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Strong equilibrium in cost sharing connection games ☆

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Abstract

We study network games in which each player wishes to connect his source and sink, and the cost of each edge is shared among its users either equally (in Fair Connection Games—FCG's) or arbitrarily (in General Connection Games—GCG's). We study the existence and quality of strong equilibria (SE)—strategy profiles from which no coalition can improve the cost of each of its members—in these settings. We show that SE always exist in the following games: (1) Single source and sink FCG's and GCG's. (2) Single source multiple sinks FCG's and GCG's on *series parallel* graphs. (3) Multi source and sink FCG's on *extension parallel* graphs. As for the quality of the SE, in any FCG with n players, the cost of any SE is bounded by $H(n)$ (i.e., the harmonic sum), contrasted with the $\Theta(n)$ price of anarchy. For *any* GCG, any SE is optimal.

JEL classification: C62; C72

Keywords: Strong equilibrium; Price of anarchy; Strong price of anarchy; Coalitions; Cost sharing; Network design

1. Introduction

Computational game theory has introduced the issue of incentives to many of the classical combinatorial optimization problems. The view that the actions taken by the players often are not under the control of a central authority that optimizes the global performance, but rather influenced by strategic considerations of the individual players, has led already to many important insights. In this paper we study the problem of *network construction* from a game theoretic perspective.

Consider the problem of constructing a large network serving strategic users, who wish to connect their terminal as cheaply as possible. Examples of such networks are water distribution systems, telecommunication services and multicast transmission. In such cases, users make global structural decisions about building and maintaining edges throughout the network and incur the associated costs. Such problems can naturally arise also in the design of large computer networks, such as the Internet, which are built, operated and used by a large number of independent autonomous systems, attempting to optimize their own performance.

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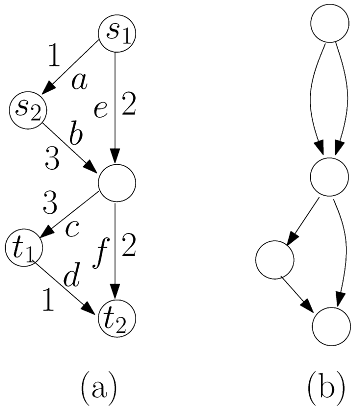

Fig. 1. Graph topologies. In (a), each edge is labeled by a letter, and the associate number is its cost.

Here, we study network design games in which users choose paths to connect their terminals, and incur a cost that is determined by some pre-defined sharing rule. More formally, there is an underlying directed graph with edge costs, and each of n users has a specified source node and sink node he wishes to connect, while paying as little as possible. For example, in the underlying network depicted in Fig. 1, user 1 wishes to connect node s_1 with node t_1, and needs to choose between the paths $\{e, c\}$ and $\{a, b, c\}$. Similarly, user 2 wishes to connect nodes s_2 and t_2, and needs to choose between the paths $\{b, f\}$ and $\{b, c, d\}$. We assume that each user has to guarantee the connectivity between his source and sink nodes.

A crucial question in such settings is how to allocate the edge costs among the users. Sharing of joint costs among heterogeneous agents is a common problem, and a large number of mechanisms (sharing rules) have been proposed for this problem, each associated with different incentives and fairness properties (Moulin and Shenker, 1992, 2001; Herzog et al., 1997). Here, our focus is not on the mechanism design point of view. Rather, we analyze two specific sharing rules with respect to *equilibrium existence* and *equilibrium quality*. The first model we study is the fair connection game (Anshelevich et al., 2004), which corresponds to the fair cost-sharing rule (Herzog et al., 1997). The second is the general connection game (Anshelevich et al., 2003), corresponding to an arbitrary allocation of costs between the users.

In fair connection games, each player selects a path from its source to its sink, and the cost of each edge is split evenly amongst all players whose path contains it. The total cost of each player is the sum of its costs on the edges it selected. For example, in the example depicted in Fig. 1, suppose player 1 chooses the path $\{a, b, c\}$ and player 2 chooses the path $\{b, c, d\}$, then, players 1 and 2 pay fully for edges a and d respectively, and the costs of edges b and c are split among them equally, such that each player pays 1.5 to each of these edges. The total cost of each player in this case is 4. The fair cost sharing scheme is simple and has several basic economic motivations. It corresponds to the Shapley value (Shapley, 1953) of the problem, and is also attractive from a mechanism design point of view (Moulin and Shenker, 2001). In addition, the resulting game is a *congestion game*, and thus always possesses a pure Nash equilibrium (Rosenthal, 1973; Monderer and Shapley, 1996).

The *general connection game* allows each player to offer prices for edges. In this game an edge is bought if the sum of the offers at least covers its cost, and the cost of a player is the sum of its offers on the bought edges. We assume that each player can only offer prices for edges on one path from its source to its destination.[1] For example, in Fig. 1, suppose player 1 offers to pay $1, 2.5, 1$ for edges a, b, c, respectively, and player 2 offers to pay $0.5, 2, 1$ for edges b, c, d, respectively. Then, the edges a, b, c, d are bought, and each player pays the sum of its offers on the edges. Note that in this model the strategy space of a player is more complex than in the fair connection game, where a player should only specify his chosen path.

When considering individual incentives, one needs to discuss the appropriate *solution concept*. Much of the research in game theory has focused on the classical notion of *Nash equilibrium* (NE), where the strategy of each player is a best response to the strategies of all the other players. Indeed, Nash equilibrium is a powerful tool for predicting

[1] Without this assumption, some of the results on the equilibrium quality are trivial, as will be specified below.

outcomes of competitive games and has many benefits, most importantly, it always exists (in mixed strategies). Yet, a natural objection to Nash equilibrium is that it is only resilient to unilateral deviations, and thus not sustainable in the face of coordinated strategies. In many reasonable situations of network construction, agents may be able to coordinate their actions, and thus NE might not be an appropriate solution concept. To address the issue of coordinated actions, we adopt the *strong equilibrium* solution concept.

A strong equilibrium (SE) (Aumann, 1959) is a state from which no coalition (of any size) can deviate and improve the utility of *every* member of the coalition (while possibly lowering the utility of players outside the coalition). This resilience to deviations by coalitions of the players is highly attractive, and one can hope that once a strong equilibrium is reached it is highly likely to sustain.

A major downside of strong equilibrium is that most games do not admit any strong equilibrium. While some families of games have been recently shown to always possess a SE (Andelman et al., 2007), even simple classical games like the *prisoner's dilemma* do not possess any strong equilibrium (which is also an example of a congestion game that does not possess a strong equilibrium). This unfortunate fact is perhaps one of the reasons for the relatively limited literature on strong equilibrium (or the related notion of coalition-proof Nash equilibrium; Bernheim et al., 1987), despite its highly attractive properties.

Our approach is to identify under what conditions the existence of a strong equilibrium in cost sharing games is guaranteed. We identify families of *acyclic graph topologies* that possess a strong equilibrium for any assignment of edge costs. One can view this separation between the graph topology and the edge costs as a separation between the underlying infrastructure and the costs the players observe to purchase edges. While one expects the infrastructure to be stable over long periods of time, the costs the players observe can be easily modified over short time periods. Thus, if a network designer restricts the network topology of his underlying infrastructure to the topologies characterized here, he ensures the existence of a SE (and thus stability) in his network, independent of the specific costs on the edges.

