, and classification are given in the paper.

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1 Introduction

It is apparent that for an image processing algorithm to be efficient, it must take into account features that characterize the images. Such features are edges oriented in various directions, texture patterns, that can be approximated by patches oscillating in various directions, and smooth regions. The ability to extract such features even from degraded images is a key element in schemes for image denoising, inpainting, deblurring, classification, and target detection. This stems from the fact that practically all the processed images have a sparse representation in a proper transform domain. The sparse representation of an image means that it can be approximated by a linear combination of a relatively small number of 2D “basic” elements selected from a versatile collection called dictionary, while retaining the above-mentioned components of the image.

Such a dictionary should comprise a variety of waveforms which are able to capture edges oriented in any direction, texture patches oscillating with any frequency and to represent smooth regions by very few elements. The latter requirement can be fulfilled if the dictionary elements possess vanishing moments, at least locally. In order to meet the former two, the dictionary elements have to be oriented in multiple directions and to have oscillating structures with multiple frequencies.

Image processing applications are the field of extensive research, especially in the recent decades. Naturally, a number of dictionaries are reported in the literature and applied to image processing. We mention brushlets [1], contourlets [2], curvelets [3, 4], pseudo-polar Fourier transforms [5, 6], and related to them shearlets [7, 8]. These dictionaries are used in various image processing applications such as, for example, Affine Shear transforms (DAS-1) [9]. However, while successfully capturing edges in images, these dictionaries did not demonstrate a satisfactory texture restoration due to the lack of oscillating waveforms in the dictionaries.

A number of publications [10–16], to name a few, design directional dictionaries by the tensor multiplication of complex wavelets [17, 18], wavelet frames, and wavelet packets (WPs). The tight tensor-product complex wavelet frames (TP_CTF_n)¹ with a different number of directions are designed in [12–14] and some of them, in particular cptTP_CTF_6 , TP_CTF_6 , and $\text{TP_CTF}_6^\downarrow$,² demonstrate impressive performance for image denoising and inpainting. The waveforms in these frames

¹The index n refers to the number of filters in the underlying one-dimensional complex tight framelet filter bank.

²Here TP_CTF_6 means the tight tensor-product complex wavelet tight frame with six filters, cptTP_CTF_6 means the six-filters frame with compactly supported framelets, and $\text{TP_CTF}_6^\downarrow$ means the six-filters frame with low redundancy.

are oriented in 14 directions and, due to the 2-layer structure of their spectra, they possess certain, although limited, oscillatory properties.

In [19] (algorithm *Digital Affine Shear Filter Transform with 2-Layer Structure* (DAS-2)), the two-layer structure, which is inherent in the TP_CTF₆ frames, is incorporated into shearlet-based directional filter banks introduced in [9]. This improves the performance of DAS-2 in comparison to TP_CTF₆ on texture-rich images such as “Barbara,” “Bridge,” which is not the case for smoother images like “Lena.”

In [20–22], a feature extraction scheme based on rotating Morlet wavelets is designed and used for the construction of scattering Neural Networks, which proved to be efficient in image classification.

Our motivation is to design a family of dictionaries that maximally meet requirements for image processing applications. As a base for such a design, we have a library of orthonormal WPs originating from the so-called discrete-time splines³ of multiple orders (see [24]). The waveforms in the library are symmetric, well localized in time domain, their shapes vary from low-frequency smooth curves to high-frequency oscillating transients. They can have any number of local vanishing moments (to be defined in Section 2.4). Their spectra provide a variety of refined splits of the frequency domain and the shapes of the magnitude spectra tend to a rectangular as the spline’s order increases. Their tensor products possess similar properties that are extended to 2D setting while the directionality, which is of crucial importance for image processing, does not exist.

The following steps are used to design directional WPs: 1. Apply the Hilbert transform (HT) to the set $\{\psi\}$ of orthonormal WPs thus producing the set $\{\theta \stackrel{\text{def}}{=} H(\psi)\}$. 2. A slight correction of the lowest- and the highest-frequency waveforms from the set $\{\theta\}$ provides an orthonormal set $\{\varphi\}$ of the so-called complimentary WPs (cWPs), which are anti-symmetric and whose magnitude spectra coincides with the magnitude spectra of the respective WPs from the set $\{\psi\}$. 3. Define two sets of complex quasi-analytic WPs (qWPs) $\{\Psi_+ \stackrel{\text{def}}{=} \psi + i\varphi\}$ and $\{\Psi_- \stackrel{\text{def}}{=} \psi - i\varphi\}$ whose spectra are localized in the positive and negative half-bands of the frequency domain, respectively. 4. Define two sets of 2D complex qWPs by tensor multiplication of the qWPs $\{\Psi_{\pm}\}$ as: $\{\Psi_{++} \stackrel{\text{def}}{=} \Psi_+ \otimes \Psi_+\}$ and $\{\Psi_{+-} \stackrel{\text{def}}{=} \Psi_+ \otimes \Psi_-\}$. 5. The dictionaries we are looking for are obtained as real parts of these qWPs: $\{\vartheta_+ \stackrel{\text{def}}{=} \Re(\Psi_{++})\}$ and $\{\vartheta_- \stackrel{\text{def}}{=} \Re(\Psi_{+-})\}$.

The DFT spectra of elements from the dictionaries $\{\vartheta_+\}$ and $\{\vartheta_-\}$ form various tiling of pairs of quadrants $\mathbf{Q}_0 \cup \mathbf{Q}_3$ and $\mathbf{Q}_1 \cup \mathbf{Q}_2$ (see Eq. (1.1)) of the frequency domain, respectively, by squares of different size depending on the decomposition level. The waveforms’ shapes are close to windowed cosine waves with multiple frequencies oriented in multiple directions. Combinations of waveforms from the sets

³The discrete-time splines are derived by the discretization of polynomial splines. They are referred to as the discrete splines in [23].

$\{\vartheta_+\}$ and $\{\vartheta_-\}$ provide a variety of frames in the 2D signals space. The transforms are executed in a fast way by using the FFT.

Main contribution of the paper is the design of a versatile library of directional oscillating waveforms and development of fast transform schemes. The library's members (qWPs) possess an exclusive collection of properties that make them invaluable for image processing applications. To be specific, the qWP waveforms

1. Are oriented in multiple directions;
2. Have oscillating structure with multiple frequencies;
3. Have local vanishing moments;
4. Well localized in the spatial domain;
5. DFT spectra of the qWPs produce a refined frequency separation;
6. The corresponding transforms are implemented in a fast way using the FFT;
7. Transforms have a number of free parameters which enable flexible adaptation to the objects under processing;
8. The transform coefficients have a clear (explainable) physical meaning.

Remark 1 Directional WPs in our scheme are derived, similarly to the brushlets [1] and the tight frames TP_CTF_n ([12], by tensor multiplication of 1D complex waveforms. However, our approach to the design of 1D waveforms is, to some extent, opposite to the approach for brushlets and TP_CTF_n . In those schemes, the waveforms are derived by the inverse Fourier transforms of some prescribed windowed partitions of the frequency domain. On the contrary, we start from the symmetric $\{\psi\}$ and antisymmetric $\{\varphi\}$ waveforms, whose DFT spectra form a variety of partitions of the frequency domain. Such an approach allows much flexibility in the design and provides versatile libraries of waveforms in one and several dimensions with exclusive properties listed above.

The above properties make the designed qWPs a powerful tool for a variety of image processing applications, which is confirmed by multiple numerical experiments. In particular, application of qWPs to the denoising and inpainting problems [25, 26] produced state-of-the-art results. However, qWPs have a strong potential to handle a new important class of problems. Namely, the above properties of qWPs provide a perfect tool for feature extraction from images and, in that capacity, can serve as a significant unit of deep neural networks (DNNs). Due to versatility of testing waveforms and, most important, to the explainable physical meaning of the transform coefficients, which are results of convolution of the image with a variety of qWPs waveforms, it is possible to replace at least some of the convolution layers in the convolution DNNs by convolving the image with qWPs. This will lead to a significant reduction of the training dataset. Our first experiments demonstrated the feasibility of qWPs for such a task.

The paper is organized as follows: Section 2 briefly outlines the orthonormal WPs originated from polynomial splines and the corresponding transforms that serve as a basis for the design of qWPs. Section 3 presents the design of 1D qWPs and Section 4 describes the implementation of the transforms. Section 5 extends the design of 1D qWPs to 2D case and discusses the qWPs' directionality. Section 6 describes the implementation of 2D transforms. Section 7 presents a couple of examples of image

denoising, inpainting and classification using qWPs. Section 8 comprises a brief discussion. Appendix provides proofs for two propositions and a Matlab code. Tables 1, 2 and 3 present notations and abbreviations used in the paper.

Quadrants of the frequency domain $\mathbf{F} : [-N/2, N/2-1] \times [-N/2, N/2-1]$ are:

$$\begin{aligned} \mathbf{Q}_0 &\stackrel{\text{def}}{=} [0, N/2-1] \times [0, N/2-1], \quad \mathbf{Q}_1 \stackrel{\text{def}}{=} [0, N/2-1] \times [-N/2, -1], \\ \mathbf{Q}_2 &\stackrel{\text{def}}{=} [-N/2, -1] \times [0, N/2-1], \quad \mathbf{Q}_3 \stackrel{\text{def}}{=} [-N/2, -1] \times [-N/2, -1]. \end{aligned} \quad (1.1)$$

2 Preliminaries: outline of orthonormal WPs originated from discrete-time splines

It was mentioned in Section 1 that the quasi-analytic wavelet packets $\Psi_{\pm[m],l}^p$ and $\Psi_{+\pm[m],l,j}^p$ are designed on the base of the periodic discrete-time wavelet packets (dWPs). This section presents a brief outline of properties of the dWPs and the corresponding transforms. For details, see Chapter 4 in [24]. Notations and abbreviations for Section 2 are given in Table 2.

2.1 Periodic discrete-time splines and first-level wavelet packets decomposition

The centered N -periodic polynomial B-spline $B^p(t)$ of order p is an N -periodization of the function

$$b^p(t) \stackrel{\text{def}}{=} \frac{1}{(p-1)!} \sum_{k=0}^p (-1)^k \binom{p}{k} \left(t + \frac{p}{2} - k\right)_+^{p-1}, \quad x_+ \stackrel{\text{def}}{=} \max\{x, 0\}. \quad (2.1)$$

The B-spline $B^p(t)$ is supported on the interval $(-p/2, p/2)$, up to periodization. It is strictly positive inside this interval and symmetric about zero, where it has its single maximum, and has $p-2$ continuous derivatives. The Fourier coefficients of the B-spline are

$$c_n(B^p) = \int_{-N/2}^{N/2} B^p(t) e^{-2\pi i n t / N} dt = \left(\frac{\sin \pi n / N}{\pi n / N} \right)^p.$$

The functions $S^p(t) \stackrel{\text{def}}{=} \sum_{l=0}^{N-1} q[l] B^p[t-l]$, where $\{q[l]\}$ is a real N -periodic sequence, are referred to as the order- p periodic splines. The following two sequences (Eqs. (2.2) and (2.3)) will be repeatedly used in the paper:

$$\begin{aligned} u^p[n] &\stackrel{\text{def}}{=} \sum_{k=-N/2}^{N/2-1} \omega^{-kn} b^p(k) = \sum_{l \in \mathbb{Z}} \left(\frac{\sin \pi (n/N + l)}{\pi (n/N + l)} \right)^p \\ &= \sin^p \frac{\pi n}{N} \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp}}{(\pi (n/N + l))^p}, \end{aligned} \quad (2.2)$$

$$v^p[n] \stackrel{\text{def}}{=} \omega^{-n/2} \sum_{k=-N/2}^{N/2-1} \omega^{-kn} b^p\left(k + \frac{1}{2}\right) = \sin^p \frac{\pi n}{N} \sum_{l \in \mathbb{Z}} \frac{(-1)^{l(p+1)}}{(\pi (n/N + l))^p}. \quad (2.3)$$

Table 1 Notations and abbreviations

$N = 2^J$, $\delta[n]$ – N -periodic Kronecker delta	$\omega \stackrel{\text{def}}{=} e^{2\pi i / N}$	.* – complex conjugate
$\Pi[N]$ – space of N -periodic signals	HT–Hilbert transform	$H(\mathbf{x})$ – HT of $\mathbf{x} \in \Pi[N]$
$\Pi[N, N]$ – space of 2D N -periodic arrays	2D – two-dimensional	WP – wavelet packet
DFT – Discrete Fourier transform	$\hat{x}[n]_m \stackrel{\text{def}}{=} \sum_{k=0}^{2^m N - 1} \omega^{-2^m k n} x[k]$	WPT – WP transform
cWP – complimentary WP	qWP – quasi-analytic WP	HT – Hilbert transform
$H(\mathbf{x})$ – HT of $\mathbf{x} \in \Pi[N]$	DTS – discrete-time spline	dWP – WP from DTS
SSIM – Structural Similarity Index	p-filter – periodic filter	PSNR – peak signal-to-noise ratio

Table 2 Notations for Section 2

$b^p(t)$ – periodic polynomial B-spline of order p —Eq. (2.1)	$\mathbf{b}_{[1]}^p$ – discrete-time order- p B-spline
$u^p[n]$ and $v^p[n]$ – DFTs of sampled B-spline – Eqs. (2.2) and (2.3)	Sequences $\alpha[n]$, $\beta[n]$ – Eq. (2.6)
$\psi_{[m],l}^p, l = 0, \dots, 2^m - 1$, – dWPs of order p from level m – Eq. (2.6)	$\tilde{\mathbf{H}} = \mathbf{H} = \{\mathbf{h}_0, \mathbf{h}_1\}$ – one-level p -filter bank
LDVM – local discrete vanishing moment	$\psi_{[m],j,l}^p[k, n] \stackrel{\text{def}}{=} \psi_{[m],j}^p[k] \psi_{[m],l}^p[n]$ – 2D dWP
$\mathbf{M}[n]$ and $\tilde{\mathbf{M}}[n]$ –modulation matrices for dWP transform Eq. (2.8)	$y_{[m],j,l}^p[k, v]$ – 2D dWp transform coefficients from level m

Remark 2 It is well known (for example, [27]) that the N -periodic sequence $u^p[n]$ is strictly positive and symmetric about $N/2 \pmod{N}$, where it attains its single minimum. The sequence $v^p[n]$ is $2N$ -periodic and $v^p[n+N] = -v^p[n]$.

Denote by $b_d^p(t) \stackrel{\text{def}}{=} b^p(t/2)/2$, which is the two-times dilation of the B-spline $b^p(t)$.

Definition 1 The discrete-time B-spline $\mathbf{b}_{[1]}^p$ ⁴ of order p is defined as an N -periodization of the sampled B-spline $b_d^p(t)$ ⁵: $b_{[1]}^p[k] \stackrel{\text{def}}{=} b_d^p(k)$, $k = -N/2, \dots, N/2 - 1 \pmod{N}$.

The discrete-time B-spline $\mathbf{b}_{[1]}^p$ is an N -periodic signal from $\Pi[N]$. The DFT of the B-spline $b_{[1]}^p$ is

$$\begin{aligned} \hat{b}_{[1]}^p[n] &= \sum_{k=-N/4}^{N/4-1} \omega^{-2kn} b_d^p(2k) + \omega^{-n} \sum_{k=-N/4}^{N/4-1} \omega^{-2kn} b_d^p(2k+1) \\ &= \frac{1}{2} \sum_{k=-N/4}^{N/4-1} \omega^{-2kn} b^p(k) + \frac{\omega^{-n}}{2} \sum_{k=-N/4}^{N/4-1} \omega^{-2kn} b^p\left(k + \frac{1}{2}\right) \quad (2.4) \\ &= \frac{u^p[2n] + v^p[2n]}{2}. \end{aligned}$$

The sequences $u^p[n]$ and $v^p[n]$ are defined in Eqs. (2.2) and (2.3). The samples of B-splines $b^p(t)$ of different orders at points $\{k\}$ and $\{k + 1/2\}$ can be easily computed using Eq. (2.1). We used here the fact that $b^p(t)$ is supported on the interval $(-p/2, p/2) \subset (-N/4, N/4 - 1) \cup (-N/2, N/2 - 1)$.

Remark 3 By referring to Remark 2, we claim that

$$\begin{aligned} u^p[2n+N] &= u^p[2n], \quad v^p[2n+N] = -v^p[2n], \quad u^p[0] = v^p[0] = u^p[N] = 1, \\ v^p[N] &= -1, \quad \hat{b}_{[1]}^p[0] = 1, \quad \hat{b}_{[1]}^p[N/2] = 0. \end{aligned} \quad (2.5)$$

Linear combinations of two-sample shifts of the B-splines $s_{[1]}^p[k] = \sum_{l=0}^{N/2-1} q[l] b_{[1]}^p[k-2l]$, where $\{q[l]\}$ is a real $N/2$ -periodic sequence, are referred to as the order- p periodic discrete-time splines (DTSs). Their DFTs are $\hat{s}_{[1]}^p[n] = \hat{q}[n] \hat{b}_{[1]}^p[n]$. The $N/2$ -dimensional space of the order- p DTSs is denoted by ${}^p\mathcal{S}_{[1]}^0 \subset \Pi[N]$. Two-sample shifts of the discrete-time B-spline $\mathbf{b}_{[1]}^p$ form a basis in the space ${}^p\mathcal{S}_{[1]}^0$. Denote by ${}^p\mathcal{S}_{[1]}^1$ the orthogonal complement of the subspace ${}^p\mathcal{S}_{[1]}^0$ in the signal space $\Pi[N]$. Thus, $\Pi[N] = {}^p\mathcal{S}_{[1]}^0 \oplus {}^p\mathcal{S}_{[1]}^1$.

⁴Here and in what follows the subscript $\cdot_{[m]}$ indicates the m th multiresolution level. In particular, $\mathbf{b}_{[1]}^p$ means the spline from the first level.

⁵Compare with the definition of the discrete B-spline in [23].

