# Limits of Efficiency in Sequential Auctions

Michal Feldman[1], Brendan Lucier[2], and Vasilis Syrgkanis[3]

[1] Hebrew Univsersity
mfeldman@huji.ac.il
[2] Microsoft Research
brlucier@microsoft.com
[3] Dept of Computer Science, Cornell University
vasilis@cs.cornell.edu

**Abstract.** We study the efficiency of sequential first-price item auctions at (subgame perfect) equilibrium. This auction format has recently attracted much attention, with previous work establishing positive results for unit-demand valuations and negative results for submodular valuations. This leaves a large gap in our understanding between these valuation classes. In this work we resolve this gap on the negative side. In particular, we show that even in the very restricted case in which each bidder has either an additive valuation or a unit-demand valuation, there exist instances in which the inefficiency at equilibrium grows linearly with the minimum of the number of items and the number of bidders. Moreover, these inefficient equilibria persist even under iterated elimination of weakly dominated strategies. Our main result implies linear inefficiency for many natural settings, including auctions with gross substitute valuations, capacitated valuations, budget-additive valuations, and additive valuations with hard budget constraints on the payments. For capacitated valuations, our results imply a lower bound that equals the maximum capacity of any bidder, which is tight following the upper-bound technique established by Paes Leme et al. [20].

## 1 Introduction

Consider the following natural auction setting. An auction house has a number of items that will go up for auction on a particular day. To orchestrate this, the auction house publishes a list of the items to be sold and the order in which they will be auctioned off. The items are then sold one at a time in the given order. A group of bidders attends this session of auctions, with each bidder being allowed to participate in any or all of the single-item auctions that will be run throughout the day. Since the auctions are run one at a time, in sequence, this format is referred to as a sequential auction.

This way of auctioning multiple items is prevalent in practice, due to its relative simplicity and transparency. This model is also related to electronic markets, such as eBay, due to the asynchronous nature of the multiple single-item auctions that are executed on the platform. A natural question, then, is how well such a sequential auction performs in practice. While the auction of a single

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item is relatively simple, equilibria of the larger game may be significantly more complex. For instance, a bidder who views two of the items as substitutes might prefer to win whichever sells at the lower price, and hence when bidding on the first item he must look ahead to the anticipated outcome of the second auction. What's more, the sequential nature of the mechanism implies that the outcome of one auction can influence the behavior of bidders in subsequent auctions. This gives rise to complex reasoning about the value of individual outcomes, with the potential to undermine the efficiency of the overall auction.

In this work we study the efficiency of sequential single-item first-price auctions, where items are sold sequentially using some predefined order and each item is sold by means of a first-price auction. We study the efficiency of outcomes at subgame perfect equilibrium, which is the natural solution concept for a dynamic, sequential game. Theoretical properties of these sequential auctions have been long studied in the economics literature starting from the seminal work of Weber [23]. However, most of the prior literature has focused on very restricted settings, such as unit-demand valuations, identical items, and symmetrically distributed player valuations. The few exceptions that have attempted to study equilibria when bidders have more complex valuations tend to have other restrictions, such as a very limited number of players or items [12,10,3,2]. Much of the difficulty in studying these auctions under complex environments and/or valuations stems from the inherent complexity of the equilibrium structure, which (as alluded to above) can involve complex reasoning about future auction outcomes.

Paes Leme et al. [20] and Syrgkanis and Tardos [21] circumvented this difficulty by performing an indirect analysis on efficiency using the price of anarchy framework. They showed that when bidders have unit-demand valuations (UD), items are heterogeneous, and bidders' valuations are arbitrarily asymmetrically distributed, then the social welfare at every equilibrium is a constant fraction of the optimal welfare. Syrgkanis and Tardos [22] extended this result to no-regret learning outcomes and to settings with budget constraints. On the negative side, Paes Leme et al. [20] showed that this result does not extend to submodular valuations (SM): there exists an instance where the unique "natural" subgame perfect equilibrium has inefficiency that scales linearly with the number of items.

The above results leave a large gap between the positive regime (unit-demand bidders) and the negative (submodular bidders). Many natural and heavily-studied classes of valuations fall in the range between UD and SM valuations. Among them are the following, arranged roughly from most to least general:

- *Gross-substitutes valuations (GS):* Whenever the cost of one item increases, this cannot reduce the demand for another item whose price did not increase.
- *$k$-capacitated valuations ($k$-CAP):* Each player $i$ has a capacity $k_i \leq k$ and a value for each item; the value for a set of items is then the value of the $k_i$ highest-valued items in the set.
- *Budget-additive valuations (BA):* The value of a player $i$ is additive up to a player-specific budget $B_i$ and then remains constant.

The class of GS valuations is motivated by the fact that it is (in a certain sense) the largest class of valuations for which a Walrasian equilibrium is guaranteed