Topological characterizations for network games have been recently provided for various equilibrium properties, including equilibrium existence (Milchtaich, 2006b; Holzman and Law-Yone, 1997, 2003), equilibrium uniqueness (Milchtaich, 2005) and equilibrium efficiency (Roughgarden, 2002; Milchtaich, 2006a; Epstein et al., 2007). Milchtaich (2006b) studied the existence of pure Nash equilibrium in single-commodity network congestion games with player-specific costs or weights. The existence of strong equilibrium was studied in both utility-decreasing (e.g., routing) and utility-increasing (e.g., fair cost-sharing) congestion games. Holzman and Law-Yone (1997) and Holzman and Law-Yone (2003) provided a full topological characterization for a SE existence in single-commodity utility-*decreasing* congestion games, and showed that a SE always exists if and only if the underlying graph is extension-parallel. Rozenfeld and Tennenholtz (2006) showed that in single-commodity utility-*increasing* congestion games, the topological characterization is essentially equivalent to parallel links. In addition, they showed that these results hold for correlated strong equilibria as well (in contrast to the decreasing setting, where correlated strong equilibria might not exist at all). Note that the fair connection game we study is a utility increasing network congestion game as well. Yet, we obtain different conditions than Rozenfeld and Tennenholtz (2006) due to the different assumptions regarding the players' action space. More specifically, Rozenfeld and Tennenholtz (2006) allow to restrict some players from using certain links, even though the links exist in the graph, while we assume that the available strategies for players are fully represented by the underlying graph.

Our results are as follows. We first study the existence of strong equilibrium in acyclic networks. For the single commodity case (all the players have the same source and sink), there is a strong equilibrium in any graph (in both fair and general connection games). Moreover, the strong equilibrium is also the optimal solution (namely, the players share a shortest path from the common source to the common sink). For the case of a single source and multiple sinks (for example, in a multicast tree, where a single sender broadcasts a message to many receivers), we show that in both fair and general connection games, there is a strong equilibrium if the underlying graph is a series parallel graph, and we show an example of a non-series parallel graph that does not admit a strong equilibrium. For the case of multi-commodity (multiple sources and sinks), we show that in a fair connection game, if the graph is an extension parallel graph then there is always a strong equilibrium, and we show an example of a series parallel graph (which is not an extension parallel graph) that does not have a strong equilibrium. As far as we know, we are the first to study the existence of strong equilibrium in multi-commodity and single-source network games.

The second issue we study here is the quality of the obtained solution. When attempting to quantify the performance of some solution, one needs to define an objective function. A natural objective function in our case is the total cost of the constructed network. In the studied models, this cost function has a particularly natural meaning, as it is exactly the sum of the individual costs of all the players. Once an objective function is defined, we can quantify the inefficiency of equilibria. Papadimitriou (2001) and Koutsoupias and Papadimitriou (1999) coined the term *price of anarchy* (PoA), referring to the ratio between the cost of the worst-case Nash equilibrium and the optimal solution. This notion has been extensively studied in various settings, including job scheduling (Koutsoupias and Papadimitriou, 1999; Christodoulou et al., 2004; Czumaj and Vöcking, 2002; Awerbuch et al., 2003), network design (Albers et al., 2006; Anshelevich et al., 2004, 2003; Fabrikant et al., 2003), network routing (Roughgarden and Tardos, 2002; Roughgarden, 2002; Awerbuch et al., 2005; Christodoulou and Koutsoupias, 2005) and more. Similarly, Andelman et al. (2007) defined the *strong price of anarchy* (SPoA) as the ratio between the cost of the worst-case *strong* equilibrium and the optimal solution. In addition, the k-SPoA is defined analogously for the cost of a k-SE (i.e., a profile that is resilient to deviations of coalitions of size up to k). Since every k-SE is a NE but not vice versa, k-SE has a potential to reduce the PoA. The notion of SPoA has been studied in job scheduling and network formation settings (Andelman et al., 2007; Fiat et al., 2007; Leonardi and Sankowski, 2007), and it was shown that in many of these settings the SPoA is significantly lower than the PoA. Here, we show that for any fair connection game, if there exists a strong equilibrium, it is at most a factor of $H(n) = \sum_{i=1}^{n} \leqslant 1 + \ln n$ from the optimal solution, where n is the number of players. This should be contrasted with the $\Theta(n)$ bound that exists for the price of anarchy (Anshelevich et al., 2004).[2] We also show that the k-SPoA is at most $\frac{n}{k} \cdot H(k)$, and provide an almost tight lower bound. We prove that these results can be extended to any fair connection game with nondecreasing concave edge cost functions. For general connection games, we show that any strong equilibrium is optimal. Here, the contrast with the $\Theta(n)$ price of anarchy is even more significant than in fair connection games.

It should be noted that other approaches have been taken to study the effect of coalition formation in network games (Hayrapetyan et al., 2006; Fotakis et al., 2006; Kuniavsky and Smorodinsky, 2007). For example, Hayrapetyan et al. (2006) studied the effect of collusion among fixed sets of coalitions (according to an exogenous partition over the set of agents) that attempt to maximize their collective welfare, and found that in such settings, the quality of the solution can deteriorate by an arbitrarily high factor in the presence of coalitions.

2. Model

2.1. Game theory definitions

A game $\Lambda = \langle N, (\Sigma_i), (c_i) \rangle$ has a finite set $N = \{1, \ldots, n\}$ of players. Player $i \in N$ has a set Σ_i of actions, the joint action set is $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ and a joint action $S \in \Sigma$ is also called a *profile*. The cost function of player i is $c_i : \Sigma \to \mathbb{R}^+$, which maps the joint action $S \in \Sigma$ to a non-negative real number. Let $S = (S_1, \ldots, S_n)$ denote the profile of actions taken by the players, and let $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ denote the profile of actions taken by all players other than player i. Note that $S = (S_i, S_{-i})$. The *social cost* of a game Λ is the sum of the costs of the players, and we denote by $OPT(\Lambda)$ the minimal social cost of a game Λ. That is, $OPT(\Lambda) = \min_{S \in \Sigma} cost_\Lambda(S)$, where $cost_\Lambda(S) = \sum_{i \in N} c_i(S)$.

A joint action $S \in \Sigma$ is a *pure Nash equilibrium* if no player $i \in N$ can benefit from unilaterally deviating from its action to another action, i.e., $\forall i \in N\ \forall S_i' \in \Sigma_i$: $c_i(S_{-i}, S_i') \geqslant c_i(S)$. We denote by $NE(\Lambda)$ the set of pure Nash equilibria in the game Λ.