Define the DTS $\psi_{[1],0}^p$ and the signal $\psi_{[1],1}^p \in \Pi[N]$ by their DFTs (See Eq. (2.4)):

$$\begin{aligned}\hat{\psi}_{[1],0}^p[n] &\stackrel{\text{def}}{=} \frac{\hat{b}_{[1]}^p[n]}{\sqrt{\Upsilon^p[n]}} \stackrel{\text{def}}{=} \beta[n], \quad \hat{\psi}_{[1],1}^p[n] \stackrel{\text{def}}{=} \omega^n \frac{\hat{b}_{[1]}^p[n + N/2]}{\sqrt{\Upsilon^p[n]}} \stackrel{\text{def}}{=} \alpha[n], \\ \Upsilon^p[n] &\stackrel{\text{def}}{=} \frac{u^p[2n]^2 + v^p[2n]^2}{4}.\end{aligned}\quad (2.6)$$

The real-valued signals $\psi_{[1],0}^p[k]$ and $\psi_{[1],1}^p[k]$ are symmetric about $k = 0$ and $k = -1$, respectively.

Proposition 1 *Two-sample shifts of the signals $\psi_{[1],\lambda}^p[k]$, $\lambda = 0, 1$ form orthonormal bases of the mutually orthogonal subspaces ${}^p\mathcal{S}_{[1]}^\lambda$, $\lambda = 0, 1$, respectively, such that their inner products in the space $\Pi[N]$ are $\langle \psi_{[1],\lambda}^p[\cdot - 2l], \psi_{[1],\lambda}^p[\cdot - 2m] \rangle = \delta(l - m)$, $\lambda = 0, 1$. The set $\left\{ \psi_{[1],0}^p[\cdot - 2l] \right\} \oplus \left\{ \psi_{[1],1}^p[\cdot - 2l] \right\}$, where $l = 0, \dots, N/2 - 1$, form an orthonormal basis of the space $\Pi[N]$.*

Proof Equation (2.5) implies that the B-spline's DFT

$$\hat{b}_{[1]}^p[n + N/2] = \frac{1}{2} (u^p[2n + N] + v^p[2n + N]) = \frac{1}{2} (u^p[2n] - v^p[2n]).$$

Thus the inner product of the DTSs is

$$\begin{aligned}\langle \psi_{[1],0}^p[\cdot - 2l], \psi_{[1],0}^p[\cdot - 2m] \rangle &= \frac{1}{N} \sum_{n=0}^{N-1} \omega^{2(m-l)n} \left(\hat{\psi}_{[1],0}^p[n] \right)^2 \\ &= \frac{1}{N} \sum_{n=0}^{N/2-1} \omega^{2(m-l)n} \left(\left(\hat{\psi}_{[1],0}^p[n] \right)^2 + \left(\hat{\psi}_{[1],0}^p[n + N/2] \right)^2 \right) \\ &= \frac{1}{N} \sum_{n=0}^{N/2-1} \omega^{2(m-l)n} \frac{(u^p[2n] + v^p[2n])^2 + (u^p[2n] - v^p[2n])^2}{u^p[2n]^2 + v^p[2n]^2} \\ &= \frac{2}{N} \sum_{n=0}^{N/2-1} \omega^{2(m-l)n} = \delta[l - m].\end{aligned}$$

Consequently, the set $\psi_{[1],0}^p[\cdot - 2l]$, $l = 0, \dots, N/2 - 1$, form an orthonormal basis for the $N/2$ -dimensional subspace ${}^p\mathcal{S}_{[1]}^0 \subset \Pi[N]$. The proof for the signals $\psi_{[1],1}^p[\cdot - 2l]$, $l = 0, \dots, N/2 - 1$, is quite similar. $\square$

Corollary 1 *The orthogonal projections of a signal $\mathbf{x} \in \Pi[N]$ onto the subspaces ${}^p\mathcal{S}_{[1]}^\lambda$, $\lambda = 0, 1$, are the signals $\mathbf{x}_{[1]}^\lambda \in \Pi[N]$, respectively, such that*

$$\begin{aligned}x_{[1]}^\lambda[k] &= \sum_{l=0}^{N/2-1} y_{[1]}^\lambda[l] \psi_{[1],\lambda}^p[k - 2l] = \sum_{l=0}^{N/2-1} y_{[1]}^\lambda[l] h_{[1]}^\lambda[k - 2l], \\ y_{[1]}^\lambda[l] &= \langle \mathbf{x}, \psi_{[1],\lambda}^p[\cdot - 2l] \rangle = \sum_{k=0}^{N-1} h_{[1]}^\lambda[k - 2l] x[k], \quad h_{[1]}^\lambda[k] \stackrel{\text{def}}{=} \psi_{[1],\lambda}^p[k], \quad \lambda = 0, 1, \quad k \in \mathbb{Z}.\end{aligned}$$

Remark 4 The sets $\{y_{[1]}^0[l]\}$ and $\{y_{[1]}^1[l]\}$, $l = 0, \dots, N/2 - 1$, of the orthogonal projection coefficients are regarded as results of filtering the signal $\mathbf{x}$ by the time-reversed low- and high-pass p-filters $\mathbf{h}_{[1]}^0$ and $\mathbf{h}_{[1]}^1$, respectively, which is followed by downsampling by factor 2. The signal $\mathbf{x}$ is restored by filtering the transform coefficients $\{y_{[1]}^0[l]\}$ and $\{y_{[1]}^1[l]\}$, $l = 0, \dots, N/2 - 1$, by the same p-filters $\mathbf{h}_{[1]}^0$ and $\mathbf{h}_{[1]}^1$, respectively, which is followed by upsampling by factor 2. Thus, the pair $\mathbf{H}_{[1]} \stackrel{\text{def}}{=} \{\mathbf{h}_{[1]}^0, \mathbf{h}_{[1]}^1\}$ is a critically sampled perfect reconstruction p-filter bank.

The impulse responses of the p-filters $\mathbf{h}_{[1]}^\lambda$, $\lambda = 0, 1$, coincide with the signals $\psi_{[1],\lambda}^p[k]$, respectively. Their frequency responses are $\hat{h}_{[1]}^0[n] = \beta[n]$, $\hat{h}_{[1]}^1[n] = \alpha[n]$. The p-filters are compactly supported neither in the time nor in the frequency domain but are well localized in both (see Fig. 1).

Definition 2 The signals $\psi_{[1],0}^p$ and $\psi_{[1],1}^p$ are referred to as the discrete-time-spline wavelet packets (dWPs) of order p from the first decomposition level.

Figure 1 displays the dWPs $\psi_{[1],0}^p$ and $\psi_{[1],1}^p$ (which are the p-filters' $\mathbf{h}_{[1]}^0$ and $\mathbf{h}_{[1]}^1$ impulse responses) and magnitudes of their DFTs (which are the p-filters' $\mathbf{h}_{[1]}^0$ and $\mathbf{h}_{[1]}^1$ magnitude responses) of different orders. It is seen that the WPs are well localized in time domain. Their spectra are flat and their shapes tend to rectangular as their orders increase.

The one-level dWP transform of a signal $\mathbf{x}$ and its inverse are represented in a matrix form:

$$\begin{pmatrix} \hat{y}_{[1]}^0[n] \\ \hat{y}_{[1]}^1[n] \end{pmatrix} = \frac{1}{2} \tilde{\mathbf{M}}[-n] \cdot \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\tilde{n}] \end{pmatrix}, \quad \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\tilde{n}] \end{pmatrix} = \mathbf{M}[n] \cdot \begin{pmatrix} \hat{y}_{[1]}^0[n] \\ \hat{y}_{[1]}^1[n] \end{pmatrix}, \quad (2.7)$$

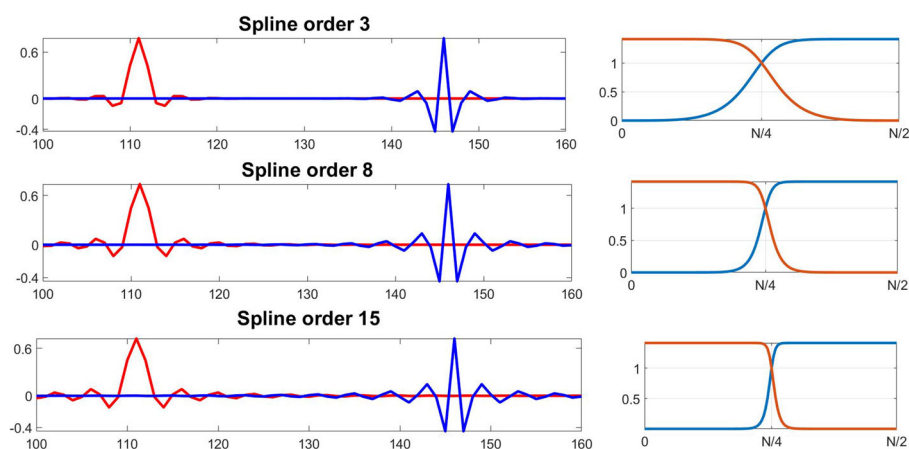


Fig. 1 Left: dWPs $\psi_{[1],0}^p$ (red lines) and $\psi_{[1],1}^p$ (blue lines), $p = 3, 8, 15$. Right: magnitude spectra of $\psi_{[1],0}^p$ (red lines) and $\psi_{[1],1}^p$ (blue lines)

where $\tilde{n} = n + N/2$ and $\tilde{\mathbf{M}}[n]$ and $\mathbf{M}[n]$ are the modulation matrices of the analysis and synthesis p-filter banks, respectively.

The modulation matrices are related as $\mathbf{M}[n] = \tilde{\mathbf{M}}[n]^T$, and

$$\mathbf{M}[n] \stackrel{\text{def}}{=} \sqrt{2} \begin{pmatrix} \hat{h}_{[1]}^0[n] & \hat{h}_{[1]}^1[n] \\ \hat{h}_{[1]}^0[n + \frac{N}{2}] & \hat{h}_{[1]}^1[n + \frac{N}{2}] \end{pmatrix} = \sqrt{2} \begin{pmatrix} \beta[n] & \alpha[n] \\ \beta[n + \frac{N}{2}] & \alpha[n + \frac{N}{2}] \end{pmatrix}, \quad (2.8)$$

where $\beta[n]$ and $\alpha[n]$ are defined in Eq. (2.6).

The synthesis p-filter bank $\mathbf{H}_{[1]} = \{\mathbf{h}_{[1]}^0, \mathbf{h}_{[1]}^1\}$ coincides with the analysis p-filter bank and, together, they form a perfect reconstruction p-filter bank.

2.2 Extension of transforms to deeper decomposition levels

2.2.1 Second-level wavelet packet transforms (WPTs)

The WPT from the first to the second decomposition level is implemented by application of the analysis p-filter bank $\tilde{\mathbf{H}}_{[2]} \stackrel{\text{def}}{=} \{\mathbf{h}_{[2]}^0, \mathbf{h}_{[2]}^1\}$, which operates in the space $\Pi[N/2]$, to the signals $\mathbf{y}_{[1]}^\lambda$, $\lambda = 0, 1$. The frequency responses of the p-filters are $\hat{h}_{[2]}^0[n]_1 = \beta[2n]$, $\hat{h}_{[2]}^1[n]_1 = \alpha[2n]$, where $\beta[n]$ and $\alpha[n]$ are defined in Eq. (2.6). Consequently, the modulation matrices of the p-filter bank $\mathbf{H}_{[2]}$ are $\tilde{\mathbf{M}}_{[2]}[n] = \tilde{\mathbf{M}}[2n]$, and $\mathbf{M}_{[2]}[n] = \mathbf{M}[2n]$, where the matrices $\tilde{\mathbf{M}}[n]$ and $\mathbf{M}[n]$ are defined in Eq. (2.8).

Define the second-level dWPs $\psi_{[2],\rho}^p \in \Pi[N]$ by their DFT

$$\hat{\psi}_{[2],\rho}^p[n] \stackrel{\text{def}}{=} \hat{\psi}_{[1],\lambda}^p[n] \hat{h}_{[2]}^\mu[n]_1 = \hat{\psi}_{[1],\lambda}^p[n] \hat{\psi}_{[1],\mu}^p[2n]_1, \quad (2.9)$$

where $\rho \stackrel{\text{def}}{=} \begin{cases} \mu, & \text{if } \lambda = 0; \\ 3 - \mu, & \text{if } \lambda = 1. \end{cases}$ Equation (2.9) means that the dWPs $\psi_{[2],\rho}^p$ are derived from the first-level dWPs $\psi_{[1],\lambda}^p$ by filtering the latter with the p-filters $\mathbf{h}_{[2]}^\mu$, $\lambda, \mu = 0, 1$. Figure 2 displays the second-level WPs originating from DTSs of orders 3, 8 and 15 and their DFTs. One can observe that the wavelet packets are symmetric and well localized in time domain. Their spectra are flat and their shapes tend to rectangular as their orders increase. They split the frequency domain into four quarter-bands.

Proposition 2 [24, Chapter 4] *The norms of the signals $\psi_{[2],\rho}^p \in \Pi[N]$ are equal to one. The 4-sample shifts $\{\psi_{[2],\rho}^p[\cdot - 4l]\}$, $l = 0, \dots, N/4 - 1$, of this signal are mutually orthogonal and signals with different indices ρ are orthogonal to each other.*

Thus, the signal space $\Pi[N]$ splits into four mutually orthogonal subspaces $\Pi[N] = \bigoplus_{\rho=0}^3 {}^p\mathcal{S}_{[2]}^\rho$ whose orthonormal bases are formed by 4-sample shifts $\{\psi_{[2],\rho}^p[\cdot - 4l]\}$, $l = 0, \dots, N/4 - 1$, of the signals $\psi_{[2],\rho}^p$, which are referred to as the second-level dWPs of order p .

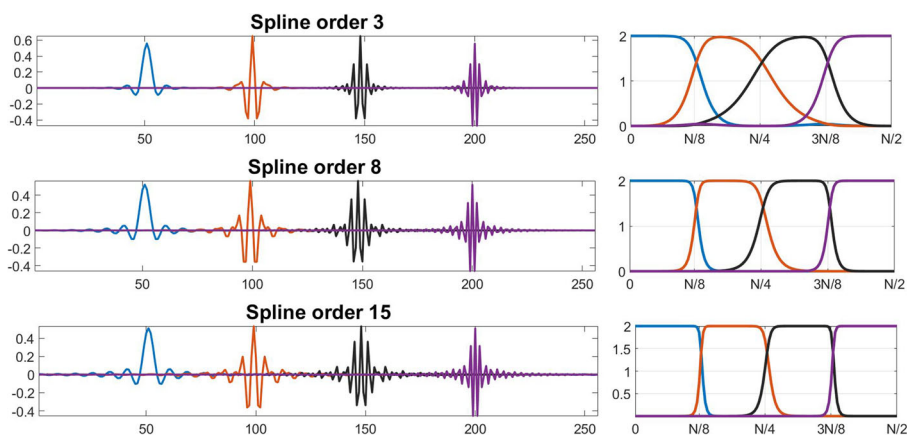


Fig. 2 Left: second-level dWPs of different orders; left to right: $\psi_{[2],0}^p \rightarrow \psi_{[2],1}^p \rightarrow \psi_{[2],2}^p \rightarrow \psi_{[2],3}^p$. Right: Their magnitude DFT spectra

The orthogonal projection of a signal $\mathbf{x} \in \Pi[N]$ onto the subspace ${}^p\mathcal{S}_{[2]}^\rho$ is the signal

$$x_{[2]}^\rho[k] = \sum_{l=0}^{N/4-1} \langle \mathbf{x}, \psi_{[2],\rho}^p[\cdot - 4l] \rangle \psi_{[2],\rho}^p[k - 4l] = \sum_{l=0}^{N/4-1} y_{[2]}^\rho[l] \psi_{[2],\rho}^p[k - 4l], \quad k = 0, \dots, N-1.$$

Derivation of the wavelet packet transform coefficients $\mathbf{y}_{[2]}^\rho$, $\rho = 0, 1, 2, 3$, from $\mathbf{y}_{[1]}^\lambda$, $\lambda = 0, 1$, and the inverse operation are implemented using the modulation matrices of the p-filter bank $\mathbf{H}_{[2]}$:

$$\begin{pmatrix} \hat{y}_{[2]}^{\rho 0}[n]_2 \\ \hat{y}_{[2]}^{\rho 1}[n]_2 \end{pmatrix} = \frac{1}{2} \tilde{\mathbf{M}}[-2n] \cdot \begin{pmatrix} \hat{y}_{[1]}^\lambda[n]_1 \\ \hat{y}_{[1]}^\lambda[\bar{n}]_1 \end{pmatrix}, \quad \begin{pmatrix} \hat{y}_{[1]}^\lambda[n]_1 \\ \hat{y}_{[1]}^\lambda[\bar{n}]_1 \end{pmatrix} = \mathbf{M}[2n] \cdot \begin{pmatrix} \hat{y}_{[2]}^{\rho 0}[n]_2 \\ \hat{y}_{[2]}^{\rho 1}[n]_2 \end{pmatrix}, \quad (2.10)$$

where $\bar{n} = n + N/4$ and $\begin{cases} \rho 0 = 0, \rho 1 = 1, & \text{if } \lambda = 0; \\ \rho 0 = 3, \rho 1 = 2, & \text{if } \lambda = 1. \end{cases}$

2.2.2 Transforms to deeper levels

The WPTs to deeper decomposition levels are implemented iteratively, while the transform coefficients $\{\mathbf{y}_{[m+1]}^\rho\}$ are derived by filtering the coefficients $\{\mathbf{y}_{[m]}^\lambda\}$ with the p-filters $\mathbf{h}_{[m+1]}^\mu$, where $\lambda = 0, \dots, 2^m - 1$, $\mu = 0, 1$ and $\rho \stackrel{\text{def}}{=} \begin{cases} 2\lambda + \mu, & \text{if } \lambda \text{ is even;} \\ 2\lambda + (1 - \mu), & \text{if } \lambda \text{ is odd.} \end{cases}$ The frequency responses of the p-filters are $\hat{h}_{[m+1]}^0[n]_1 = \beta[2^m n]$, $\hat{h}_{[2]}^1[n]_1 = \alpha[2^m n]$, where $\beta[n]$ and $\alpha[n]$ are defined in Eq. (2.6).