Resilience to coalitions: A *pure deviation* of a set of players $\Gamma \subseteq N$ (also called *coalition*) specifies an action for each player in the coalition, i.e., $\gamma \in \times_{i \in \Gamma} \Sigma_i$. A joint action $S \in \Sigma$ is not resilient to a pure deviation of a coalition Γ if there is a pure joint action γ of Γ such that $c_i(S_{-\Gamma}, \gamma) < c_i(S)$ for every $i \in \Gamma$ (i.e., the players in the coalition can deviate in such a way that *each* player in the coalition reduces its cost). A pure Nash equilibrium $S \in \Sigma$ is a *k-strong equilibrium*, if there is no coalition Γ of size at most k, such that S is not resilient to a pure deviation by Γ. We denote by k-$SE(\Lambda)$ the set of k-strong equilibria in the game Λ. We denote by $SE(\Lambda)$ the set of n-strong equilibria, and call $S \in SE(\Lambda)$ a *strong equilibrium* (SE).

[2] In what follows, as well as in the remainder of this paper, we use the Big-Theta, Big-O and Big-Omega notations, which are used to describe the asymptotic behavior of functions. We denote $g(n) = O(f(n))$ if $\exists c$ such that $g(n) \leqslant c \cdot f(n)\ \forall n \geqslant 1$. We denote $g(n) = \Omega(f(n))$ if $\exists c$ such that $g(n) \geqslant c \cdot f(n)\ \forall n \geqslant 1$. Finally, we denote $g(n) = \Theta(f(n))$ if $\exists c_1, c_2$ such that $c_1 \cdot f(n) \leqslant g(n) \leqslant c_2 \cdot f(n)\ \forall n \geqslant 1$.

Next we define the price of anarchy (PoA) (Koutsoupias and Papadimitriou, 1999) and the k-strong price of anarchy (k-SPoA) for the game Λ. The *Price of Anarchy* (PoA) is the ratio between the *maximal* cost of a pure Nash equilibrium (assuming one exists) and the social optimum, i.e., $\max_{S\in NE(\Lambda)} \frac{cost_\Lambda(S)}{OPT(\Lambda)}$. Similarly, the *Price of Stability* (PoS) is the ratio between the *minimal* cost of a pure Nash equilibrium and the social optimum, i.e., $\min_{S\in NE(\Lambda)} \frac{cost_\Lambda(S)}{OPT(\Lambda)}$. The *$k$-Strong Price of Anarchy* (k-SPoA) is the ratio between the *maximal* cost of a k-strong equilibrium (assuming one exists) and the social optimum, i.e., $\max_{S\in k\text{-}SE(\Lambda)} \frac{cost_\Lambda(S)}{OPT(\Lambda)}$. The SPoA is the n-SPoA. Similarly, the *Strong Price of Stability* (SPoS) is the ratio between the *minimal* cost of a pure strong equilibrium and the social optimum, i.e., $\min_{S\in SE(\Lambda)} \frac{cost_\Lambda(S)}{OPT(\Lambda)}$. Note that both k-SPoA and SPoS are defined only if some strong equilibrium exists.

2.2. Cost sharing connection games

A cost sharing connection game has an underlying directed graph $G = (V, E)$ where each edge $e \in E$ has an associated finite cost $c_e \geqslant 0$. In a connection game each player $i \in N$ has an associated source s_i and sink t_i.

In a *fair connection game*, the actions Σ_i of player i include all the paths from s_i to t_i. The cost of each edge is shared equally by the set of all players whose paths contain it. Given a joint action, the cost of a player is the sum of its costs on the edges it selected. More formally, the cost function of each player on an edge e, in a joint action S, is $f_e(n_e(S)) = \frac{c_e}{n_e(S)}$, where $n_e(S)$ is the number of players that selected a path containing edge e in S. The cost of player i, when selecting path $Q_i \in \Sigma_i$ is $c_i(S) = \sum_{e\in Q_i} f_e(n_e(S))$.

In a *general connection game*, the actions Σ_i of player i is a payment vector p_i, where $p_i(e)$ is how much player i is offering to contribute to the cost of edge e.[3] Given a profile p, any edge e such that $\sum_i p_i(e) \geqslant c_e$ is considered *bought*, and E_p denotes the set of bought edges. Let $G_p = (V, E_p)$ denote the graph bought by the players for profile $p = (p_1, \ldots, p_n)$. Clearly, each player tries to minimize its total payment which is $c_i(p) = \sum_{e\in E_p} p_i(e)$ if s_i is connected to t_i in G_p, and infinity otherwise.[4] We denote by $c(p) = \sum_i c_i(p)$ the total cost under the profile p. For a subgraph H of G we denote the total cost of the edges in H by $c(H)$.

A *symmetric* connection game is a game in which all the players share the same source and sink nodes. We also call a symmetric connection game a *single source single sink* connection game, or a *single commodity* connection game. A *single source* connection game is a game in which all the players share the same source. Finally, a *multi-commodity* connection game implies that each player has its own source and his own sink.

2.3. Extension parallel and series parallel directed graphs

Our directed graphs would be acyclic, and would have a *source* node (from which all nodes are reachable) and a *sink* node (which every node can reach). We first define the following actions for composition of directed graphs.

- Identification: The *identification* operation allows to collapse two nodes to one. More formally, given graph $G = (V, E)$, we define the *identification* of a node $v_1 \in V$ and $v_2 \in V$ forming a new node $v \in V$ as creating a new graph $G' = (V', E')$, where $V' = V - \{v_1, v_2\} \cup \{v\}$ and E' includes the edges of E where the edges of v_1 and v_2 are now connected to v.
- Parallel composition: Given two directed graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new graph $G = G_1 \parallel G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union graph. To create $G = G_1 \parallel G_2$, we identify the sources s_1 and s_2, forming a new source node s, and identify the sinks t_1 and t_2, forming a new sink t.
- Series composition: Given two directed graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new graph $G = G_1 \to G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union graph. To create $G = G_1 \to G_2$ we identify the vertices t_1 and s_2, forming a new vertex u. The graph G has a source $s = s_1$ and a sink $t = t_2$.
- Extension composition: A series composition when one of the graphs, G_1 or G_2, is composed of a single directed edge is an extension composition, and we denote it by $G = G_1 \to_e G_2$.

[3] We limit the players to select a path connecting s_i to t_i and payments only on those edges.

[4] This implies that in equilibrium every player has its sink and source connected by a path in G_p.

An *extension parallel graph* (*EPG*) is a graph G consisting of either: (1) a single directed edge (s,t), (2) a graph $G = G_1 \parallel G_2$ or (3) a graph $G = G_1 \to_e G_2$, where G_1 and G_2 are extension parallel graphs. A *series parallel graph* (*SPG*) is a graph G consisting of either: (1) a single directed edge (s,t), (2) a graph $G = G_1 \parallel G_2$ or (3) a graph $G = G_1 \to G_2$, where G_1 and G_2 are series parallel graphs.

An undirected SPG is defined similarly to a directed SPG, with the difference that the edges are undirected. Thus, an underlying undirected graph of a directed SPG is an undirected SPG.

Given a path Q and two vertices u, v on Q, we denote the subpath of Q from u to v by $Q_{u,v}$. The following lemma would be the main topological tool in the case of single source graphs. Its proof appears in Appendix A.

Lemma 2.1. *Let G be an SPG with source s and sink t. Given a path Q, from s to t, and a vertex t', there exists a vertex $y \in Q$, such that for any path Q' from s to t', the path Q' contains y and the paths $Q'_{y,t'}$ and Q are edge disjoint.* (*We call the vertex y the* intersecting vertex *of Q and t'.*)

3. Fair connection games

This section derives our results for fair connection games.