The transform coefficients are $y_{[m]}^\lambda[l] = \langle \mathbf{x}, \psi_{[m],\lambda}^p[\cdot, -2^m l] \rangle$, where the signals $\psi_{[m],\lambda}^p$ are normalized, orthogonal to each other in the space $\Pi[N]$, and their $2^m l$ -sample shifts are mutually orthogonal. They are referred to as the level- m dWPs of order p . The set $\{\psi_{[m],\lambda}^p[\cdot, -2^m l]\}$, $\lambda = 0, \dots, 2^m - 1$, $l = 0, \dots, N/2^m - 1$, constitutes an orthonormal basis of the space $\Pi[N]$ and generates its split into 2^m orthogonal subspaces. The next-level dWPs $\psi_{[m+1],\rho}^p$ are derived by filtering the dWPs $\psi_{[m],\lambda}^p$ with the p-filters $\mathbf{h}_{[m+1]}^\mu$ such that

$$\psi_{[m+1],\rho}^p[n] = \sum_{k=0}^{N/2^m-1} h_{[m+1]}^\mu[k] \psi_{[m],\lambda}^p[n - 2^m k], \quad \lambda = 0, \dots, 2^m - 1, \quad \mu = 0, 1. \quad (2.11)$$

Note that the frequency response of an m -level p-filter is $\hat{h}_{[m]}^\mu[n] = \hat{h}_{[1]}^\mu[2^{m-1}n]$.

The transforms are executed in the spectral domain using the Fast Fourier transform (FFT) by the application of critically sampled two-channel filter banks to the half-band spectral components of a signal. For example, the Matlab execution of the 8-level 13th-order WPT of a signal comprising 245760 samples takes 0.2324 s.

2.3 2D WPTs

A standard way to extend the one-dimensional (1D) WPTs to multiple dimensions is the tensor-product extension. The 2D one-level WPT of a signal $\mathbf{x} = \{x[k, n]\}$, $k, n = 0, \dots, N - 1$, which belongs to $\Pi[N, N]$, consists of the application of 1D WPT to columns of $\mathbf{x}$, which is followed by the application of the transform to rows of the coefficient array. As a result of the 2D WPT of signals from $\Pi[N, N]$, the space becomes split into four mutually orthogonal subspaces $\Pi[N, N] = \bigoplus_{j,l=0}^1 {}^p\mathcal{S}_{[1]}^{j,l}$.

The 2D dWPs are $\psi_{[1],j,l}^p[n, m] \stackrel{\text{def}}{=} \psi_{[1],j}^p[n] \psi_{[1],l}^p[m]$, $j, l = 0, 1$. They are normalized and orthogonal to each other in the space $\Pi[N, N]$. It means that their inner products are $\sum_{n,m=0}^{N-1} \psi_{[1],j_1,l_1}^p[n, m] \psi_{[1],j_2,l_2}^p[n, m] = \delta[j_1 - j_2] \delta[l_1 - l_2]$. Their two-sample shifts in both directions are mutually orthogonal. The subspace ${}^p\mathcal{S}_{[1]}^{j,l}$ is a linear hull of two-sample shifts of the 2D dWPs $\{\psi_{[1],j,l}^p[k - 2\kappa, n - 2\nu]\}$, $\kappa, \nu = 0, \dots, N/2 - 1$, that form an orthonormal basis of ${}^p\mathcal{S}_{[1]}^{j,l}$. The orthogonal projection of the signal $\mathbf{x} \in \Pi[N, N]$ onto the subspace ${}^p\mathcal{S}_{[1]}^{j,l}$ is the signal $\mathbf{x}_{[1]}^{j,l} \in \Pi[N, N]$ such that $x_{[1]}^{j,l}[k, n] = \sum_{\kappa,\nu=0}^{N/2-1} y_{[1]}^{j,l}[\kappa, \nu] \psi_{[1],j,l}^p[k - 2\kappa, n - 2\nu]$, $j, l = 0, 1$. The transform coefficients are

$$y_{[1]}^{j,l}[\kappa, \nu] = \langle \mathbf{x}, \psi_{[1],j,l}^p[\cdot - 2\kappa, \cdot - 2\nu] \rangle = \sum_{n,m=0}^{N-1} \psi_{[1],j,l}^p[n - 2\kappa, m - 2\nu] x[n, m].$$

By the application of the above transforms iteratively to blocks of the transform coefficients down to m th level, we get the space $\Pi[N, N]$ decomposed into 4^m mutually orthogonal subspaces $\Pi[N, N] = \bigoplus_{j,l=0}^{2^m-1} {}^p\mathcal{S}_{[m]}^{j,l}$.

The 2D tensor-product wavelet packets $\psi_{[m],j,l}^p$ are well localized in the spatial domain, and their 2D DFT spectra provide a refined split of the frequency domain of signals from $\Pi[N, N]$.⁶ The disadvantage for the image processing is that the WPs are oriented in either horizontal or vertical directions or are not oriented at all.

2.4 Local discrete vanishing moments

One of fundamental features of wavelets and wavelet packets is their vanishing moment property. In a conventional setting, it means the annihilation of polynomials of a certain degree by a continuous wavelet or wavelet packet $\psi(t)$. To be specific, if for any polynomial $P_{m-1}(t)$ of degree $m-1$ the relation $\int \psi(t - \tau) P_{m-1}(\tau) d\tau \equiv 0$ holds, then it is said that $\psi(t)$ has m vanishing moments.

We modify the vanishing moment property for the discrete periodic setting.

Proposition 3 [28, Chapter 15] Assume that the frequency response of the high(band)-pass p -filter $\mathbf{g}$ can be represented as $\hat{g}[n] = \sin\left(\frac{\pi n}{N}\right)^m \xi[n]$, where m is some natural number; and $\xi[n]$ is an N -periodic sequence. Assume that $\mathbf{p}$ is a signal from $\Pi[N]$, and it coincides with a sampled polynomial $\mathbf{P}_{m-1}$ of degree $m-1$ at some interval $p[k] = P_{m-1}(k)$ as $k = k_0, k_0 + 1, \dots, k_m$, where $m < k_m - k_0 < N$. Then, $\sum_{l=0}^{N-1} g[k-l] p[l] = 0$, as $k = k_0, \dots, k_m - m - 1$.

Definition 3 If a high(band)-pass p -filter $\mathbf{g}$ satisfies the conditions of Proposition 3, we say that the p -filter $\mathbf{g}$ locally eliminates sampled polynomials of degree $m-1$. If the periodic filter $\mathbf{g}$ is generated by a wavelet packet $\psi_{[l],j}^p$, that is $g[k] \stackrel{\text{def}}{=} \psi_{[l],j}^p[k]$, $k \in \mathbb{Z}$, then we say that the wavelet packet $\psi_{[l],j}^p$ has m local discrete vanishing moments (LDVMs).

Proposition 4 Assume that $\psi_{[l],j}^p$, $j = 1, \dots, 2^l - 1$ is a dWP from the decomposition level l , which is derived from the spline of order p . If p is equal to either $2r-1$ or $2r$, then the wavelet packet $\psi_{[l],j}^p$ has $2r$ LDVMs.

Proof In [Appendix](#). □

3 (Quasi-)analytic and complementary WPs

In this section, we define analytic and the so-called quasi-analytic WPs related to the dWPs discussed in Section 2 and introduce an orthonormal set of waveforms which are complementary to the above dWPs. Notations for Sections 3–6 are given in Table 3.

⁶Especially it is true for WPs derived from higher-order DTSSs.

Table 3 Notations for Sections 3, 4, 5, and 6

$\mathbf{h} = H(\mathbf{x})$ – HT of the signal $\mathbf{x}$	$\mathbf{x}_{\pm} = \mathbf{x} \pm i\mathbf{h}$ – analytic signal related to the signal $\mathbf{x}$ Eq. (3.1)
$\theta_{[m],l}^p = H(\psi_{[m],l}^p)$ – HT of dWP $\psi_{[m],l}^p$	$\psi_{\pm[m],l}^p = \psi_{[m],l}^p \pm i\theta_{[m],l}^p$ – analytic dWPs
$\varphi_{[m],l}^p$ – complementary WP (cWP) –Eq. (3.2)	$\Psi_{\pm[m],l}^p = \psi_{[m],l}^p \pm i\varphi_{[m],l}^p$ – complex quasi-analytic WP (qWP)
$\{\tilde{\mathbf{M}}^c[n], \mathbf{M}^c[n]\}$ – modulation matrices for one-level transform with $\varphi_{[1],l}^p$	$\{\tilde{\mathbf{M}}_{\pm}^q[n], \mathbf{M}_{\pm}^q[n]\}$ – modulation matrices for one-level qWP transform
$\Psi_{++[m],j,l}^p[k, n]$ and $\Psi_{+-[m],j,l}^p[k, n]$ – 2D complex qWPs	$\vartheta_{+[m],j,l}^p[k, n]$ and $\vartheta_{-[m],j,l}^p[k, n]$ – directional qWPs, Eq. (5.1)

3.1 Analytic periodic signals

A signal $\mathbf{x} \in \Pi[N]$ is represented by its inverse DFT which can be written as follows:

$$x[k] = \frac{\hat{x}[0] + (-1)^k \hat{x}[N/2]}{N} + \frac{2}{N} \sum_{n=1}^{N/2-1} \frac{\hat{x}[n] \omega^{kn} + (\hat{x}[n] \omega^{kn})^*}{2}.$$

Define the real-valued signal $\mathbf{h} \in \Pi[N]$ and two complex-valued signals $\mathbf{x}_+$ and $\mathbf{x}_-$ such that

$$\begin{aligned} h[k] &\stackrel{\text{def}}{=} \frac{2}{N} \sum_{n=1}^{N/2-1} \frac{\hat{x}[n] \omega^{kn} - \hat{x}[n]^* \omega^{-kn}}{2i}, \\ x_{\pm}[k] &\stackrel{\text{def}}{=} x[k] \pm i h[k] = \frac{\hat{x}[0] + (-1)^k \hat{x}[N/2]}{N} \\ &\quad + \frac{2}{N} \sum_{n=1}^{N/2-1} \begin{cases} \hat{x}[n] \omega^{kn}, & \text{for } \bar{x}_+; \\ \hat{x}[-n] \omega^{-kn} = \hat{x}[N-n] \omega^{-k(N-n)}, & \text{for } \bar{x}_-. \end{cases} \end{aligned} \quad (3.1)$$

The spectrum of $\mathbf{x}_+$ comprises only non-negative frequencies and vice versa for $\mathbf{x}_-$. We have $\mathbf{x} = \Re(\mathbf{x}_{\pm})$ and $\Im(\mathbf{x}_{\pm}) = \pm \mathbf{h}$. The signals $\mathbf{x}_{\pm}$ are referred to as discrete periodic analytic signals.

Thus, the signal $\mathbf{h}$ can be regarded as a discrete periodic version of the Hilbert transform (HT) of a discrete-time periodic signal $\mathbf{x}$, that is $\mathbf{h} = H(\mathbf{x})$ (see [29], for example).

Proposition 5

1. If a signal $\mathbf{x} \in \Pi[N]$ is symmetric about a grid point $k = K$, then the HT $\mathbf{h} = H(\mathbf{x})$ is antisymmetric about K and $h[K] = 0$.
2. Assume that a signal $\mathbf{x} \in \Pi[N]$ and $\hat{x}[0] = \hat{x}[N/2] = 0$. Then,
 - (a) The norm of its HT is $\|H(\mathbf{x})\| = \|\mathbf{x}\|$.
 - (b) The magnitude spectra of the signals $\mathbf{x}$ and $\mathbf{h} = H(\mathbf{x})$ coincide.

Proof straightforward. □

3.2 Analytic WPs

Denote $l_0 \stackrel{\text{def}}{=} 0$, $l_m \stackrel{\text{def}}{=} 2^m - 1$.

The analytic dWPs and their DFT spectra are derived from the corresponding dWPs $\{\psi_{[m],l}^p\}$, $m = 1, \dots, M$, $l = 0, \dots, 2^m - 1$, in line with the scheme in Section 3.1. Recall that for all $l \neq l_0$, the DFT $\hat{\psi}_{[m],l}^p[0] = 0$ and for all $l \neq l_m$, the DFT $\hat{\psi}_{[m],l}^p[N/2] = 0$.

Denote by $\theta_{[m],l}^p = H(\psi_{[m],l}^p)$ the HT of the wavelet packet $\psi_{[m],l}^p$, such that the DFT is

$$\hat{\theta}_{[m],l}^p[n] = \begin{cases} -i \hat{\psi}_{[m],l}^p[n], & \text{if } 0 < n < N/2; \\ i \hat{\psi}_{[m],l}^p[n], & \text{if } -N/2 < n < 0; \\ 0, & \text{if } n = 0, \text{ or } n = N/2. \end{cases}$$

Then, the corresponding analytic dWPs are $\psi_{\pm[m],l}^p \stackrel{\text{def}}{=} \psi_{[m],l}^p \pm i\theta_{[m],l}^p$.

Properties of the analytic WPs

1. The DFT spectra of the analytic WPs $\psi_{+[m],l}^p$ and $\psi_{-[m],l}^p$ are located within the bands $[0, N/2]$ and $[N/2, N] \iff [-N/2, 0]$, respectively.
2. The real component $\psi_{[m],l}^p$ is the same for both WPs $\psi_{+[m],l}^p$ and $\psi_{-[m],l}^p$. It is a symmetric oscillating waveform.
3. The HT WPs $\theta_{[m],l}^p = H(\psi_{[m],l}^p)$ are antisymmetric oscillating waveforms.
4. For all $l \neq l_0$, l_m , the norms $\|\theta_{[m],l}^p\| = 1$. Their magnitude spectra $|\hat{\theta}_{[m],l}^p[n]|$ coincide with the magnitude spectra of the respective WPs $\psi_{[m],l}^p$.
5. When $l = l_0$ or $l = l_m$, the magnitude spectra of $\theta_{[m],l}^p$ coincide with that of $\psi_{[m],l}^p$ everywhere except for the points $n = 0$ or $N/2$, respectively, and the waveforms' norms are no longer equal to 1.

Properties in items 3–5 follow directly from Proposition 5.

Proposition 6 For all $l \neq l_0, l_m$, the shifts of the HT WPs $\{\theta_{[m],l}^p[\cdot - 2^m l]\}$ are orthogonal to each other in the space $\Pi[N]$. The orthogonality does not take place for $\theta_{[m],l_0}^p$ and $\theta_{[m],l_m}^p$.

Proof Assume that $l \neq l_0, l_m$. The inner product is

$$\begin{aligned} \langle \theta_{[m],l}^p, \theta_{[m],l}^p[\cdot - 2^m r] \rangle &= \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \omega^{2^m r n} |\hat{\theta}_{[m],l}^p[n]|^2 \\ &= \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \omega^{2^m r n} |\hat{\psi}_{[m],l}^p[n]|^2 = \langle \psi_{[m],l}^p, \psi_{[m],l}^p[\cdot - 2^m r] \rangle = 0. \end{aligned}$$

□

3.3 Complementary set of wavelet packets and quasi-analytic WPs

3.3.1 Complementary orthonormal WPs

The values $\hat{\theta}_{[m],l_0}^p[0]$ and $\hat{\theta}_{[m],l_m}^p[N/2]$ are missing in the DFT spectra of the HT waveforms $\theta_{[m],l_0}^p$ and $\theta_{[m],l_m}^p$, respectively, which prevents the set $\{\theta_{[m],l}^p\}_{l=0, \dots, 2^m-1}$ from forming orthonormal bases in the corresponding subspaces.

This keeping in mind, we define a set $\{\varphi_{[m],l}^p\}$, $m = 1, \dots, M$, $l = 0, \dots, 2^m - 1$, of signals from the space $\Pi[N]$ via their DFTs:

$$\hat{\varphi}_{[m],l}^p[n] \stackrel{\text{def}}{=} \hat{\theta}_{[m],l}^p[n] + \hat{\psi}_{[m],l}^p[0] + \hat{\psi}_{[m],l}^p[N/2]. \quad (3.2)$$

For all $l \neq l_0, l_m$,⁷ the signals $\varphi_{[m],l}^p$ coincide with $\theta_{[m],l}^p = H(\psi_{[m],l}^p)$.

Properties of signals $\{\varphi_{[m],l}^p\}$, $m = 1, \dots, M$, $l = 0, \dots, 2^m - 1$:

1. The magnitude spectra $|\hat{\varphi}_{[m],l}^p[n]|$ coincide with those of the respective WPs $\psi_{[m],l}^p$.
2. For any $m = 1, \dots, M$, and $l = 1, \dots, 2^m - 2$, the signals $\varphi_{[m],l}^p$ are antisymmetric oscillating waveforms. For $l = l_0, l_m$, the shapes of the signals are near antisymmetric.
3. The orthonormality properties that are similar to the properties of WPs $\psi_{[m],l}^p$ hold for the signals $\varphi_{[m],l}^p$ such that $\langle \varphi_{[m],l}^p[\cdot - r 2^m], \varphi_{[m],\lambda}^p[\cdot - s 2^m] \rangle = \delta[\lambda, l] \delta[r, s]$.

Figure 3 displays the signals $\varphi_{[3],l}^9$ and $\varphi_{[3],l}^9$, $l = 0, \dots, 7$, from the third decomposition level and their magnitude spectra. Addition of $\hat{\psi}_{[3],l}^p[0]$ and $\hat{\psi}_{[3],l}^p[N/2]$ to the spectra of $\varphi_{[3],l}^9$, $l = 0, 7$ results in an antisymmetry distortion.

We call the signals $\{\varphi_{[m],l}^p\}$, $m = 1, \dots, M$, $l = 0, \dots, 2^m - 1$, the *complementary wavelet packets* (cWPs). Similarly to the dWPs $\{\psi_{[m],l}^p\}$, different combinations of the cWPs can provide different orthonormal bases for the space $\Pi[N]$. These can be, for example, the wavelet bases or a type of Best Basis [30, 31].

3.3.2 Quasi-analytic WPs

The sets of complex-valued WPs, which we refer to as the quasi-analytic wavelet packets (qWPs), are defined by $\Psi_{\pm[m],l}^p \stackrel{\text{def}}{=} \psi_{[m],l}^p \pm i\varphi_{[m],l}^p$, $m = 1, \dots, M$, $l = 0, \dots, 2^m - 1$, where $\varphi_{[m],l}^p$ are the cWPs from Eq. (3.2). The qWPs $\Psi_{\pm[m],l}^p$ differ from the analytic WPs $\psi_{\pm[m],l}^p$ by the addition of the two values $\pm i\hat{\psi}_{[m],l}^p[0]$ and

⁷Recall that $l_0 \stackrel{\text{def}}{=} 0$, $l_m \stackrel{\text{def}}{=} 2^m - 1$.

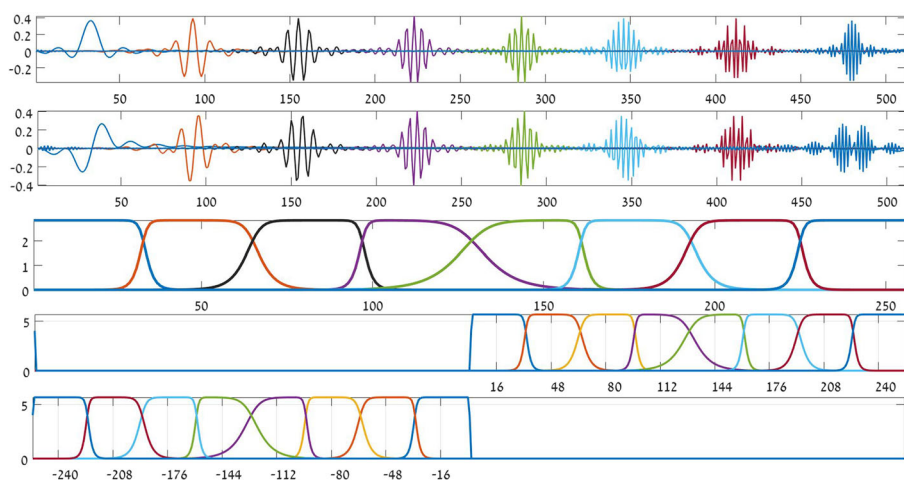


Fig. 3 Top to bottom: signals $\psi_{[3],l}^9$; signals $\varphi_{[3],l}^9$, $l = 0, \dots, 7$; their magnitude DFT spectra, respectively; magnitude DFT spectra of complex qWPs $\Psi_{+[3],l}^9$; same for $\Psi_{-[3],l}^9$, $l = 0, \dots, 7$

$\pm i \hat{\psi}_{[m],l}^p[N/2]$ into their DFT spectra, respectively. For a given decomposition level m , these values are zero for all l except for $l_0 = 0$ and $l_m = 2^m - 1$. It means that for all l except for l_0 and l_m , the qWPs $\Psi_{\pm[m],l}^p$ are analytic. The DFTs of qWPs are

$$\hat{\Psi}_{+[m],l}^p[n] = \begin{cases} (1 + i) \hat{\psi}_{[m],l}^p[n], & \text{if } n = 0, N/2; \\ 2 \hat{\psi}_{[m],l}^p[n], & \text{if } 0 < n < N/2; \\ 0 & \text{if } N/2 < n < N, \end{cases} \quad (3.3)$$

$$\hat{\Psi}_{-[m],l}^p[n] = \begin{cases} (1 - i) \hat{\psi}_{[m],l}^p[n], & \text{if } n = 0, N/2; \\ 0 & \text{if } 0 < n < N/2; \\ 2 \hat{\psi}_{[m],l}^p[n], & \text{if } N/2 < n < N. \end{cases}$$

3.3.3 Design of cWPs and qWPs

The DFTs of the first-level dWPs are $\hat{\psi}_{[1],0}^p[n] = \beta[n]$, $\hat{\psi}_{[1],1}^p[n] = \omega^n \beta[n + N/2] = \alpha[n]$, where the sequences $\beta[n]$ and $\alpha[n]$ are defined in Eq. (2.6). Equation (2.5) implies that $\hat{\psi}_{[1],0}^p[0] = \sqrt{2}$ and $\hat{\psi}_{[1],1}^p[N/2] = -\sqrt{2}$.

Consequently, the DFTs of the first-level cWPs are

$$\hat{\phi}_{[1],0}^p[n] = \begin{cases} -i \beta[n], & \text{if } 0 < n < N/2; \\ i \beta[n], & \text{if } N/2 < n < N; \\ \sqrt{2}, & \text{if } n = 0; \\ 0, & \text{if } n = N/2, \end{cases} \quad \hat{\phi}_{[1],1}^p[n] = \begin{cases} -i \alpha[n], & \text{if } 0 < n < N/2; \\ i \alpha[n], & \text{if } N/2 < n < N; \\ 0, & \text{if } n = 0; \\ -\sqrt{2}, & \text{if } n = N/2. \end{cases} \quad (3.4)$$

Figure 3 displays the magnitude spectra of complex qWPs $\Psi_{\pm[3],l}^9$, $l = 0, \dots, 7$, from the third decomposition level.

Remark 5 The following Proposition 7 and Corollary 2 establish an important fact that in order to derive the $m + 1$ -level cWPs and qWPs from the m -level ones, the same p-filters are used that are used for deriving the corresponding $(m + 1)$ -level dWPs from the m -level ones.

Proposition 7 Assume that for a dWP $\psi_{[m+1],\rho}^p$ the relation in Eq. 2.11 holds. Then, for the cWP $\varphi_{[m+1],\rho}^p$ we have

$$\begin{aligned}\varphi_{[m+1],\rho}^p[n] &= \sum_{k=0}^{N/2^m-1} h_{[m+1]}^\mu[k] \varphi_{[m],\lambda}^p[n - 2^m k] \iff \hat{\varphi}_{[m+1],\rho}^p[v] = \hat{h}_{[1]}^\mu[2^m v]_m \hat{\varphi}_{[m],\lambda}^p[v], \\ \hat{h}_{[1]}^0[v] &= \hat{\psi}_{[1],0}^p[v] = \beta[v], \quad \hat{h}_{[1]}^1[v] = \hat{\psi}_{[1],1}^p[v] = \alpha[v].\end{aligned}$$

Proof Due to Eq. (2.9), the DFT of the second-level dWPs are

$$\begin{aligned}\hat{\psi}_{[2],\rho}^p[n] &= \hat{\psi}_{[1],\lambda}^p[n] \hat{h}_{[2]}^\mu[n]_1, \quad \lambda, \mu = 0, 1, \quad \rho = 2\lambda + \begin{cases} \mu, & \text{if } \lambda = 0; \\ 1 - \mu, & \text{if } \lambda = 1. \end{cases} \\ \hat{h}_{[2]}^0[n]_1 &= \beta[2n], \quad \hat{h}_{[2]}^1[n]_1 = \alpha[2n].\end{aligned}$$

For example, assume that $\lambda = \mu = 0$. Then we have $\hat{\psi}_{[2],0}^p[n] = \hat{\psi}_{[1],0}^p[n] \hat{h}_{[2]}^0[n]_1 = \beta[n] \beta[2n]$. Keeping in mind that the sequence $\beta[2n]$ is $N/2$ -periodic, we have that the DFT of the corresponding cWP is

$$\begin{aligned}\hat{\varphi}_{[2],0}^p[n] &= \beta[0]^2 + \widehat{H(\psi_{[2],0}^p)}[n] = \beta[2n] \begin{cases} -i\beta[n], & \text{if } 0 < n < N/2; \\ i\beta[n], & \text{if } N/2 < n < N; \\ \sqrt{2}, & \text{if } n = 0; \\ 0, & \text{if } n = N/2, \end{cases} \\ &= \hat{\varphi}_{[1],0}^p[n] \hat{h}_{[2]}^0[n]_1 = \hat{\varphi}_{[1],0}^p[n] \hat{h}_{[1]}^0[2n]_1.\end{aligned}$$

A similar reasoning is applicable to all the second-level cWPs and to the cWPs from further decomposition levels. $\square$

Corollary 2 Assume that for a dWP $\psi_{[m+1],\rho}^p$ the relation in Eq. 2.11 holds. Then, for the qWP $\Psi_{\pm[m+1],\rho}^p$ we have

$$\begin{aligned}\Psi_{\pm[m+1],\rho}^p[n] &= \sum_{k=0}^{N/2^m-1} h_{[m+1]}^\mu[k] \Psi_{\pm[m],\lambda}^p[n - 2^m k] \\ &\iff \hat{\Psi}_{\pm[m+1],\rho}^p[v] = \hat{h}_{[1]}^\mu[2^m v]_m \hat{\Psi}_{\pm[m],\lambda}^p[v].\end{aligned}$$

4 Implementation of cWP and qWP transforms

Implementation of transforms with dWPs $\psi_{[m],\lambda}^p$ was discussed in Section 2. In this section, we extend the transform scheme to the transforms with cWPs $\varphi_{[m],\lambda}^p$ and

qWPs $\Psi_{[m],\lambda}^p$. Note that the structure of the filter bank for one-level transforms is completely different from that for deeper-levels transforms.

4.1 One-level transforms

Denote by ${}^p\mathcal{C}_{[1]}^0$ the subspace of the signal space $\Pi[N]$, which is the linear hull of the set $\Phi_{[1]}^0 \stackrel{\text{def}}{=} \{\varphi_{[1],0}^p[\cdot - 2k]\}$, $k = 0, \dots, N/2 - 1$. The signals from the set $\Phi_{[1]}^0$ form an orthonormal basis of the $N/2$ -dimensional subspace ${}^p\mathcal{C}_{[1]}^0$. Denote by ${}^p\mathcal{C}_{[1]}^1$ the orthogonal complement of the subspace ${}^p\mathcal{C}_{[1]}^0$ in the space $\Pi[N]$. The signals from the set $\Phi_{[1]}^1 \stackrel{\text{def}}{=} \{\varphi_{[1],1}^p[\cdot - 2k]\}$, $k = 0, \dots, N/2 - 1$ form an orthonormal basis of the subspace ${}^p\mathcal{C}_{[1]}^1$.

Proposition 8 *The orthogonal projections of a signal $\mathbf{x} \in \Pi[N]$ onto the spaces ${}^p\mathcal{C}_{[1]}^\mu$, $\mu = 0, 1$ are the signals $\mathbf{x}_{[1]}^\mu \in \Pi[N]$ such that*

$$x_{[1]}^\lambda[k] = \sum_{l=0}^{N/2-1} c_{[1]}^\lambda[l] \varphi_{[1],\lambda}^p[k - 2l], \quad c_{[1]}^\lambda[l] = \langle \mathbf{x}, \varphi_{[1],\lambda}^p[\cdot - 2l] \rangle = \sum_{k=0}^{N-1} g_{[1]}^\lambda[k - 2l] x[k],$$

$$g_{[1]}^\lambda[k] = \varphi_{[1],\lambda}^p[k], \quad \hat{g}_{[1]}^\lambda[n] = \hat{\varphi}_{[1],\lambda}^p[n], \quad \lambda = 0, 1.$$

The DFTs $\hat{\varphi}_{[1],\lambda}^p[n]$ of the first-level cWPs are given in Eq. 3.4.

The transforms $\mathbf{x} \rightarrow \mathbf{c}_{[1]}^0 \cup \mathbf{c}_{[1]}^1$ and back are implemented using the analysis $\tilde{\mathbf{M}}^c[n]$ and the synthesis $\mathbf{M}^c[n]$ modulation matrices:

$$\tilde{\mathbf{M}}^c[n] \stackrel{\text{def}}{=} \sqrt{2} \begin{pmatrix} \hat{g}_{[1]}^0[n] & \hat{g}_{[1]}^0\left[n + \frac{N}{2}\right] \\ \hat{g}_{[1]}^1[n] & \hat{g}_{[1]}^1\left[n + \frac{N}{2}\right] \end{pmatrix} = \sqrt{2} \begin{pmatrix} \check{\beta}[n] & -\check{\beta}\left[n + \frac{N}{2}\right] \\ \check{\alpha}[n] & -\check{\alpha}\left[n + \frac{N}{2}\right] \end{pmatrix},$$

$$\mathbf{M}^c[n] \stackrel{\text{def}}{=} \tilde{\mathbf{M}}^c[n]^T, \quad n = 0, \dots, N/2,$$

$$\check{\beta}[n] \stackrel{\text{def}}{=} \begin{cases} \beta[0], & \text{if } n = 0; \\ -i\beta[n], & \text{otherwise,} \end{cases} \quad \check{\alpha}[n] \stackrel{\text{def}}{=} \begin{cases} \alpha[N/2], & \text{if } n = N/2; \\ -i\alpha[n], & \text{otherwise.} \end{cases} \quad (4.1)$$

The sequences $\beta[n]$ and $\alpha[n]$ are given in Eq. (2.6).

Similarly to Eq. (2.7), the one-level cWP transform of a signal $\mathbf{x}$ and its inverse are:

$$\begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix} = \frac{1}{2} \tilde{\mathbf{M}}^c[-n] \cdot \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\bar{n}] \end{pmatrix}, \quad \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\bar{n}] \end{pmatrix} = \mathbf{M}^c[n] \cdot \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix},$$

where $\bar{n} = n + N/2$.

Define the p-filters $\mathbf{q}_{\pm[1]}^l \stackrel{\text{def}}{=} \mathbf{h}_{[1]}^j \pm i \mathbf{g}_{[1]}^j = \psi_{[1],l}^p \pm i \varphi_{[1],l}^p = \Psi_{\pm[1],l}^p$, $l = 0, 1$. Equation (3.3) implies that their frequency response s are

$$\hat{q}_{+[1]}^0[n] = \begin{cases} (1+i)\sqrt{2}, & \text{if } n = 0; \\ 2\beta[n], & \text{if } 0 < n < N/2; \\ 0, & \text{if } N/2 \leq n < N, \end{cases} \quad \hat{q}_{+[1]}^1[n] = \begin{cases} -(1+i)\sqrt{2}, & \text{if } n = N/2; \\ 2\alpha[n], & \text{if } 0 < n < N/2; \\ 0, & \text{if } N/2 < n \leq N. \end{cases}$$

$$\hat{q}_{-[1]}^0[n] = \begin{cases} (1-i)\sqrt{2}, & \text{if } n = 0; \\ 2\beta[n], & \text{if } N/2 < n < N, \\ 0, & \text{if } 0 < n \leq N/2; \end{cases} \quad \hat{q}_{-[1]}^1[n] = \begin{cases} -(1-i)\sqrt{2}, & \text{if } n = N/2; \\ 2\alpha[n], & \text{if } N/2 < n \leq N; \\ 0, & \text{if } 0 \leq n < N/2. \end{cases}$$

Thus, the analysis modulation matrices for the p-filters $\mathbf{q}_{\pm[1]}^l$ are

$$\tilde{\mathbf{M}}_+^q[n] = \sqrt{2} \begin{pmatrix} \hat{q}_{+[1]}^0[n] & 0 \\ \hat{q}_{+[1]}^1[n] & -\sqrt{2}(1+i)\delta[n-N/2] \end{pmatrix} = \tilde{\mathbf{M}}[n] + i\tilde{\mathbf{M}}^c[n], \quad (4.2)$$

$$\tilde{\mathbf{M}}_-^q[n] = \sqrt{2} \begin{pmatrix} (1-i)\sqrt{2}\delta[n] & \hat{q}_{-[1]}^0[n] \\ 0 & \hat{q}_{-[1]}^1[n] \end{pmatrix} = \tilde{\mathbf{M}}[n] - i\tilde{\mathbf{M}}^c[n], \quad (4.3)$$

where the modulation matrix $\tilde{\mathbf{M}}[n]$ is defined in Eq. (2.8) and $\tilde{\mathbf{M}}^c[n]$ is defined in Eq. (4.1). Application of the matrices $\tilde{\mathbf{M}}_{\pm}^q[n]$ to the vector $(\hat{x}[n], \hat{x}[\bar{n}])^T$ produces the vectors

$$\begin{pmatrix} \hat{z}_{\pm[1]}^0[n]_1 \\ \hat{z}_{\pm[1]}^1[n]_1 \end{pmatrix} = \frac{1}{2}(\tilde{\mathbf{M}}_{\pm}^q[n])^* \cdot \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\bar{n}] \end{pmatrix} = \begin{pmatrix} \hat{y}_{[1]}^0[n]_1 \\ \hat{y}_{[1]}^1[n]_1 \end{pmatrix} \mp i \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix}. \quad (4.4)$$

Equation (4.4) implies that the inverse DFTs of the sequences $\hat{z}_{\pm[1]}^j[n]_1$, $j = 0, 1$, which are the one-level qWP transform coefficients, are

$$z_{\pm[1]}^j[l] = \left\langle \mathbf{x}, \Psi_{\pm[1],j}^p[\cdot, -2l] \right\rangle = \sum_{k=0}^{N-1} x[k] \Psi_{\pm[1],j}^p[k-2l]^*, \quad l = 0, \dots, N/2-1. \quad (4.5)$$

Define the matrices $\mathbf{M}_{\pm}^q[n] \stackrel{\text{def}}{=} \tilde{\mathbf{M}}_{\pm}^q[n] = \mathbf{M}[n] \pm i\mathbf{M}^c[n]$ and apply these matrices to the vectors $(\hat{z}_{\pm[1]}^0[n]_1, \hat{z}_{\pm[1]}^1[n]_1)^T$.