3.1. Existence of strong equilibrium

Since any fair connection game is a congestion game, it always admits a Nash equilibrium in pure strategies. Yet, this is not necessarily the case for a strong equilibrium. In this section, we study the existence of strong equilibrium in fair connection games. We begin with a simple case, showing that every symmetric fair connection game possesses a strong equilibrium. This observation can be also derived as a special case of Proposition 3 of Rozenfeld (2007), and we give here the proof for completeness.

Theorem 3.1. *Every symmetric fair connection game admits a strong equilibrium.*

Proof. Let s' be the source and t' be the sink of all the players. We show that a profile S in which all the players choose the same shortest path Q (from the source s' to the sink t') is a strong equilibrium. Suppose by contradiction that S is not a SE. Then there is a coalition Γ that can deviate to a new profile S' such that the cost of every player $j \in \Gamma$ decreases. Let Q'_j be a new path used by player $j \in \Gamma$. Since Q is a shortest path, it holds that $c(Q'_j \setminus (Q \cap Q'_j)) \geqslant c(Q \setminus (Q \cap Q'_j))$, for any path Q'_j. Therefore for every player $j \in \Gamma$ we have that $c_j(S') \geqslant c_j(S)$. However, this contradicts the fact that all players in Γ reduce their cost. (In fact, no player in Γ has reduced its cost.) □

While every symmetric fair connection game admits a SE, it does not hold for every fair connection game. In what follows, we study the acyclic network topologies that admit a strong equilibrium for *any* assignment of edge costs, and give examples of topologies for which a strong equilibrium does not exist. The following lemma, the proof of which appears in Appendix A, plays a major role in our proofs of the existence of SE.

Lemma 3.2. *Let Λ be a fair connection game on a series parallel graph G with source s and sink t. Assume that player i has $s_i = s$ and $t_i = t$ and that Λ has some SE. Let S be a SE that minimizes the cost of player i* (*out of all SE*), *i.e., $c_i(S) = \min_{T \in SE} c_i(T)$ and let S^* be the profile that minimizes the cost of player i* (*out of all possible profiles*), *i.e., $c_i(S^*) = \min_{T \in \Sigma} c_i(T)$. Then, $c_i(S) = c_i(S^*)$.*

The next lemma considers a parallel composition.

Lemma 3.3. *Let Λ be a fair connection game on a graph $G = G_1 \parallel G_2$, where G_1 and G_2 are series parallel graphs. If every fair connection game on the graphs G_1 and G_2 possesses a strong equilibrium, then the game Λ possesses a strong equilibrium.*

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have sources s_1 and s_2 and sinks t_1 and t_2, respectively. Let T_i be the set of players with an endpoint in $V_i \setminus \{s, t\}$, for $i \in \{1, 2\}$. (An endpoint is either a source or a sink of a player.) Let

T_3 be the set of players j such that $s_j = s$ and $t_j = t$. Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players $T_1 \cup T_3$ and $T_2 \cup T_3$, respectively.

Let S' and S'' be the SE in Λ_1 and Λ_2 that minimizes the cost of players in T_3, respectively. Assume w.l.o.g. that $c_i(S') \leqslant c_i(S'')$, where player $i \in T_3$. (Note that all the players in T_3, in a SE, follow the same path and therefore have the same cost.) In addition, let Λ_2' be the game on the graph G_2 with players T_2 and let $\bar{S}$ be a SE in Λ_2'.

We will show that the profile $S = S' \cup \bar{S}$ is a SE in Λ. Suppose by contradiction that S is not a SE. Then, there is a coalition Γ that can deviate such that the cost of every player $j \in \Gamma$ decreases. By Lemma 3.2 and the assumption that $c_i(S') \leqslant c_i(S'')$, a player $j \in T_3$ cannot improve its cost. Therefore, $\Gamma \subseteq T_1 \cup T_2$. But this is a contradiction to S' being a SE in Λ_1 or $\bar{S}$ being a SE in Λ_2'. □

Lemma 3.3 will serve us in our analysis of fair connection games. We first state our existence result for the case of single source fair connection games.

Theorem 3.4. *Every single source fair connection game on a series-parallel graph possesses a strong equilibrium.*

Proof. We prove the theorem by induction on the network size $|V|$. The claim obviously holds if $|V| = 2$. We show the claim for a series composition, i.e., $G = G_1 \to G_2$, and for a parallel composition, i.e., $G = G_1 \parallel G_2$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are SPG's with sources s_1, s_2, and sinks t_1, t_2, respectively.

Series composition. Let $G = G_1 \to G_2$. Let T_1 be the set of players j such that $t_j \in V_1$, and T_2 be the set of players j such that $t_j \in V_2 \setminus \{s_2\}$.

Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players $T_1 \cup T_2$ and T_2, respectively. For every player $i \in T_2$ with action S_i in the game Λ, let $S_i \cap E_1$ be its induced action in the game Λ_1, and let $S_i \cap E_2$ be its induced action in the game Λ_2. Thus, in the game Λ_1 the sinks of all players in T_2 are set to t_1 and in the game Λ_2 the sources of all players in T_2 are set to s_2.

Let S' be a SE in Λ_1 that minimizes the cost of players in T_2 (such a SE exists by the induction hypothesis and Lemma 3.2). Let S'' be any SE in Λ_2. We will show that the profile $S = S' \cup S''$ is a SE in the game Λ, i.e., for player $j \in T_2$ we use the profile $S_j = S_j' \cup S_j''$.

Suppose by contradiction that S is not a SE. Then, there is a coalition Γ that can deviate such that the cost of every player $j \in \Gamma$ decreases. Next, we distinguish between two cases:

Case 1: $\Gamma \subseteq T_1$. This is a contradiction to S' being a SE.

Case 2: There exists a player $j \in \Gamma \cap T_2$. By Lemma 3.2, player j cannot improve its cost in Λ_1 so the improvement is due to Λ_2. Consider the coalition $\Gamma \cap T_2$, it would still improve its cost. However, this contradicts the fact that S'' is a SE in Λ_2.

Parallel composition. Follows directly from Lemma 3.3. □

We next provide sufficient topological conditions for the existence of SE in multi-commodity fair connection games.

Theorem 3.5. *Every fair connection game on an extension parallel graph possesses a strong equilibrium.*

Proof. We prove the theorem by induction on the network size $|V|$. Let Λ be a fair connection game on an EPG $G = (V, E)$. The claim obviously holds if $|V| = 2$. If the graph G is a parallel composition of two EPG graphs G_1 and G_2, then the claim follows from Lemma 3.3. It remains to prove the claim for extension composition. Suppose the graph G is an extension composition of the graph G_1 consisting of a single edge $e = (s_1, t_1)$ and an EPG $G_2 = (V_2, E_2)$ with terminals s_2, t_2, such that $s = s_1$ and $t = t_2$. (The case that G_1 is an EPG and G_2 is a single edge is similar.) Let T_1 be the set of players whose source and sink are s_1 and t_1, respectively (i.e., their path is in G_1). Let T_2 be the set of players whose source and sink are in G_2. Finally, let T_3 be the set of players whose source is s_1 and whose sink is in $V_2 \setminus t_1$.

Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players $T_1 \cup T_3$ and $T_2 \cup T_3$, respectively. Let S', S'' be SE in Λ_1 and Λ_2, respectively. We will show that the profile $S = S' \cup S''$ is a SE in the game Λ. Suppose by contradiction that S is not a SE. Then, there is a coalition Γ of minimal size that can deviate

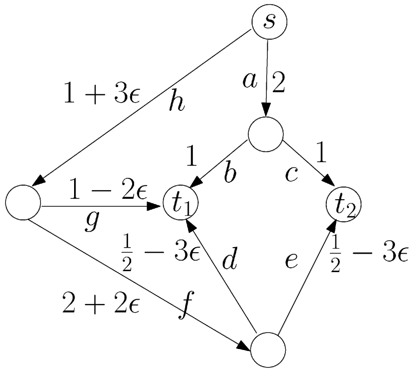

Fig. 2. Example of a single source fair connection game that does not admit a SE.

such that the cost of every player $j \in \Gamma$ decreases. Clearly, $T_1 \cap \Gamma = \emptyset$, since players in T_1 have a single strategy. Hence, $\Gamma \subseteq T_2 \cup T_3$. Since no player $j \in T_2 \cup T_3$ can improve its cost in Λ_1, any player $j \in T_2 \cup T_3$ improves its cost in Λ_2. However, this contradicts the fact that S'' is a SE in Λ_2. □

In the following theorem we provide a few examples of slightly more general topologies in which a strong equilibrium does not exist.

Theorem 3.6. *The following connection games exist*: (1) *There exists a multi-commodity fair connection game on a series parallel graph that does not possess a strong equilibrium.* (2) *There exists a single source fair connection game that does not possess a strong equilibrium.*

Proof. For claim (1), consider the graph depicted in Fig. 1(a). This game has a unique NE,[5] where $S_1 = \{e, c\}$, $S_2 = \{b, f\}$, and each player has a cost of 5. However, consider the following coordinated deviation S', where $S'_1 = \{a, b, c\}$, and $S'_2 = \{b, c, d\}$. In this profile, each player pays a cost of 4, and thus improves its cost.

For claim (2), consider a single source fair connection game on the graph depicted in Fig. 2. There are two players. Player $i = 1, 2$ wishes to connect the source s to its sink t_i and the unique NE[6] is $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, and each player has a cost of 2. Then, if both players can deviate to $S'_1 = \{h, f, d\}$ and $S'_2 = \{h, f, e\}$ they decrease their costs to $2 - \epsilon/2$. □

Unfortunately, our topological conditions for the existence of SE are only sufficient, but not necessary. The Braess graph in Fig. 5 (where the edges are directed from the source to the sink and the middle edge is arbitrarily directed) with a single source s and multiple sinks is an example of a non-series parallel graph with a single source that always admits a SE. Similarly, the graph in Fig. 1(b) is an example of a non-extension parallel graph which always admits a strong equilibrium.

3.2. Strong price of anarchy

The price of anarchy in fair connection games can be as high as n. The following theorem shows that the strong price of anarchy is bounded by $H(n) = \sum_{i=1}^{n} \frac{1}{i} \leqslant 1 + \ln n$.

Theorem 3.7. *The strong price of anarchy of a fair connection game with n players is at most $H(n)$.*

---

[5] In every NE of the game, player 1 will buy the edge e and player 2 will buy the edge f. This is since the alternate path, in the respective part, will cost the player 2.5. Thus, player 1 (player 2) will buy the edge c (edge b) alone, and each player will have a cost of 5.

[6] We can show that this is the unique NE by a simple case analysis: (i) If $S_1 = \{h, f, d\}$ and $S_2 = \{h, f, e\}$, then player 1 can deviate to $S'_1 = \{h, g\}$ and decrease its cost. (ii) If $S_1 = \{h, g\}$ and $S_2 = \{h, f, e\}$, then player 2 can deviate to $S'_2 = \{a, c\}$ and decrease its cost. (iii) If $S_1 = \{h, g\}$ and $S_2 = \{a, c\}$, then player 1 can deviate to $S'_1 = \{a, b\}$ and decrease its cost.

Proof. Let Λ be a fair connection game on the graph G. We denote by $\Lambda(\Gamma)$ the game played on the graph G by a set of players Γ, where the action space of player $i \in \Gamma$ remains Σ_i (the same as in Λ). Let $S = (S_1, \ldots, S_n)$ be a profile in the game Λ. We denote by $S(\Gamma) = S_\Gamma$ the induced profile of players in Γ in the game $\Lambda(\Gamma)$. Let $n_e(S(\Gamma))$ denote the load of edge e under the profile $S(\Gamma)$ in the game $\Lambda(\Gamma)$, i.e., $n_e(S(\Gamma)) = |\{j \mid j \in \Gamma,\ e \in S_j\}|$. We use the potential function for congestion games specified in (Rosenthal, 1973; Monderer and Shapley, 1996). We denote by $\Phi(S(\Gamma))$ the potential function of the profile $S(\Gamma)$ in the game $\Lambda(\Gamma)$, where $\Phi(S(\Gamma)) = \sum_{e\in E}\sum_{j=1}^{n_e(S(\Gamma))} f_e(j)$, and define $\Phi(S(\phi)) = 0$. In our case, it holds that

$$\Phi(S) = \sum_{e\in E} c_e \cdot H\big(n_e(S)\big). \tag{1}$$

Let S be a SE, and let S^* be the profile of the optimal solution. We define an order on the players as follows. Let $\Gamma_n = \{1, \ldots, n\}$ be the set of all the players. For each $k = n, \ldots, 1$, since S is a SE, there exists a player in Γ_k, w.l.o.g. call it player k, such that

$$c_k(S) \leqslant c_k\big(S_{-\Gamma_k}, S^*_{\Gamma_k}\big). \tag{2}$$

In this way, Γ_k is defined recursively, such that for every $k = n, \ldots, 2$ it holds that $\Gamma_{k-1} = \Gamma_k \setminus \{k\}$. (That is, after the renaming, $\Gamma_k = \{1, \ldots, k\}$.)

Let $c_k(S(\Gamma_k))$ denote the cost of player k in the game $\Lambda(\Gamma_k)$ under the induced profile $S(\Gamma_k)$. It is easy to see that $c_k(S(\Gamma_k)) = \Phi(S(\Gamma_k)) - \Phi(S(\Gamma_{k-1}))$.[7] Therefore,

$$c_k(S) \leqslant c_k\big(S_{-\Gamma_k}, S^*_{\Gamma_k}\big) \leqslant c_k\big(S^*(\Gamma_k)\big) = \Phi\big(S^*(\Gamma_k)\big) - \Phi\big(S^*(\Gamma_{k-1})\big). \tag{3}$$

Summing over all players, we obtain:

$$\begin{aligned}\sum_{i\in N} c_i(S) &\leqslant \Phi\big(S^*(\Gamma_n)\big) - \Phi\big(S^*(\phi)\big)\\ &= \Phi\big(S^*(\Gamma_n)\big) = \sum_{e\in S^*} c_e \cdot H\big(n_e(S^*)\big)\\ &\leqslant \sum_{e\in S^*} c_e \cdot H(n) = H(n)\cdot OPT(\Lambda),\end{aligned}$$

where the first inequality follows since the sum of the right-hand side of Eq. (3) telescopes, and the second equality follows from Eq. (1). □

Next we bound the SPoA when coalitions of size at most k are allowed.