Proposition 9 *The following relations hold*

$$\begin{aligned} \mathbf{M}_{\pm}^q[n] \cdot \begin{pmatrix} \hat{z}_{\pm[1]}^0[n]_1 \\ \hat{z}_{\pm[1]}^1[n]_1 \end{pmatrix} &= \mathbf{M}[n] \cdot \begin{pmatrix} \hat{y}_{[1]}^0[n]_1 \\ \hat{y}_{[1]}^1[n]_1 \end{pmatrix} + \mathbf{M}^c[n] \cdot \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix} \\ \pm i \left(\mathbf{M}^c[n] \cdot \begin{pmatrix} \hat{y}_{[1]}^0[n]_1 \\ \hat{y}_{[1]}^1[n]_1 \end{pmatrix} - \mathbf{M}[n] \cdot \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix} \right) \\ &= 2 \left(\begin{pmatrix} \hat{x}[n] \\ \hat{x}[n+N/2] \end{pmatrix} \pm i \begin{pmatrix} \hat{h}[n] \\ \hat{h}[n+N/2] \end{pmatrix} \right) = 2 \begin{pmatrix} \hat{x}_{\pm}[n] \\ \hat{x}_{\pm}[n+N/2] \end{pmatrix}, \end{aligned}$$

where $\mathbf{h}$ is the HT of the signal $\mathbf{x} \in \Pi[N]$ and $\mathbf{x}_{\pm}$ are the analytic signals associated with $\mathbf{x}$.

Proof In [Appendix](#). $\square$

Definition 4 The matrices $\tilde{\mathbf{M}}_{\pm}^q[n]$ and $\mathbf{M}_{\pm}^q[n]$ are called the analysis and synthesis modulation matrices for the one-level qWP transforms, respectively.

Remark 6 Successive application of the filter banks $\tilde{\mathbf{H}}_{\pm}^q$ and $\mathbf{H}_{\pm}^q$ defined by the analysis and synthesis modulation matrices $\tilde{\mathbf{M}}_{\pm}^q[n]$ and $\mathbf{M}_{\pm}^q[n]$, respectively, to a signal $\mathbf{x} \in \Pi[N]$ produces the analytic signals $\tilde{\mathbf{x}}_{\pm}$ associated with $\mathbf{x}$:

$$\mathbf{H}_{\pm}^q \cdot \tilde{\mathbf{H}}_{\pm}^q \cdot \mathbf{x} = 2\tilde{\mathbf{x}}_{\pm} \implies \mathbf{x} = \frac{1}{2} \Re(\mathbf{H}_{\pm}^q \cdot \tilde{\mathbf{H}}_{\pm}^q \cdot \mathbf{x}). \quad (4.6)$$

Corollary 3 A signal $\mathbf{x} \in \Pi[N]$ is represented by the redundant system

$$x[k] = \frac{1}{2} \sum_{j=0}^1 \sum_{l=0}^{N/2-1} \left(y_{[1]}^j[l] \psi_{[1],j}^p[k-2l] + c_{[1]}^j[l] \varphi_{[1],j}^p[k-2l] \right),$$

$$y_{[1]}^j[l] = \left\langle \mathbf{x}, \psi_{[1],j}^p[\cdot - 2l] \right\rangle, \quad c_{[1]}^j[l] = \left\langle \mathbf{x}, \varphi_{[1],j}^p[\cdot - 2l] \right\rangle.$$

Thus, the system

$$\mathbf{F} \stackrel{\text{def}}{=} \left\{ \left\{ \psi_{[1],0}^p[\cdot - 2l] \right\} \oplus \left\{ \psi_{[1],1}^p[\cdot - 2l] \right\} \right\} \cup \left\{ \left\{ \varphi_{[1],0}^p[\cdot - 2l] \right\} \oplus \left\{ \varphi_{[1],1}^p[\cdot - 2l] \right\} \right\},$$

whose components are orthonormal, form a tight frame of the space $\Pi[N]$. Here $l = 0, \dots, N/2 - 1$.

4.2 Multi-level transforms

It was explained in [Section 2.2.2](#) that the second-level transform coefficients $\mathbf{y}_{[2]}^{\rho}$ are

$$y_{[2]}^{\rho}[l] = \sum_{n=0}^{N-1} x[n] \psi_{[2],\rho}^p[n-4l], \quad \psi_{[2],\rho}^p[n] = \sum_{k=0}^{N/2-1} h_{[2]}^{\mu}[k] \psi_{[1],\lambda}^p[n-2k] \implies$$

$$y_{[2]}^{\rho}[l] = \sum_{k=0}^{N/2-1} h_{[2]}^{\mu}[k-2l] y_{[1]}^{\lambda}[k], \quad \lambda, \mu = 0, 1, \quad \rho = \begin{cases} \mu, & \text{if } \lambda = 0; \\ 3 - \mu, & \text{if } \lambda = 1. \end{cases}$$

The frequency responses of the p-filters are $\hat{h}_{[2]}^0[n] = \beta[2n]$ and $\hat{h}_{[2]}^1[n] = \alpha[2n]$. The direct and inverse transforms $\mathbf{y}_{[1]}^{\lambda} \longleftrightarrow \mathbf{y}_{[2]}^{2\lambda} \cup \mathbf{y}_{[2]}^{2\lambda+1}$ are implemented using the analysis and synthesis modulation matrices $\tilde{\mathbf{M}}[2n]$ and $\mathbf{M}[2n]$, respectively (see [Eq. \(2.8\)](#)).

Proposition 7 implies that the second-level transform coefficients $\mathbf{c}_{[2]}^\rho$ are

$$\begin{aligned} c_{[2]}^\rho[l] &= \sum_{n=0}^{N-1} x[n] \varphi_{[2],\rho}^p[n-4l], \quad \varphi_{[2],\rho}^p[n] = \sum_{k=0}^{N/2-1} h_{[2]}^\mu[k] \varphi_{[1],\lambda}^p[n-2k] \implies \\ c_{[2]}^\rho[l] &= \sum_{k=0}^{N/2-1} h_{[2]}^\mu[k-2l] c_{[1]}^\lambda[k], \quad \lambda, \mu = 0, 1, \rho = \begin{cases} \mu, & \text{if } \lambda = 0; \\ 3 - \mu, & \text{if } \lambda = 1. \end{cases} \end{aligned}$$

We emphasize that the p-filters $\mathbf{h}_{[2]}^\mu$ for the transform $\mathbf{c}_{[1]}^\lambda \longleftrightarrow \mathbf{c}_{[2]}^{2\lambda} \cup \mathbf{c}_{[2]}^{2\lambda+1}$ are the same that the p-filters for the transform $\mathbf{y}_{[1]}^\lambda \longleftrightarrow \mathbf{y}_{[2]}^{2\lambda} \cup \mathbf{y}_{[2]}^{2\lambda+1}$. Therefore, the direct and inverse transforms $\mathbf{c}_{[1]}^\lambda \longleftrightarrow \mathbf{c}_{[2]}^{2\lambda} \cup \mathbf{c}_{[2]}^{2\lambda+1}$ are implemented using the same analysis and synthesis modulation matrices $\tilde{\mathbf{M}}[2n]$ and $\mathbf{M}[2n]$. Apparently, it is the case also for the transforms $\mathbf{z}_{\pm[1]}^\lambda \longleftrightarrow \mathbf{z}_{\pm[2]}^{2\lambda} \cup \mathbf{z}_{\pm[2]}^{2\lambda+1}$ (see Corollary 2). The transforms to subsequent decomposition levels are implemented in an iterative way:

$$\begin{aligned} \begin{pmatrix} \hat{z}_{\pm[m+1]}^{\rho 0}[n]_{m+1} \\ \hat{z}_{\pm[m+1]}^{\rho 1}[n]_{m+1} \end{pmatrix} &= \frac{1}{2} \tilde{\mathbf{M}}[-2^m n] \cdot \begin{pmatrix} \hat{z}_{\pm[m]}^\lambda[n]_m \\ \hat{z}_{\pm[m]}^\lambda[\bar{n}]_m \end{pmatrix}, \\ \begin{pmatrix} \hat{z}_{\pm[m]}^\lambda[n]_m \\ \hat{z}_{\pm[m]}^\lambda[\bar{n}]_m \end{pmatrix} &= \mathbf{M}[2^m n] \cdot \begin{pmatrix} \hat{z}_{\pm[m+1]}^{\rho 0}[n]_{m+1} \\ \hat{z}_{\pm[m+1]}^{\rho 1}[n]_{m+1} \end{pmatrix}, \end{aligned}$$

where $\rho 0 \stackrel{\text{def}}{=} \begin{cases} 2\lambda, & \text{if } \lambda \text{ is even;} \\ 2\lambda + 1, & \text{if } \lambda \text{ is odd,} \end{cases}$ and vice versa for $\rho 1$; $\bar{n} = n + N/2^{m+1}$ and $m = 1, \dots, M$. By the application of the inverse DFT to the arrays $\{\hat{z}_{\pm[m+1]}^{\rho}[n]_{m+1}\}$, we get the arrays $\{z_{\pm[m+1]}^\rho[k] = y_{[m+1]}^\rho[k] \pm i c_{[m+1]}^\rho[k]\}$ of the transform coefficients with the qWPs $\Psi_{\pm[m+1],\rho}^p$.

The transforms are executed in the spectral domain using the FFT by the application of critically sampled two-channel filter banks to the half-band spectral components $(\hat{x}[n], \hat{x}[n + N/2])^T$ of a signal.

The diagrams in Fig. 4 illustrate the three-level forward and inverse qWP transforms of a signal with quasi-analytic wavelet packets, which use the analysis $\tilde{\mathbf{M}}^q[n]$ and the synthesis $\mathbf{M}^q[n]$ modulation matrices, respectively, for the transforms to and from the first decomposition level, respectively, and the modulation matrices $\tilde{\mathbf{M}}[2^m n]$ and $\mathbf{M}[2^m n]$ for the subsequent levels.

Remark 7 The decomposition of a signal $\mathbf{x} \in \Pi[N]$ down to the M th level produces $2MN$ transform coefficients $\{y_{[m]}^\rho[k]\} \cup \{c_{[m]}^\rho[k]\}$. Such a redundancy provides many options for the signal reconstruction. Some of them are listed below.

- A basis compiled from either WPs $\{\psi_{[m]}^p\}$ or $\{\varphi_{[m]}^p\}$.
 - Wavelet basis.
 - Best basis [30], Local Discriminant basis [31].
 - WPs from a single decomposition level.

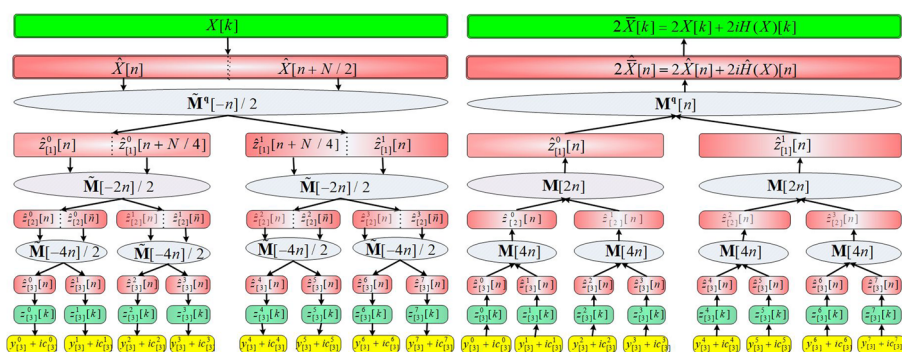


Fig. 4 Left: Forward qWTP of a signal $\mathbf{X}$ down to the third decomposition level. Right: Inverse transform that results in restoration of the signal $\mathbf{X}$ and its HT $H(\mathbf{X})$. Here $\bar{n}$ means $n + N/8$

- Combination of bases compiled from both $\{\psi_{[m]}^p\}$ and $\{\varphi_{[m]}^p\}$ WPs generates a tight frame of the space $\Pi[N]$ with redundancy rate 2. The bases for $\{\psi_{[m]}^p\}$ and $\{\varphi_{[m]}^p\}$ can have a different structure.
- Frames with increased redundancy rate. For example, a combined reconstruction from several decomposition levels.

The collection of dWPs $\{\psi_{[m]}^p\}$ and cWPs $\{\varphi_{[m]}^p\}$, which originate from DTSs of different orders p , provides a variety of waveforms that are (anti)symmetric, well localized in time domain. Any number of the discrete local vanishing moments can be achieved. The DFT spectra of the WPs are flat and the magnitude spectra' shapes tend to rectangles when the order p increases. Therefore, they can be utilized as a collection of band-pass filters which produce a refined split of the frequency domain into bands of different widths. The (c)WPs can be used as testing waveforms for the signal $\hat{\mathbf{a}}$, such as a dictionary for the Matching Pursuit procedures [32, 33].

Remark 8 Since the magnitude spectra of the WPs $\psi_{[m],\lambda}^p$ and $\varphi_{[m],\lambda}^p$ coincide, they have the same number of the discrete local vanishing moments.

5 Two-dimensional complex wavelet packets

The 2D dWPs are defined by the tensor products of 1D dWPs such that $\psi_{[m],j,l}^p[k,n] \stackrel{\text{def}}{=} \psi_{[m],j}^p[k] \psi_{[m],l}^p[n]$. The 2^m -sample shifts of the dWPs $\{\psi_{[m],j,l}^p\}$, $j, l = 0, \dots, 2^m - 1$, in both directions form an orthonormal basis for the

space $\Pi[N, N]$ of arrays that are N -periodic in both directions. The DFT spectrum of such a WP is concentrated in four symmetric spots in the frequency domain.

Similar properties are inherent to the 2D cWPs such that $\varphi_{[m],j,l}^p[k, n] \stackrel{\text{def}}{=} \varphi_{[m],j}^p[k] \varphi_{[m],l}^p[n]$.

5.1 Design of 2D directional WPs

5.1.1 2D complex WPs and their spectra

The dWPs $\{\psi_{[m],j,l}^p\}$ as well as the cWPs $\{\varphi_{[m],j,l}^p\}$ lack the directionality property which is needed in many applications that process 2D data. However, real-valued 2D wavelet packets oriented in multiple directions can be derived from tensor products of complex qWPs $\Psi_{\pm[m],\rho}^p$.

The complex 2D qWPs are defined as follows:

$$\Psi_{++[m],j,l}^p[k, n] \stackrel{\text{def}}{=} \Psi_{+[m],j}^p[k] \Psi_{+[m],l}^p[n], \quad \Psi_{+-[m],j,l}^p[k, n] \stackrel{\text{def}}{=} \Psi_{+[m],j}^p[k] \Psi_{-[m],l}^p[n],$$

where $m = 1, \dots, M$, $j, l = 0, \dots, 2^m - 1$, and $k, n = -N/2, \dots, N/2 - 1$. The real parts of these 2D qWPs are

$$\begin{aligned} \vartheta_{+[m],j,l}^p[k, n] &\stackrel{\text{def}}{=} \Re(\Psi_{++[m],j,l}^p[k, n]) = \psi_{[m],j,l}^p[k, n] - \varphi_{[m],j,l}^p[k, n], \\ \vartheta_{-[m],j,l}^p[k, n] &\stackrel{\text{def}}{=} \Re(\Psi_{+-[m],j,l}^p[k, n]) = \psi_{[m],j,l}^p[k, n] + \varphi_{[m],j,l}^p[k, n], \end{aligned} \quad (5.1)$$

The DFT spectra of the 2D qWPs $\Psi_{++[m],j,l}^p$, $j, l = 0, \dots, 2^m - 1$, are the tensor products of the one-sided spectra of the qWPs: $\hat{\Psi}_{++[m],j,l}^p[p, q] = \hat{\Psi}_{+[m],j}^p[p] \hat{\Psi}_{+[m],l}^p[q]$, and, as such, they fill the quadrant $\mathbf{Q}_0$ of the frequency domain, while the spectra of $\Psi_{+-[m],j,l}^p$, $j, l = 0, \dots, 2^m - 1$, fill the quadrant $\mathbf{Q}_1$

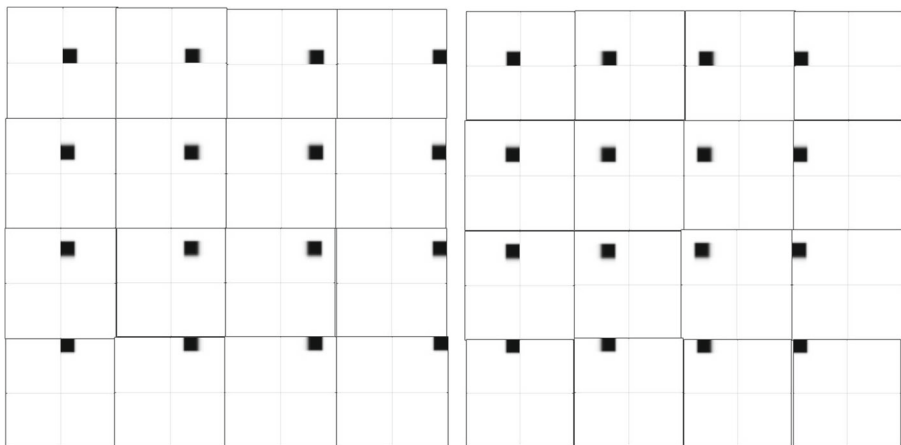


Fig. 5 Magnitude spectra of 2D qWPs $\Psi_{++[2],j,l}^9$ (left block of pictures) and qWPs $\Psi_{+-[2],j,l}^9$ (right block) from the second decomposition level

(see Eq. (1.1)). Figure 5 displays the magnitude spectra of the ninth-order 2D qWPs $\Psi_{++[2],j,l}^9$ and $\Psi_{+-[2],j,l}^9$ from the second decomposition level, respectively.

5.1.2 Directionality of real-valued 2D WPs

It is seen in Fig. 5 that the DFT spectra of the qWPs $\Psi_{\pm[m],j,l}^9$ effectively occupy relatively small squares in the frequency domain. For deeper decomposition levels, sizes of the corresponding squares decrease on geometric progression. Such configurations of the spectra lead to the directionality of the real-valued 2D WPs $\vartheta_{\pm[m],j,l}^p$.

Assume, for example, that $N = 512$, $m = 3$, $j = 2$, $l = 5$ and denote $\Psi[k, n] \stackrel{\text{def}}{=} \Psi_{++[3],2,5}^9[k, n]$ and $\vartheta[k, n] \stackrel{\text{def}}{=} \Re\{\Psi[k, n]\}$. The magnitude spectrum $|\hat{\Psi}[\kappa, \nu]|$, displayed in Fig. 6 (left), effectively occupies the square of size 40×40 pixels centered around the point $\mathbf{C} = [\kappa_0, \nu_0]$, where $\kappa_0 = 78$, $\nu_0 = 178$. Thus, the WP Ψ is represented by

$$\Psi[k, n] = \frac{1}{N^2} \sum_{\kappa, \nu=0}^{N/2-1} \omega^{k\kappa+n\nu} \hat{\Psi}[\kappa, \nu] \approx \omega^{\kappa_0 k + \nu_0 n} \underline{\Psi}[k, n],$$

$$\underline{\Psi}[k, n] \stackrel{\text{def}}{=} \frac{1}{N^2} \sum_{\kappa, \nu=-20}^{19} \omega^{k\kappa+n\nu} \hat{\Psi}[\kappa + \kappa_0, \nu + \nu_0].$$

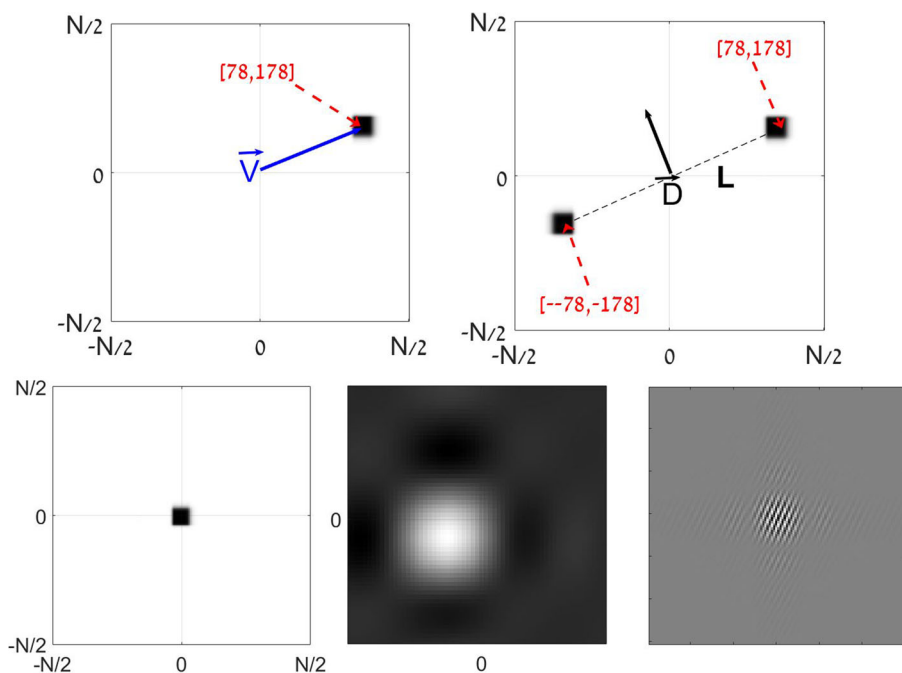


Fig. 6 Top: Magnitude spectra of 2D qWP $\Psi[k, n]$ (left) and $\Re\{\Psi\} = \vartheta[k, n]$ (right). Bottom: Left—magnitude spectrum of low-frequency signal $\underline{\vartheta}[k, n]$. Center—signal $\underline{\vartheta}[k, n]$. Right—2D WP $\vartheta[k, n]$ (magnified)

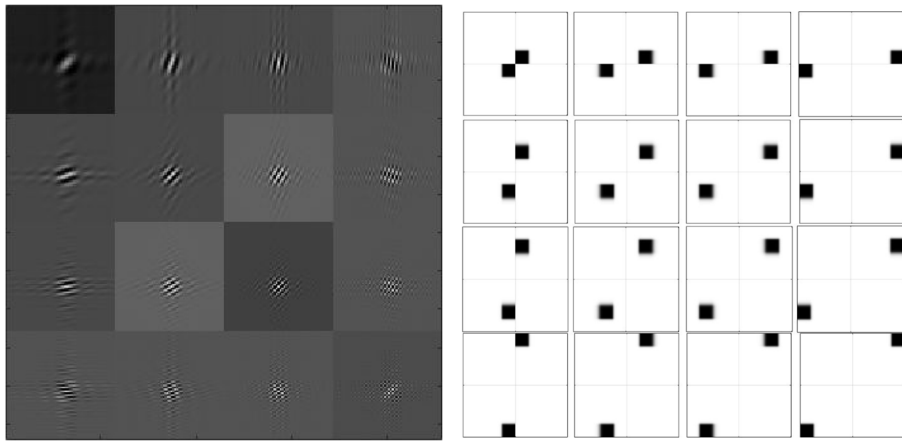


Fig. 7 WPs $\vartheta_{+[2],j,l}^9$ from the second decomposition level and their magnitude spectra

Consequently, the real-valued WP ϑ , whose magnitude spectrum is displayed in Fig. 6 (second from left), is represented as follows:

$$\vartheta[k, n] \approx \cos \frac{2\pi(\kappa_0 k + \nu_0 n)}{N} \Re(\underline{\Psi}[k, n]) + \cos \left(\frac{2\pi(\kappa_0 k + \nu_0 n)}{N} - \frac{\pi}{2} \right) \Im(\underline{\Psi}[k, n]).$$

The spectrum of the 2D signal $\underline{\Psi}$ comprises only low frequencies in both directions and its real and imaginary parts do not have a directionality. But the 2D signal $\cos \frac{2\pi(\kappa_0 k + \nu_0 n)}{N}$ is oscillating in the direction of the vector $\vec{D}$, which is orthogonal to the vector $\vec{V}_{++[2],2,5} = 178\vec{i} + 78\vec{j}$. The 2D WP $\vartheta[k, n]$ is well localized in the spatial domain as is seen from Eq. (5.1) and the same is true for the low-frequency signal $\underline{\vartheta}$. Therefore, WP $\vartheta[k, n]$ can be regarded as the directional cosine wave modulated by the localized low-frequency signals $\Re(\underline{\Psi}[k, n])$ and $\Im(\underline{\Psi}[k, n])$.

The same arguments are applicable to the 2D WPs $\vartheta_{-[m],j,l}^P[k, n] = \Re(\Psi_{+-[m],j,l}^P[k, n])$. Figure 6 displays the low-frequency signal $\Re(\underline{\Psi})$, its magnitude spectrum, and the 2D WP $\vartheta[k, n]$.

Figure 7 displays WPs $\vartheta_{+[2],j,l}^9$, $j, l = 0, 1, 2, 3$, from the second decomposition level and their magnitude spectra.

Figure 8 displays WPs $\vartheta_{-[2],j,l}^9$, $j, l = 0, 1, 2, 3$, from the second decomposition level and their magnitude spectra.

Orientation of a wavelet packet $\vartheta_{+[m],j,l}^P$ is determined by the orientation of the vector $\vec{V}_{++[m],j,l} = \kappa_0\vec{i} + \nu_0\vec{j}$, which is directed from the origin to the center $\mathbf{C}_{++[m],j,l} = [\kappa_0, \nu_0]$ of the spot occupied by the DFT spectrum of the qWP $\Psi_{++[m],j,l}^P = \Psi_{+[m],j}^P \otimes \Psi_{+[m],l}^P$. The spectra of qWPs $\Psi_{+[m],k}^P$, $k = 0, \dots, 2^m - 1$, split the positive frequency half-band $\mathbf{B} = [0, N/2]$ into 2^m approximately equal subbands $\mathbf{b}_k$, $k = 0, \dots, 2^m - 1$ (see Fig. 3). Thus, the center of the subband $\mathbf{b}_j$ is $\mathbf{c}_j \approx N/2^m(j + 1/2)$. We can say that location of the point $\mathbf{c}_j$ is (roughly) proportional of the index $j + 1$. Consequently, the coordinates κ_0, ν_0 of the point $\mathbf{C}_{++[m],j,l}$

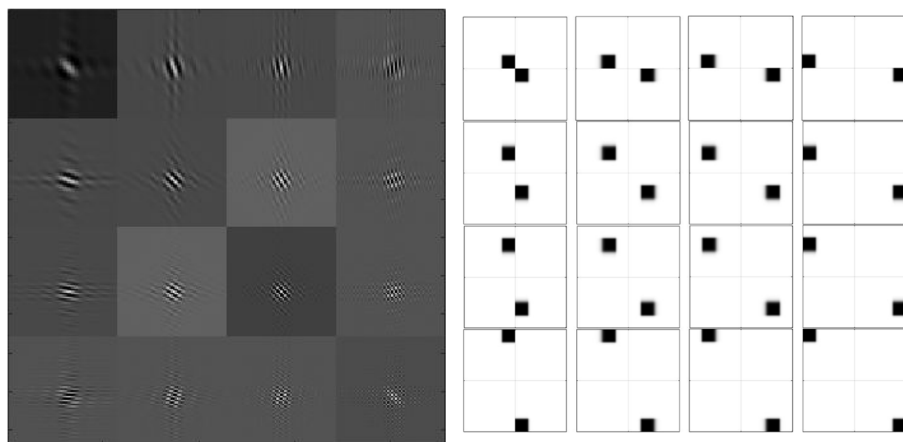


Fig. 8 WPs $\vartheta_{-[2],j,l}^9$ from the second decomposition level and their magnitude spectra

are, approximately, proportional to the indices $j + 1, l + 1$, respectively. Thus the vector $\vec{V}_{++[m],j,l} \approx P((j + 1)\vec{i} + (l + 1)\vec{j})$, where P is a proportionality coefficient.

Surely, the vector $\vec{V}_{++[m],j,l}^r \stackrel{\text{def}}{=} P(r(j + 1)\vec{i} + r(l + 1)\vec{j})$ (r is an integer), which is related to the qWP $\Psi_{++[m],rj,rl}^P$, has approximately the same orientation as the vector $\vec{V}_{++[m],j,l}$. Consequently, the real qWPs $\vartheta_{++[m],j,l}^P$ and $\vartheta_{++[m],rj,rl}^P$ have approximately the same orientation, provided $\max\{rj, rl\} \leq N/2$. Therefore, there is a number of groups within the set of 2^{2m} m -level qWPs $\{\vartheta_{++[m],j,l}^P\}$, $j, l = 0, \dots, 2^m - 1$, whose members have approximately the same orientation. We must take this into account when calculating the number of different orientations within the sets of qWP from different levels. These numbers are calculated by a simple Matlab function `dir_count.m`, which is placed in [Appendix](#). Apparently, all the above considerations are applicable to “negative” qWPs $\{\vartheta_{--[m],j,l}^P\}$, $j, l = 0, \dots, 2^m - 1$, which have the same number of different orientations as the “positive” qWPs (Table 4).

For example, all the “diagonal” qWPs $\{\vartheta_{\pm[m],j,j}^P\}$, $j = 0, \dots, 2^m - 1$, are oscillating with different frequencies in the directions of 135° and 45° , respectively. It is seen in Figs. 7, 8, and 9.

Remark 9 It is seen in Figs. 7, 8, and 9 that besides the well-localized oriented central parts of the WPs originated from the ninth-order splines, visible sidelobes present. On the other hand, the DFT spectra of these WPs are concentrated within small

Table 4 Numbers of different orientations of qWPs $\{\vartheta_{\pm[m],j,l}^P\}$, $j, l = 0, \dots, 2^m - 1$, for different levels

Level m	1	2	3	4	5	6	...
# of directions	6	22	86	314	1218	4606	...

squares. We illustrate this observation in Fig. 10, which displays the 3D image of the ninth-order WP $\vartheta_{+[2],3,2}^9$ and its magnitude spectrum versus WP $\vartheta_{+[2],3,2}^2$ originated from the second-order spline s and its magnitude spectrum. One can see that, while the WP $\vartheta_{+[2],3,2}^9$ has distinguishable sidelobes, the shape of its magnitude spectrum is close to a pair of thin parallelepipeds. The situation is quite opposite for the WP $\vartheta_{+[2],3,2}^2$: It is almost compactly supported in the spatial domain but the shape of its magnitude spectrum is close to a pair of wide-base cones. Such properties of WPs originated from the splines of different orders, which are due to the uncertainty principle, provide tools for adaptation of the qWPs to the problem under consideration. An additional adaptation tool is changing of the decomposition levels of the qWPs. Note that change of the originating spline order and decomposition level can be done locally within a single image.

6 Implementation of 2D qWP transforms

The spectra of the real-valued 2D WPs $\{\vartheta_{+[m],j,l}^p\}$, $j, l = 0, \dots, 2^m - 1$, and $\{\vartheta_{-[m],j,l}^p\}$ fill the pairs of quadrants $\mathbf{Q}_+ \stackrel{\text{def}}{=} \mathbf{Q}_0 \cup \mathbf{Q}_3$ and $\mathbf{Q}_- \stackrel{\text{def}}{=} \mathbf{Q}_1 \cup \mathbf{Q}_2$ (see Eq. (1.1)), respectively (Figs. 7 and 8).

By this reason, none linear combination of the WPs $\{\vartheta_{+[m],j,l}^p\}$ and their shifts can serve as a basis for the signal space $\Pi[N, N]$. The same is true for WPs $\{\vartheta_{-[m],j,l}^p\}$. However, combinations of the WPs $\{\vartheta_{\pm[m],j,l}^p\}$ provide frames of the space $\Pi[N, N]$.

6.1 One-level 2D transforms

The one-level 2D qWP transforms of a signal $\mathbf{X} = \{X[k, n]\} \in \Pi[N, N]$ are implemented by a tensor-product scheme. Denote by $H^v(\mathbf{S})$ and $H^h(\mathbf{S})$ results of application of the Hilbert transforms to columns and rows of a 2D signal $\mathbf{S}$, respectively.

Denote by $\tilde{\mathbf{T}}_{\pm}^h$ the 1D transforms of row signals from $\Pi[N]$ with the analysis modulation matrices $\tilde{\mathbf{M}}_{\pm}^q$ which are defined in Eq. (4.2). Application of these transforms to rows of a signal $\mathbf{X}$ produces the coefficient arrays

$$\begin{aligned}\tilde{\mathbf{T}}_+^h \cdot \mathbf{X} &= (\zeta_+^0, \zeta_+^1), \quad \zeta_+^j[k, n] = \eta^j[k, n] - i\xi^j[k, n], \\ \tilde{\mathbf{T}}_-^h \cdot \mathbf{X} &= (\zeta_-^0, \zeta_-^1), \quad \zeta_-^j[k, n] = \eta^j[k, n] + i\xi^j[k, n] = (\zeta_+^j[k, n])^*, \\ \eta^j[k, n] &= \langle \mathbf{X}[k, \cdot], \psi_{[1],j}^p[\cdot - 2n] \rangle, \quad \xi^j[k, n] = \langle \mathbf{X}[k, \cdot], \varphi_{[1],j}^p[\cdot - 2n] \rangle, \quad j = 0, 1.\end{aligned}$$

Here η^j and ξ^j are real-valued arrays of size $N \times N/2$. Obviously we have

$$\tilde{\mathbf{T}}_{\pm}^h \cdot H^v(\mathbf{X}) = (H^v(\zeta_{\pm}^0), H^v(\zeta_{\pm}^1)), \quad H^v(\zeta_{\pm}^j) = H^v(\eta^j) \mp H^v(\xi^j). \quad (6.1)$$

Denote by $\tilde{\mathbf{T}}_+^v$ the direct 1D transform determined by the modulation matrix $\tilde{\mathbf{M}}_+^q$ applicable to columns of the corresponding signals. The next step of the tensor prod-

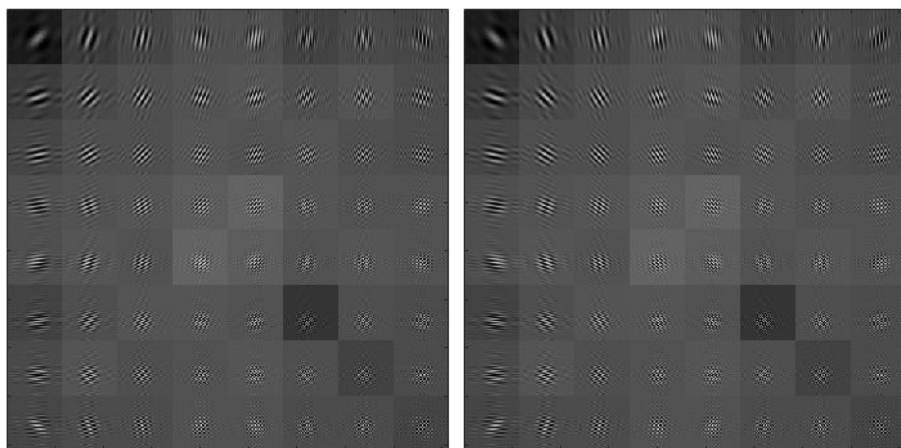


Fig. 9 WPs $\vartheta_{+[3],j,l}^9$ (left) and $\vartheta_{-[3],j,l}^9$ (right) from the third decomposition level

uct transform consists of the application of the 1D transform $\tilde{\mathbf{T}}_+^v$ to columns of the arrays ζ^j , $j = 0, 1$.

$$\tilde{\mathbf{T}}_+^v \cdot \zeta_+^j = \tilde{\mathbf{T}}_+^v \cdot \eta^j - i\tilde{\mathbf{T}}_+^v \cdot \xi^j = \mathbf{Z}_{+[1]}^j, \quad \tilde{\mathbf{T}}_+^v \cdot \zeta_-^j = \tilde{\mathbf{T}}_+^v \cdot \eta^j + i\tilde{\mathbf{T}}_+^v \cdot \xi^j = \mathbf{Z}_{-[1]}^j.$$