Theorem 3.8. *The k-SPoA of a fair connection game with n players is at most $\frac{n}{k} \cdot H(k)$.*

Proof. Let S be a SE of Λ, and S^* be the profile of the optimal solution of Λ. To simplify the proof, we assume that n/k is an integer. We partition the players to n/k groups $T_1, \ldots, T_{n/k}$ each of size k. Let the game Λ_j be the subgame induced by the profile S on the graph G played by the set of players T_j, i.e., the profile of the players in $N \setminus T_j$ is fixed to S_{-T_j} and only the players in T_j play. The obtained edge cost functions in the game Λ_j are concave and nondecreasing. Let $S(T_j)$ denote the profile of the k players in T_j in the game Λ_j induced by the profile S of the game Λ. Since Theorem 3.7 holds for nondecreasing concave edge cost functions (see Theorem 3.10), it follows that for each game Λ_j, $j = 1, \ldots, n/k$,

$$cost_{\Lambda_j}\big(S(T_j)\big) = \sum_{i\in T_j} c_i(S) \leqslant H(k)\cdot OPT(\Lambda_j) \leqslant H(k)\cdot OPT(\Lambda),$$

where the first equality follows since for each game Λ_j, the cost of player $i \in T_j$ is $c_i(S)$.

[7] This follows since for any strategy profile S, if a single player k deviates to strategy S'_k, then the change in the potential value $\Phi(S) - \Phi(S'_k, S_{-k})$ is exactly the change in the cost to player k.

Summing over all games Λ_j, $j = 1, \dots, n/k$, we obtain:

$$cost_\Lambda(S) = \sum_{j=1}^{n/k} cost_{\Lambda_j}\big(S(T_j)\big) \leqslant \frac{n}{k} \cdot H(k) \cdot OPT(\Lambda). \qquad \square$$

The following theorem shows an almost matching lower bound. The lower bound is at most $H(n)$ from the upper bound and both for $k = O(1)$ and $k = \Omega(n)$ the difference is only a constant.

Theorem 3.9. *For fair connection games with n players, k-SPoA* $\geqslant \max\{\frac{n}{k}, H(n)\}$.

Proof. For the lower bound of $H(n)$, we observe that in the example presented in Anshelevich et al. (2004), the unique Nash equilibrium is also a strong equilibrium, and therefore $k\text{-}SPoA = H(n)$ for any $1 \leqslant k \leqslant n$. For the lower bound of n/k, consider a graph composed of two parallel links of costs 1 and n/k. Consider the profile S in which all n players use the link of cost n/k. The cost of each player is $1/k$, while if any coalition of size at most k deviates to the link of cost 1, the cost of each player is at least $1/k$. Therefore, the profile S is a k-SE, and $k\text{-}SPoA = n/k$. $\square$

The results of Theorems 3.7 and 3.8 can be extended to concave cost functions. Consider the extended fair connection game, where each edge has a cost which depends on the number of players using that edge, $c_e(n_e)$. We assume that the cost function $c_e(n_e)$ is a nondecreasing, concave function. Note that the cost of an edge $c_e(n_e)$ might increase with the number of players using it, but the cost per player $f_e(n_e) = \frac{c_e(n_e)}{n_e}$ decreases when $c_e(n_e)$ is concave.

Theorem 3.10. *The strong price of anarchy of a fair connection game with nondecreasing concave edge cost functions and n players is at most* $H(n)$.

Proof. The proof is analogues to the proof of Theorem 3.7. For the proof we show that $cost(S) \leqslant \Phi(S^*) \leqslant H(n) \cdot cost(S^*)$. We first show the first inequality. Since the function $c_e(x)$ is concave, the cost per player $\frac{c_e(x)}{x}$ is a non-increasing function. Therefore, inequality (3) in the proof of Theorem 3.7 holds. Summing inequality (3) over all players we obtain $cost(S) = \sum_i c_i(S) \leqslant \Phi(S^*(\Gamma_n)) - \Phi(S^*(\phi)) = \Phi(S^*)$. The second inequality follows since $c_e(x)$ is nondecreasing and therefore $\sum_{x=1}^{n_e} \frac{c_e(x)}{x} \leqslant H(n_e) \cdot c_e(n_e)$. $\square$

Using the arguments in the proof of Theorem 3.10 and the proof of Theorem 3.8 we derive,

Theorem 3.11. *The k-SPoA of a fair connection game with nondecreasing concave edge cost functions and n players is at most* $\frac{n}{k} \cdot H(k)$.

Since the set of strong equilibria is contained in the set of Nash equilibria, it must hold that $SPoA \leqslant PoA$, meaning that the SPoA can only be improved compared to the PoA. However, with respect to the price of stability, the opposite direction holds, that is, $SPoS \geqslant PoS$. We next show that there exists a fair connection game in which the inequality is strict.

Theorem 3.12. *There exists a fair connection game in which SPoS > PoS.*

Proof. Consider a single source fair connection game on the graph depicted in Fig. 3 (which is a variation of the example given in Anshelevich et al., 2004). Player $i = 1, \dots, n$ wishes to connect the source s to its sink t_i. Assume that each player $i = 1, \dots, n-2$ has its own path of cost $\frac{1}{i}$ from s to t_i and players $i = n-1, n$ have a joint path of cost $\frac{2}{n}$ from s to t_i. Additionally, all players can share a common path of cost $1+\epsilon$ for some small $\epsilon > 0$. The optimal solution connects all players through the common path of cost $1+\epsilon$, and this is also a Nash equilibrium with total cost $1+\epsilon$. It is easy to verify that the solution where each player $i = 1, \dots, n-2$ uses his own path and users $i = n-1, n$ use their joint path is the unique strong equilibrium of this game with total cost $\sum_{i=1}^{n-2} \frac{1}{i} + \frac{2}{n} = \Theta(\log n)$. $\square$

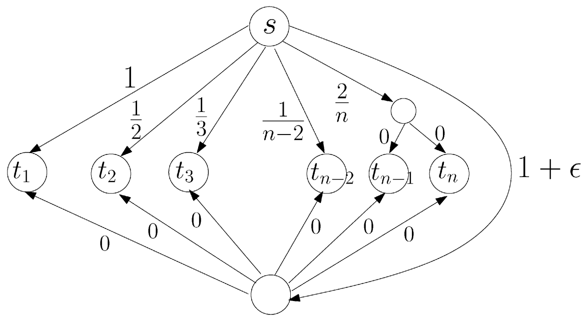

Fig. 3. Example of a network topology in which *SPoS* > *PoS*.

While the example above shows that the SPoS may be greater than the PoS, the upper bound of $H(n) = \Theta(\log n)$, proven for the PoS (Anshelevich et al., 2004), serves as an upper bound for the SPoS as well. This is a direct corollary from Theorem 3.7, as $SPoS \leqslant SPoA$ by definition.