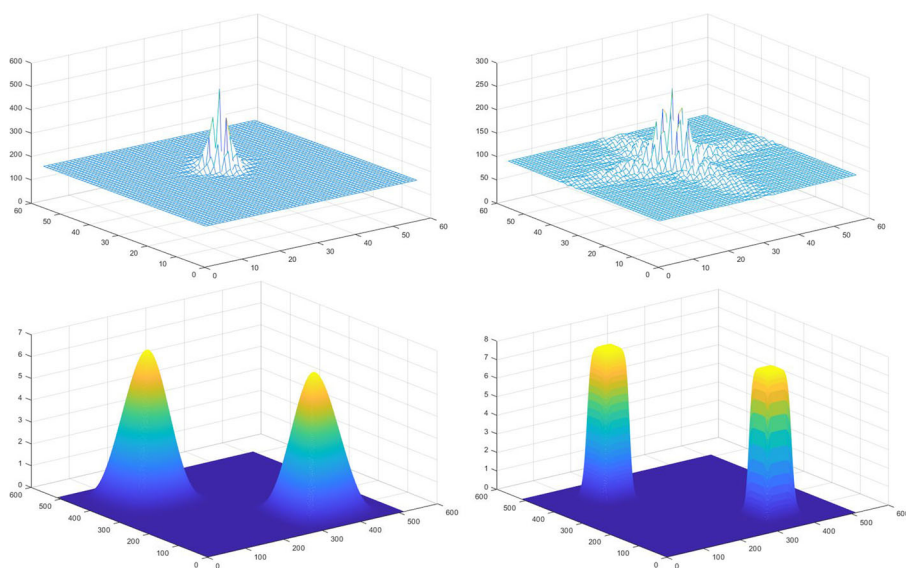


Fig. 10 Top left: Second-order WP $\vartheta_{-[2],3,2}^2$ from the second decomposition level; Top right: Ninth-order WP $\vartheta_{+[2],3,2}^9$. Bottom left: Magnitude spectrum of WP $\vartheta_{+[2],3,2}^2$; Bottom right: Magnitude spectrum of WP $\vartheta_{-[2],3,2}^9$

Denote by $\mathbf{T}_+^v$ the 1D inverse transform with the synthesis modulation matrix $\mathbf{M}_+^q$ applicable to columns of the coefficient arrays.

$$\begin{aligned}\mathbf{T}_+^v \cdot \mathbf{Z}_{+[1]}^j &= 2(\eta^j + iH^v(\eta^j)) - 2i(\xi^j + iH^v(\xi^j)) = 2(\zeta_+^j + iH^v(\zeta_+^j)), \\ \mathbf{T}_+^v \cdot \mathbf{Z}_{-[1]}^j &= 2(\eta^j + iH^v(\eta^j) + 2i(\xi^j + iH^v(\xi^j))) = 2(\zeta_-^j + iH^v(\zeta_-^j)).\end{aligned}$$

Denote by $\mathbf{T}_\pm^h$ the 1D inverse transforms with the synthesis modulation matrices $\mathbf{M}_\pm^q$. Application of these transforms to rows of the coefficient arrays $\zeta_\pm = (\zeta_\pm^0, \zeta_\pm^1)$, respectively, produces the 2D analytic signals: $\mathbf{T}_\pm^h \cdot (\zeta_\pm^0, \zeta_\pm^1) = 2(\mathbf{X} \pm iH^h(\mathbf{X}))$.

Equation (6.1) implies that application of the transforms $\mathbf{T}_\pm^h$ to rows of the arrays $H^v(\zeta_\pm) \stackrel{\text{def}}{=} (H^v(\zeta_\pm^0), H^v(\zeta_\pm^1))$, respectively, produces the 2D analytic signals: $\mathbf{T}_\pm^h \cdot (H^v(\zeta_\pm^0), H^v(\zeta_\pm^1)) = 2(\mathbf{G} \pm iH^h(\mathbf{G}))$, where $\mathbf{G} = H^v(\mathbf{X})$. Consequently,

$$\begin{aligned}\mathbf{X}_+ &\stackrel{\text{def}}{=} \mathbf{T}_+^h \cdot \mathbf{T}_+^v \cdot \mathbf{Z}_{+[1]}^j = 4(\mathbf{X} + iH^h(\mathbf{X}) + i\mathbf{G} - H^h(\mathbf{G})), \\ \mathbf{X}_- &\stackrel{\text{def}}{=} \mathbf{T}_-^h \cdot \mathbf{T}_+^v \cdot \mathbf{Z}_{-[1]}^j = 4(\mathbf{X} - iH^h(\mathbf{X}) + i\mathbf{G} + H^h(\mathbf{G})). \\ \mathbf{X} &= \Re\left(\frac{\mathbf{X}_+ + \mathbf{X}_-}{8}\right).\end{aligned}\quad (6.2)$$

Figure 11 illustrates the image “Fingerprint” restoration by the 2D signals $\Re(\mathbf{X}_\pm)$. The signal $\Re(\mathbf{X}_+)$ captures oscillations directed to *north-west*, while $\Re(\mathbf{X}_-)$ captures oscillations directed to *north-east*. The signal $\tilde{\mathbf{X}} = \Re(\mathbf{X}_+ + \mathbf{X}_-)/8$ perfectly restores the image achieving PSNR=312.3538 dB.

6.2 Multi-level 2D transforms

It was established in Section 4.2 that the 1D qWP transforms of a signal $\mathbf{x} \in \Pi[N]$ to the second and further decomposition levels are implemented by the successive application of the filter banks, which are determined by their analysis modulation matrices $\tilde{\mathbf{M}}[2^m n]$, $m = 1, \dots, M - 1$, to the coefficient arrays $\mathbf{z}_{\pm[m]}^\lambda$. The transforms applied to the arrays $\mathbf{z}_{\pm[m]}^\lambda$ produce the arrays $\mathbf{z}_{\pm[m+1]}^\rho$, respectively. The inverse transform consists of the iterated application of the filter banks that are determined by their

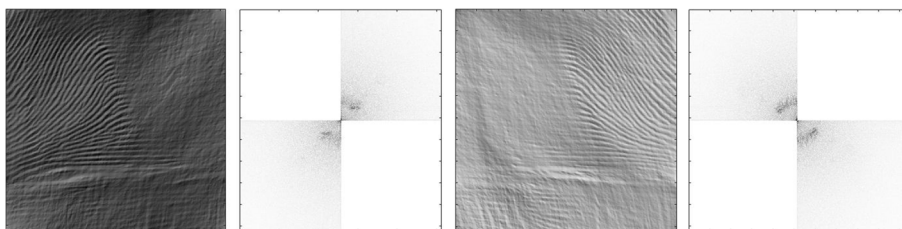


Fig. 11 Left to right: 1. Image $\Re(\mathbf{X}_+)$. 2. Its magnitude DFT spectrum. 3. Image $\Re(\mathbf{X}_-)$. 4. Its magnitude DFT spectrum

synthesis modulation matrices $\mathbf{M}[2^m n]$, $m = 1, \dots, M - 1$, to the coefficient arrays $\mathbf{z}_{\pm[m+1]}^\rho$. In that way, the first-level coefficient arrays $\mathbf{z}_{\pm[1]}^\lambda$, $\lambda = 0, 1$ are restored.⁸

The tensor-product 2D transform of a signal $\mathbf{X} \in \Pi[N, N]$ consists of the subsequent application of the 1D transforms to columns and rows of the signal and coefficient arrays. By application of filter banks, which are determined by the analysis modulation matrix $\tilde{\mathbf{M}}[2n]$ to columns and rows of the coefficient arrays $\mathbf{Z}_{\pm[1]}^{j,l}$, we derive four second-level arrays $\mathbf{Z}_{\pm[2]}^{\rho,\tau}$, $\rho = 2j, 2j + 1$; $\tau = 2l, 2l + 1$. The arrays $\mathbf{Z}_{\pm[1]}^{j,l}$ are restored by the application of the filter banks that are determined by their synthesis modulation matrices $\mathbf{M}[2n]$ to rows and columns of the coefficient arrays $\mathbf{Z}_{\pm[2]}^{\rho,\tau}$, $\rho = 2j, 2j + 1$; $\tau = 2l, 2l + 1$. The transition from the second to further levels and back are executed similarly using the modulation matrices $\tilde{\mathbf{M}}[2^m n]$ and $\mathbf{M}[2^m n]$, respectively. The inverse transforms produce the coefficient arrays $\mathbf{Z}_{\pm[1]}^{j,l}$, $j, l = 0, 1$, from which the signal $\mathbf{X} \in \Pi[N, N]$ is restored using the synthesis modulation matrices $\mathbf{M}_\pm^q[n]$ as it is explained in Section 6.1.

All the computations are implemented in the frequency domain using the FFT.

Summary The 2D qWP processing of a signal $\mathbf{X} \in \Pi[N, N]$ is implemented by a dual-tree scheme.⁹ The first step produces two sets of the coefficient arrays: $\mathbf{Z}_{+[1]} \stackrel{\text{def}}{=} \{\mathbf{Z}_{+[1]}^{j,l}\}$, $j, l = 0, 1$, which are derived using the analysis modulation matrix $\tilde{\mathbf{M}}_+^q[n]$ for the row and column transforms, and $\mathbf{Z}_{-[1]} \stackrel{\text{def}}{=} \{\mathbf{Z}_{-[1]}^{j,l}\}$, $j, l = 0, 1$, which are derived using the analysis modulation matrices $\tilde{\mathbf{M}}_+^q[n]$ for the column and $\tilde{\mathbf{M}}_-^q[n]$ for the row transforms. Further decomposition steps are implemented in parallel on the sets $\mathbf{Z}_{+[1]}$ and $\mathbf{Z}_{-[1]}$ using the same analysis modulation matrices $\tilde{\mathbf{M}}[2^m n]$, thus producing two multi-level sets of the coefficient arrays $\{\mathbf{Z}_{+[m]}^{j,l}\}$ and $\{\mathbf{Z}_{-[m]}^{j,l}\}$, $m = 2, \dots, M$, $j, l = 0, 2^m - 1$.

By parallel implementation of the inverse transforms on the coefficients from the sets $\{\mathbf{Z}_{+[m]}^{j,l}\}$ and $\{\mathbf{Z}_{-[m]}^{j,l}\}$ using the same synthesis modulation matrix $\mathbf{M}[2^m n]$, the sets $\mathbf{Z}_{+[1]}$ and $\mathbf{Z}_{-[1]}$ are restored, which, in turn, provide the signals $\mathbf{X}_+$ and $\mathbf{X}_-$, using the synthesis modulation matrices $\mathbf{M}_+^q[n]$ and $\mathbf{M}_-^q[n]$, respectively. Typical signals $\Re(\mathbf{X}_\pm)$ and their DFT spectra are displayed in Fig. 11.

Prior to the reconstruction, some structures, possibly different, are defined in the sets $\{\mathbf{Z}_{+[m]}^{j,l}\}$ and $\{\mathbf{Z}_{-[m]}^{j,l}\}$, $m = 1, \dots, M$, (for example, 2D wavelet or Best Basis structures) and some manipulations on the coefficients, (for example, thresholding, l_1 minimization) are executed.

⁸The matrices $\tilde{\mathbf{M}}[n]$ and $\mathbf{M}[n]$ are defined in Eq. (2.8).

⁹Do not confuse with the dual-tree wavelet transform by Kingsbury [17]

7 A couple of numerical examples

The 2D qWPs possess the following properties:

- The qWP transforms provide a variety of 2D waveforms oriented in multiple directions. For example, fourth-level qWPs are oriented in 314 different directions.
- The waveforms are close to directional cosine waves with a variety of frequencies modulated by spatially localized low-frequency 2D signals and can have any number of local vanishing moments.
- The DFT spectra of the waveforms produce a refined tiling of the frequency domain.
- Fast implementation of the transforms by using the FFT enables us to use the transforms with increased redundancy and iterative algorithms.

These properties of qWP transforms proved to be indispensable while dealing with image processing problems. Due to a variety of orientations, the qWPs capture edges even in severely degraded images and their oscillatory shapes with a variety of frequencies enable to recover fine structures. Multiple experiments on image denoising [25] and inpainting [26] demonstrate that qWP-based methods are quite competitive with the best state-of-the-art algorithms.

7.1 Image denoising

One of the best image denoising methods is the BM3D algorithm [34], which exploits the self-similarity of patches and sparsity of the image in a transform domain. This method is incomparable in restoration of moderately noised images. However, the BM3D tends to over-smooth and smear the image fine structure and edges when noise is strong. Also, the BM3D is not success when the image contains many edges oriented in multiple directions. On the other hand, algorithms that use directional oscillating waveforms provide an opportunity to capture lines, edges, and texture details. Therefore, it is natural to combine the qWP-based and BM3D algorithms in order to retain strong features of both algorithms and to get rid of their drawbacks. The description of two hybrid qWP–BM3D algorithms and results of multiple experiments on image denoising are presented in the paper [25]. The images in the experiments were degraded by zero-mean Gaussian noise with STD $\sigma = 5, 10, 25, 40, 50, 80, 100$ dB. The images restored by two hybrid algorithms, which are designated by **upBM3D** and **hybrid**, were compared with the images restored by BM3D and other state-of-the-art methods, such as WNNM [35] and BM3D-SAPCA [36], by the PSNR and SSIM (Structural Similarity Index) values¹⁰ and visual perception.

In practically all the experiments, the PSNR values for all compared algorithms are very close to each other but the SSIM values achieved by the hybrid algorithms

¹⁰For computation of SSIM values we use the Matlab 2020b function `ssim.m`

are significantly higher than those achieved by the other algorithms. This is especially true for texture-rich images. The hybrid algorithms succeeded in restoration of the structure of even severely degraded images. It is illustrated in Fig. 12, which displays restoration of the “Mandrill” image from the input degraded by Gaussian noise with STD $\sigma = 80$ dB. The figure comprises 12 frames, which are arranged in

a 4×3 order: $\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \\ f_{41} & f_{42} & f_{43} \end{pmatrix}$. Here frame f_{11} displays noised image; frame f_{21}

– image restored by BM3D; f_{12} – image restored by BM3D-SAPCA; f_{22} – image restored by WNNM; f_{13} – image restored by **upBM3D**; f_{23} – image restored by **hybrid**. Frame f_{31} displays a fragment of the original image. The remaining frames $\{f_{32}, f_{33}, f_{41}, f_{42}, f_{43}\}$ display the fragments of the restored images shown in frames $\{f_{12}, f_{13}, f_{21}, f_{22}, f_{23}\}$, which are arranged in the same order.

7.2 Image inpainting

The described qWPs demonstrated a high efficiency in dealing with the image inpainting problem that means restoration of images degraded by the loss of a significant share of pixels and possible addition of noise. We designed a qWP-based iterative algorithm, which combines the split Bregman iteration scheme [37, 38] with the adaptive decreasing thresholding [39, 40]. In multiple experiments on restoration of images corrupted by missing a large amount of pixels and addition of Gaussian noise with various intensities, we compared the performance of our qWP-based algorithm with the performance of a number of the state-of-the-art algorithms. The description of the algorithm and results of from multiple experiments on image inpainting are presented in the paper [26]. The results are compared according to PSNR and SSIM values and by visual perception. Similarly to denoising experiments, the qWP algorithm prevailed in restoration of edges and fine structure even in severely degraded images. This fact is reflected in highest values of SSIM. A typical example is displayed in Fig. 13.¹¹ In this example, the qWP restoration of the image degraded by missing 50% of pixels and additive Gaussian noise with $\sigma = 50$ dB is compared with the restoration by DAS-2 [19], which is one of the best algorithms for such type of images.

7.3 Image classification

We conducted a few experiments with the MNIST database of handwritten digits to test feasibility of the qWP-based feature extraction scheme for the image classification. The MNIST database contains 60,000 training images and 10,000 testing images. We explored two options of the qWPs' utilization:

¹¹The image “Bridge” did not participate in the experiments presented in [26].

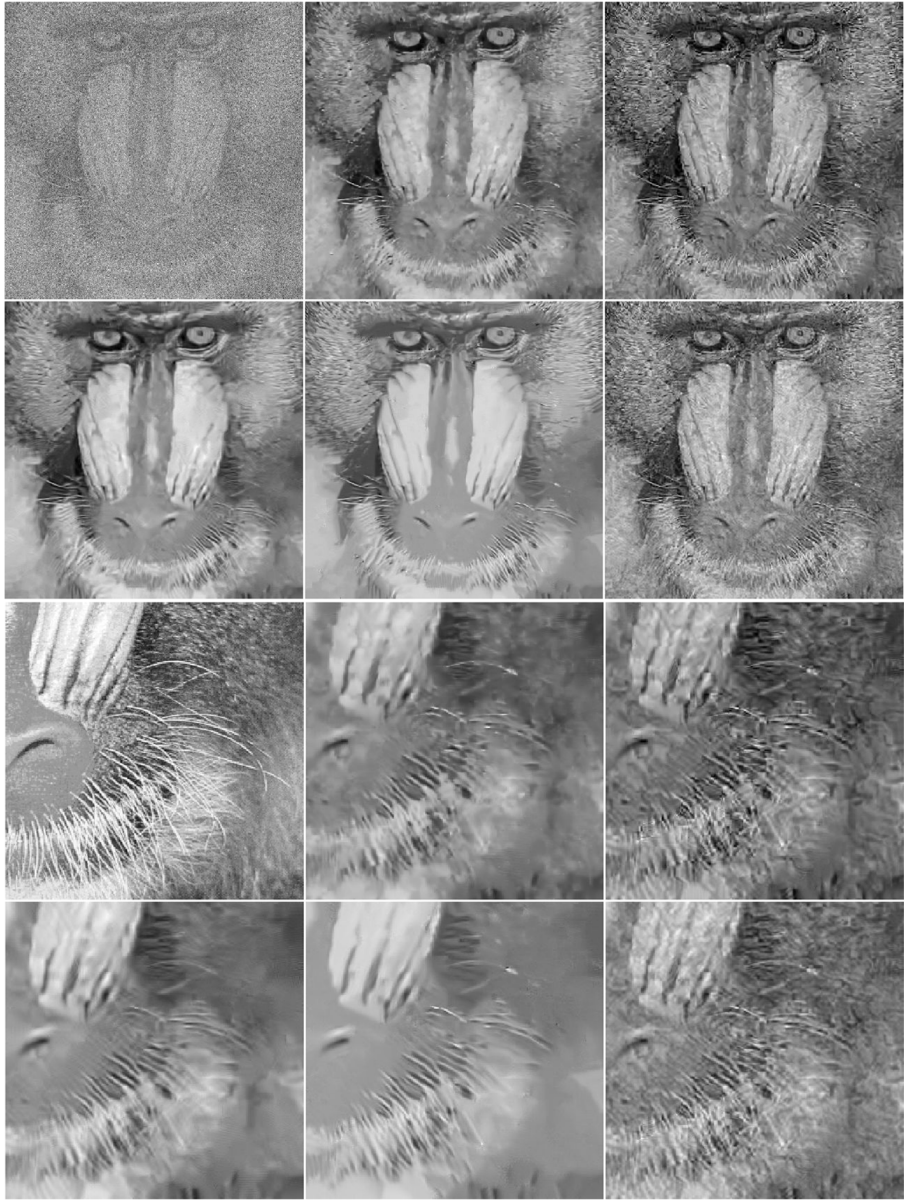


Fig. 12 Restoration of “Mandrill” image corrupted by Gaussian noise with STD $\sigma = 80$ dB. PSNR/SSIM for BM3D– 20.9/0.2439; for WNNM– 21.1/0.2656; for BM3D-SAPCA–20.92/0.2649; for **upBM3D**– 20.85/0.3205; for **hybrid**–20.56/0.3402

1. Extraction of qWP-based characteristic features from limited numbers of training images and using them for training the support vector machine (SVM, [41]) classifier. Then, the trained SVM was used for the classification of 10,000 testing

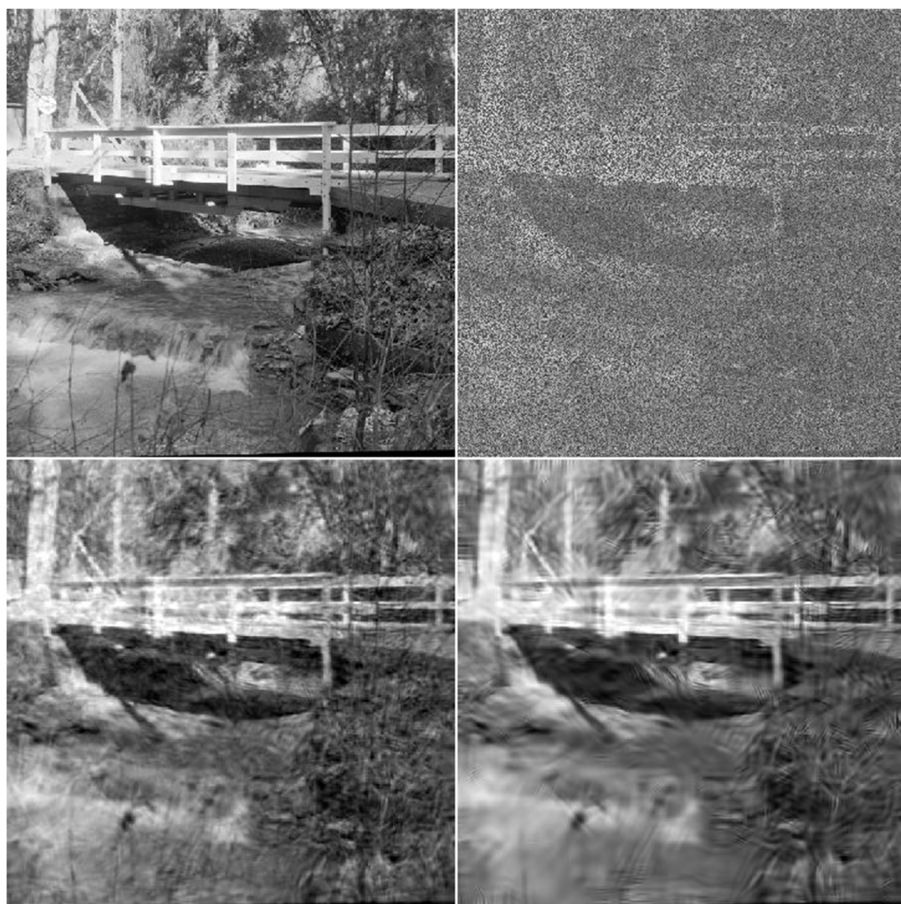


Fig. 13 Restoration of “Bridge” image. Top left: clean image. Top right: image degraded by missing 50% of pixels and Gaussian noise with $\sigma = 50$ dB. Bottom: Restoration by qWP (left), PSNR=22.08 dB, SSIM=0.3917 and by DAS-2 (right), PSNR=22.07 dB, SSIM=0.3238

images. Preliminary results of the experiments confirmed efficiency of such a scheme even with small reference datasets used for training SVM.

2. Convolutional neural networks (CNNs) achieve extraordinary success in image classification. The convolution layers in a CNN extract the characteristic features from the training data by convolving the input data with filters that are derived in the training process. Consequently, the structure of the filters and the meaning of the extracted features remain unclear. Such a scheme requires a huge amount of data and a large number of convolution layers whose number is unknown in advance. Thus, it is natural to replace at least some of the convolution layers in the CNN by filtering with predefined filters whose properties are favorable to the class of images under processing. This is done in our second series of the experiments where absolute values of the qWP transform coefficients serve as

Table 5 Classification results for the validation set of 10,000 images

Size of ref. set	60000	20000	10000	5000	2000	1000	500	300
SVM class%	98.42	98.61	98.2	97.76	97.32	96.49	94.53	94.19
CNN class%	99.51	99.18	98.79	98.26	97.38	96.08	94.12	92.67

the inputs to a CNN with a small number of convolution layers. Recall that the qWP transform coefficients have a clear physical meaning: each transform coefficient is a correlation coefficient of the image with a certain waveform located at a certain place.

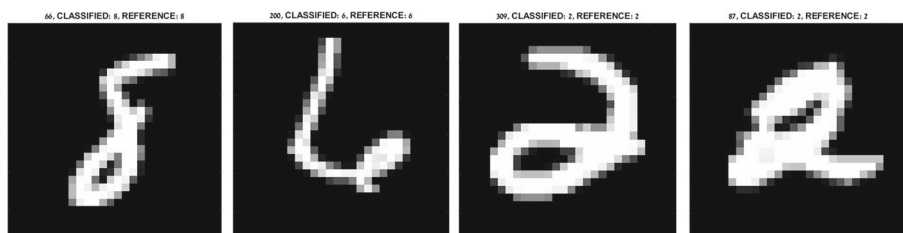
Some results, which are given in Table 5, indicate that the qWP-based feature extraction methods have a potential to handle image classification problems. The table shows percentage of correct answers from the SVM classifiers and the CNNs, which are trained on reference sets comprising different numbers of images, for the validation set comprising of 10,000 images.

Figure 14 shows four severely distorted digits that are correctly classified by the DNN trained on R_{2000} set of the reference data.

8 Discussion

The paper describes the design of one- and two-dimensional quasi-analytic WPs (qWPs) originating from polynomial splines of arbitrary order and the corresponding transforms. The qWP transforms operate in spaces of periodic signals. Seemingly, the requirement of periodicity imposes some limitations on the scope of signals available for processing, but actually these limitations are easily circumvented by symmetrical extension of images beyond the boundaries before processing and shrinkage to the original size after that. On the other hand, the periodic setting provides a lot of substantial opportunities for the design and implementation of WP transforms.

The exceptional properties of the designed qWPs such as waveforms' orientation in multiple directions combined with oscillations with multiple frequencies (to name a few) proved to be highly beneficial for the image restoration. Our experiments on image denoising and inpainting using qWP-based algorithms produced state-of-the-art results.

**Fig. 14** Identification of four digits by the DNN trained on R_{2000} set of the reference data

Summarizing, by having such a versatile and flexible tool at hand, we are in a position to address multiple data processing problems such as image deblurring, superresolution, segmentation, classification, and target detection (here the directionality is of utmost importance). The 3D directional wavelet packets, whose design is underway, may be beneficial for seismic and hyper-spectral processing.

We believe that the designed directional qWPs can boost image processing methods that are based on the deep learning by serving as a powerful tool for extraction of characteristic features from images. Our preliminary experiments confirm perfect feasibility of qWPs for that purpose and showed that a qWP-based feature extraction unit has a potential to replace at least several convolution layers in Convolutional Neural Networks. This is to be explored in our future work.

Remark 10 There is some overlap between the current paper and the denoising and inpainting papers [25] and [26]. Actually, the mentioned papers comprise a brief outline of properties of qWPs, which are placed there because currently qWPs are not of common knowledge and our image processing methodology could not be explained without that outline. This paper presents the design and theory of qWPs in full detail and in further publications, we can refer to it.

Appendix

Proof of Proposition 4 Consider the first-level wavelet packet $\psi_{[1],1}^p$ defined by Eq. 2.6. Assume that $p = 2r$. Equations (2.2) and (2.3) imply that

$$\begin{aligned} \hat{b}_{[1]}^{2r}[n + N/2] &= \frac{u^{2r}[2n + N] + v^{2r}[2n + N]}{2} = \frac{u^{2r}[2n] - v^{2r}[2n]}{2} \\ &= \sin^{2r} \frac{2\pi n}{N} q[n], \quad q[n] \stackrel{\text{def}}{=} \sum_{l \in \mathbb{Z}} \frac{1}{(\pi(2n/N + 2l + 1))^{2r}}, \quad 0 < q[n] \leq \sum_{l \in \mathbb{Z}} \frac{1}{(\pi(2l + 1))^{2r}}. \end{aligned}$$

Thus, the DFT is

$$\hat{\psi}_{[1],1}^{2r}[n] = \omega^n \frac{\hat{b}_{[1]}^{2r}[n + N/2]}{\sqrt{\Upsilon^{2r}[n]}} = \sin^{2r} \frac{\pi n}{N} \xi[n], \quad \xi[n] \stackrel{\text{def}}{=} \omega^n 4^r \cos^{2r} \frac{\pi n}{N} \frac{q[n]}{\sqrt{\Upsilon^{2r}[n]}}.$$

Consequently, the wavelet packet $\psi_{[1],1}^{2r}$ has $2r$ LDVMs.

Now assume that $p = 2r - 1$, $r > 1$. In that case, we have

$$\begin{aligned} \hat{b}_{[1]}^{2r-1}[n + N/2] &= \frac{u^{2r-1}[2n] - v^{2r-1}[2n]}{2} = \sin^{2r-1} \frac{2\pi n}{N} s[n], \\ s[n] &\stackrel{\text{def}}{=} - \sum_{l \in \mathbb{Z}} \frac{1}{(\pi(2n/N + 2l + 1))^{2r-1}}, \quad |s[n]| < \sum_{l \in \mathbb{Z}} \frac{1}{(\pi(2l + 1))^{2r-1}}. \end{aligned}$$

Unlike the previous case, $s[0] = - \sum_{l \in \mathbb{Z}} \frac{1}{(\pi(2l+1))^{2r-1}} = 0$.

To specify the structure of the sequence $s[n]$, we note that it is a discretization of the 2π -periodic function

$$s(t) = - \sum_{l \in \mathbb{Z}} \frac{1}{(t + \pi(2l + 1))^{2r-1}}, \quad t = \frac{2\pi n}{N}, \quad s'(0) \neq 0, \quad s''(0) = 0.$$

The function $s(t)$ is differentiable for $r \geq 2$ and $s(0) = 0$; thus, it can be represented by $s(t) = t s'(0) + O(t^3)$. Due to the periodicity of the function $s(t)$, we can write $s(t) = \sin t s'(0) + O(\sin^3 t)$ and $s[n] = \sin \frac{2\pi n}{N} \xi[n]$. Thus,

$$\hat{\psi}_{[1],1}^{2r-1}[n] = \omega^n \frac{\hat{b}_{[1]}^{2r-1}[n + N/2]}{\sqrt{\gamma^{2r-1}[n]}} = \sin^{2r} \frac{\pi n}{N} \eta[n].$$

Consequently, the wavelet packet $\psi_{[1],1}^{2r-1}$ has $2r$ LDVMs. It follows from Eq. 2.9 that at the second decomposition level, the DFTs of three wavelet packets $\psi_{[2],\rho}^p$, $\rho = 1, 2, 3$ comprise the factor $\sin^{2r} \frac{\pi n}{N}$ when either $p = 2r$ or $p = 2r - 1$. The same is true for $\psi_{[l],j}^p$, $j = 1, \dots, 2^l - 1$, $l > 2$. $\square$

Proof of Proposition 9 Denote $\mathbf{M}^q \stackrel{\text{def}}{=} \mathbf{M}_+^q$.

$$\begin{aligned} \mathbf{M}^q[n] \cdot \begin{pmatrix} \hat{z}_{[1]}^0[n]_1 \\ \hat{z}_{[1]}^1[n]_1 \end{pmatrix} &= (\mathbf{M}[n] + i \mathbf{M}^c[n]) \cdot \left(\begin{pmatrix} \hat{y}_{[1]}^0[n]_1 \\ \hat{y}_{[1]}^1[n]_1 \end{pmatrix} - i \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix} \right) \\ &= \mathbf{M}[n] \cdot \begin{pmatrix} \hat{y}_{[1]}^0[n]_1 \\ \hat{y}_{[1]}^1[n]_1 \end{pmatrix} + \mathbf{M}^c[n] \cdot \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix} + i \vec{P}[n] = 2 \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\vec{n}] \end{pmatrix} + i \vec{P}[n], \\ \vec{P}[n] &\stackrel{\text{def}}{=} \mathbf{M}^c[n] \cdot \begin{pmatrix} \hat{y}_{[1]}^0[n]_1 \\ \hat{y}_{[1]}^1[n]_1 \end{pmatrix} - \mathbf{M}[n] \cdot \begin{pmatrix} \hat{c}_{[1]}^0[n]_1 \\ \hat{c}_{[1]}^1[n]_1 \end{pmatrix}. \end{aligned}$$

The vector $\vec{P}[n]$ can be represented by $\vec{P}[n] = \mathbf{R}[n] \cdot \begin{pmatrix} \hat{x}[n] \\ \hat{x}[\vec{n}] \end{pmatrix}$, where

$$\mathbf{R}[n] = \frac{1}{2} \left(\mathbf{M}^c[n] \cdot \tilde{\mathbf{M}}[-n] - \mathbf{M}[n] \cdot \tilde{\mathbf{M}}^c[-n] \right).$$

For $n \neq 0$ and $\vec{n} = n + N/2$, the product $\mathbf{M}^c[n] \cdot \tilde{\mathbf{M}}[-n]$ is

$$\begin{aligned} \mathbf{M}^c[n] \cdot \tilde{\mathbf{M}}[-n] &= \begin{pmatrix} -i\beta[n] & -i\alpha[n] \\ i\beta[\vec{n}] & i\alpha[\vec{n}] \end{pmatrix} \cdot \begin{pmatrix} \beta[-n] & \beta[-\vec{n}] \\ \alpha[-n] & \alpha[-\vec{n}] \end{pmatrix} \\ &= \begin{pmatrix} -i\beta[n]\beta[-n] - i\alpha[n]\alpha[-n] & -i\beta[n]\beta[-\vec{n}] - i\alpha[\vec{n}]\alpha[-\vec{n}] \\ i\beta[\vec{n}]\beta[-n] + i\alpha[\vec{n}]\alpha[-n] & i\beta[n]\beta[-n] + i\alpha[n]\alpha[-n] \end{pmatrix}. \end{aligned}$$

Equations (2.4) and (2.6) imply that

$$\begin{aligned} \beta[n]\beta[-n] + \alpha[n]\alpha[-n] &= 2 \frac{u^p[2n]^2 + v^p[2n]^2}{u^p[2n]^2 + v^p[2n]^2} = 2, \\ \beta[\vec{n}]\beta[-n] + \alpha[\vec{n}]\alpha[-n] &= \frac{(u^p[2n] - v^p[2n])(u^p[2n] + v^p[2n])}{u^p[2n]^2 + v^p[2n]^2} \\ &\quad - \frac{(u^p[2n] + v^p[2n])(u^p[2n] - v^p[2n])}{u^p[2n]^2 + v^p[2n]^2} = 0, \quad \beta[n]\beta[-\vec{n}] + \alpha[n]\alpha[-\vec{n}] = 0. \end{aligned}$$

Thus, for $n \neq 0$, the product $\mathbf{M}^c[n] \cdot \tilde{\mathbf{M}}[-n] = 2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. Similarly, the product $\mathbf{M}[n] \cdot \tilde{\mathbf{M}}^c[-n] = 2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. When $n = 0$, the product $\mathbf{M}^c[0] \cdot \tilde{\mathbf{M}}[0] = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} = 2\mathbf{I}_2$ and, similarly, the product $\mathbf{M}[0] \cdot \tilde{\mathbf{M}}^c[0] = 2\mathbf{I}_2$, where $\mathbf{I}_2$ is the 2×2 identity matrix. As a result, we have

$$\mathbf{R}[n] = 2(1 - \delta[n]) \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \Rightarrow \tilde{P}[n] = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{if } n = 0; \\ 2 \begin{pmatrix} \hat{x}[n]/i \\ -\hat{x}[n + N/2]/i \end{pmatrix}, & \text{otherwise.} \end{cases}$$

For $\mathbf{M}^q \stackrel{\text{def}}{=} \mathbf{M}_-$, the proof is similar. $\square$

Matlab code for calculation of number of different orientations of real qWPs (Section 5.1.2)

```
function D=dir_count(m)
% m -- decomposition level
% D -- number of different orientations of real qWPs from
level ``m"
N=2^m;
M=N/2;
R=N-1;
for kk=1:(M-1)
    for nn=(kk+1):M
        for ll=2:M
            if nn*ll<N+1&&(rem(kk,2)~=0 | rem(nn,2)~=0)
                R=R+2;
            end
        end
    end
end
D=(N^2-R)*2;
end
```

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Declarations

Competing interests The authors declare no competing interests, no conflicts of interests.

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