Corollary 3.13. *The strong price of stability of a fair connection game with n players is at most* $H(n) = \Theta(\log n)$.

4. General connection games

In this section, we derive our results for general connection games.

4.1. Existence of strong equilibrium

We begin with a characterization of the existence of a strong equilibrium in symmetric general connection games. Similar to Theorem 3.1 (using a similar proof) we establish,

Theorem 4.1. *In every symmetric general connection game there exists a strong equilibrium.*

While every single source general connection game possesses a pure Nash equilibrium (Anshelevich et al., 2003), it does not necessarily admit some strong equilibrium.[8]

Theorem 4.2. *There exists a single source general connection game that does not admit any strong equilibrium.*

Proof. Consider a single source general connection game with 3 players on the graph depicted in Fig. 4. Player i wishes to connect the source s with its sink t_i. We need to consider only the NE profiles: (i) If all three players use the link of cost 3, then there must be two agents whose total sum exceeds 2, thus they can both reduce cost by deviating to an edge of cost $2 - \epsilon$. (ii) If two of the players use an edge of cost $2 - \epsilon$ jointly, and the third player uses a different edge of cost $2 - \epsilon$, then, the players with non-zero payments can deviate to the path with the edge of cost 3 and reduce their costs (since before the deviation the total payments of the players is $4 - 2\epsilon$). We showed that none of the NE are SE, and thus the game does not possess any SE. □

Next we show that for the class of series parallel graphs, there is always a strong equilibrium in the case of a single source.

Theorem 4.3. *Every single source general connection game on a series-parallel graph admits a strong equilibrium.*

Proof. Let Λ be a single source general connection game on an SPG $G = (V, E)$ with source s and sink t. We present an algorithm that constructs a specific SE. We first consider the following partial order between the players.

[8] We thank Elliot Anshelevich, whose similar topology for the fair-connection game inspired this example.

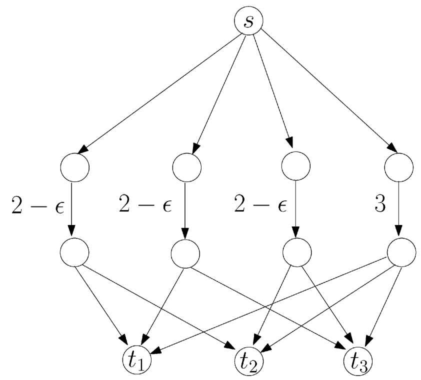

Fig. 4. Example of a single source general connection game that does not admit a strong equilibrium. The edges that are not labeled with costs have a cost of zero.

For players i and j, we have that $i \mapsto j$ if there is a directed path from t_i to t_j. We complete the partial order to a full order (in an arbitrary way), and w.l.o.g. we assume that $1 \mapsto 2 \mapsto \cdots \mapsto n$.

The algorithm COMPUTE-SE, considers the players in an increasing order, starting with player 1. Each player i will fully buy a subset of the edges, and any player $j > i$ will consider the cost of those (bought) edges as zero. When COMPUTE-SE considers player j, the cost of the edges that players 1 to $j-1$ have bought is set to zero, and player j fully buys a shortest path Q^j from s to t_j. Namely, for every edges $e \in Q^j \setminus \bigcup_{i<j} Q^i$ we have $p_j(e) = c_e$ and otherwise $p_j(e) = 0$. We next show that the algorithm COMPUTE-SE computes a SE.

Assume by way of contradiction that the profile p is not a SE. Then, there exists a coalition that can improve the costs of all its players by a deviation. Let Γ be such a coalition and let player $i = \max\{j \in \Gamma\}$. For a player $j \in \Gamma$ let $\bar{Q}^j$ and $\bar{p}^j$ be the path and payment of player j after the deviation, respectively. Let Q' be a path from the sink of player i, i.e. t_i, to the sink of G, i.e. t. Then $Q = \bar{Q}^i \cup Q'$ is a path from the source s to the sink t. For any player $j < i$, let y_j be the intersecting vertex of Q and t_j (by Lemma 2.1 one is guarantee to exist). Let y be the furthest vertex on the path Q such that $y = y_j$ for some $j < i$. The path from the source s to node y was fully paid for by players $j < i$ in p (before the deviation). We show that player i's cost after the deviation, i.e. $c_i(\bar{p})$, is at least its cost before the deviation, i.e. $c_i(p)$, contradicting the fact that player i improved its cost. Recall that given two vertices u, v on path $\bar{Q}$ we denote by $\bar{Q}_{u,v}$ the subpath of $\bar{Q}$ from u to v.

Before the deviation of the coalition Γ, a path from s to y was fully paid for by the players $j < i$. Next we show that no player $k > i$ pays for any edge on any path from s to t_i. Consider a player $k > i$ and let $Q'_k = Q^k \cup Q''_k$, where Q''_k is a path connecting t_k to t. Let y_k be the intersecting vertex of Q'_k and t_i. Since there exists a path from s to y_k that was fully paid for by players $j < k$ before the deviation, in particularly the path Q^i_{s,y_k}, player k will not pay for any edge on any path connecting s and y_k. Therefore player i fully pays for all edges on the path $\bar{Q}^i_{y,t_i}$, i.e., $\bar{p}_i(e) = c_e$ for all edges $e \in \bar{Q}^i_{y,t_i}$. Now consider the algorithm COMPUTE-SE at the step when player i selects a shortest path from the source s to its sink t_i and determines its payment p_i. At this point, player i could have bought the path $\bar{Q}^i_{y,t_i}$, since a path from s to y was already paid for by players $j < i$. Hence, $c_i(\bar{p}) \geqslant c_i(p)$. This contradicts the fact that player i improved its cost and therefore not all the players in Γ reduced their cost. This implies that p is a strong equilibrium. $\square$

4.2. Strong price of anarchy

Anshelevich et al. (2003) showed that the price of anarchy can be as large as n, even for two parallel edges. Here, we show that in contrast to Nash equilibrium, any strong equilibrium in general connection games yields the optimal cost.

Theorem 4.4 considers the case of single-source games, and Theorem 4.5 considers any general connection game (i.e., multi-commodity). Although the former is a special case of the latter, we present the two results separately due to the added value of each proof technique.

Theorem 4.4. *In single source general connection game, if there exists a strong equilibrium, then the strong price of anarchy is* 1 (*i.e., it is an optimal solution*).

Proof. Let $p = (p_1, \dots, p_n)$ be a strong equilibrium, and let T^* be the minimum cost Steiner tree of all players, rooted at the (single) source s. Let T_e^* be the subtree of T^* disconnected from s when edge e is removed, and let $\Gamma(T_e)$ be the set of players which have sinks in T_e and have non-zero payments (i.e., for any player i, $c_i(p) > 0$). For a set of edges E, let $c(E) = \sum_{e \in E} c_e$. Finally, let $P(T_e) = \sum_{i \in \Gamma(T_e)} c_i(p)$. Assume by way of contradiction that $c(p) > c(T^*)$. We will show that there exists a sub-tree T' of T^*, that connects a subset of players $\Gamma \subseteq N$, and a new set of payments $\bar{p}$, such that for each $i \in \Gamma$, $c_i(\bar{p}) < c_i(p)$. This will contradict the assumption that p is a strong equilibrium.

First we show how to find a sub-tree T' of T^*, such that for any edge $e \in T'$, the payments of players with sinks in T'_e is strictly more than the cost of $T'_e \cup \{e\}$. To build T', define an edge e to be *bad* if the cost of $T_e^* \cup \{e\}$ is at least the payments of the players with sinks in T_e^*, i.e., $c(T_e^* \cup \{e\}) \geqslant P(T_e^*)$. Let B be the set of bad edges. We define T' to be $T^* - \bigcup_{e \in B}(T_e^* \cup \{e\})$. Note that we can find a subset B' of B such that $\bigcup_{e \in B}(T_e^* \cup \{e\})$ is equal to $\bigcup_{e \in B'}(T_e^* \cup \{e\})$ and for any $e_1, e_2 \in B'$ we have $T_{e_1}^* \cap T_{e_2}^* = \emptyset$. (The set B' will include any edge $e \in B$ for which there is no other edge $e' \in B$ on the path from e to the source s.) Considering the edges in $e \in B'$ we can see that any subtree T_e^* we delete from T^* cannot decrease the difference between the payments and the cost of the remaining tree. Therefore, for every edge e in T', it holds that $c(T'_e \cup \{e\}) < P(T'_e)$.

Now we have a tree T' and our coalition will be $\Gamma(T')$. It just remains to find payments $\bar{p}$ for the players in $\Gamma(T')$ such that they will buy the tree T' and every player in $\Gamma(T')$ will lower its cost, i.e., $c_i(p) > c_i(\bar{p})$ for $i \in \Gamma(T')$. (Recall that the payments have the restriction that player i can only pay for edges on the path from s to t_i.)

We will now define the coalition payments $\bar{p}$. Let $c_i(\bar{p}, T'_e) = \sum_{e' \in T'_e} \bar{p}_i(e')$ be the payments of player i for the subtree T'_e. We will show that for every subtree T'_e, $c_i(\bar{p}, T'_e \cup \{e\}) < c_i(p)$, and hence $c_i(\bar{p}) < c_i(p)$. Consider the following bottom up process that defines $\bar{p}$. We assign the payments of edge e in T', after we assign payments to all the edges in T'_e. This implies that when we assign payments for e, we have that the sum of the payments in T'_e is equal to $c(T'_e) = \sum_{i \in \Gamma(T'_e)} c_i(\bar{p}, T'_e)$. Since e was not a bad edge, we know that $c(T'_e \cup \{e\}) = c(T'_e) + c_e < P(T'_e)$. Therefore, we can update the payments $\bar{p}$ of players $i \in \Gamma(T'_e)$, by setting $\bar{p}_i(e) = c_e \frac{\Delta_i}{\sum_{j \in \Gamma(T'_e)} \Delta_j}$, where $\Delta_j = c_j(p) - c_j(\bar{p}, T'_e)$. After the update we have for player $i \in \Gamma(T'_e)$,

$$\begin{aligned} c_i(\bar{p}, T'_e \cup \{e\}) &= c_i(\bar{p}, T'_e) + \bar{p}_i(e) \\ &= c_i(\bar{p}, T'_e) + \Delta_i \frac{c_e}{\sum_{j \in \Gamma(T'_e)} \Delta_j} \\ &= c_i(p) - \Delta_i \left(1 - \frac{c_e}{P(\Gamma(T'_e)) - c(T'_e)}\right), \end{aligned}$$

where we used the fact that $\sum_{j \in \Gamma(T'_e)} \Delta_j = P(\Gamma(T'_e)) - c(T'_e)$. Since $c_e < P(\Gamma(T'_e)) - c(T'_e)$ it follows that $c_i(\bar{p}, T'_e \cup \{e\}) < c_i(p)$. $\square$

We now present a stronger result, showing that the above result holds for multi-commodity games as well.

Theorem 4.5. *In any multi-commodity general connection game, if there exists a strong equilibrium, then the strong price of anarchy is* 1 (*i.e., it is an optimal solution*).

Proof. Let $p = (p_1, \dots, p_n)$ be a strong equilibrium, and suppose by contradiction that $cost_\Lambda(p) > OPT(\Lambda)$. For every player i and edge e, let $p'_i(e) = p_i(e) \cdot \frac{OPT(\Lambda)}{cost_\Lambda(p)}$. We slightly abuse notation and denote by $c_i(p) = \sum_{e \in E} p_i(e)$ the total payment of player i under the profile p (even if s_i is not connected to t_i). By the assumption that $cost_\Lambda(p) > OPT(\Lambda)$, $p'_i(e) < p_i(e)$ for every $i \in N$ and $\sum_i c_i(p') = OPT(\Lambda)$. We will reach a contradiction to p being a SE.

Let G denote the graph of the optimum solution, and fix a path P_i in it for each player $i \in N$. A payment vector p is said to be *valid* for G if $\forall i, e,\ p_i(e) > 0 \Rightarrow e \in P_i$ (i.e., each player pays only for edges in his path). Let $\mathcal{P}$ be the set of all payment vectors that buy G s.t. each player i pays $c_i(p')$. All the payment vectors in $\mathcal{P}$ are invalid by the assumption that p is a SE (otherwise, all the players can deviate to G and pay less than in p, contradiction). Let p'' be a payment vector in $\mathcal{P}$ that minimizes the sum of invalid payments.

Given G and p'', we construct a directed graph $\bar{G} = (\bar{V}, \bar{E})$, in which $\bar{V} = N$, and $\bar{E} = \{(i, j) \mid \exists e \in P_i \text{ s.t. } p''_j(e) > 0\}$. That is, the nodes of $\bar{G}$ are the players, and there is an edge from i to j in $\bar{G}$ if there exists an edge in i's path in G for which j pays a positive amount. An edge $(i, j) \in \bar{G}$ is said to be *bad* if $\exists e \in P_i$ s.t. $e \notin P_j$, and $p''_j(e) > 0$. Since p'' is invalid, $\bar{G}$ must have at least one bad edge.

Let us look at the graph G' representing the maximal strongly connected components (MSCC) of $\bar{G}$ (i.e., every MSCC in $\bar{G}$ is represented by a node in G'). G' is acyclic, and therefore has a sink node (according to the topological order) with only incoming edges, call it Γ (we denote by Γ both the MSCC and the set of players it represents). We will show that this component represents the deviating coalition. First, since it has no outgoing edges, no player outside of Γ pays for edges of players in Γ. It is thus left to show that there is no bad edge in Γ.

Suppose in contradiction there is a bad edge (i, j) in Γ. Since Γ is a MSCC, there must exist a path from node j to node i, thus a cycle containing edge (i, j). For every edge (k, ℓ) in this cycle, there exists an edge $e \in P_k$ s.t. $p''_\ell(e) > 0$. But for every such edge, we can update the payments of players k and ℓ to be: $p'''_\ell(e) = p''_\ell(e) - \epsilon$, and $p'''_k(e) = p''_k(e) + \epsilon$ for some small ϵ. For every node v in the cycle, $c_v(p''') = c_v(p'')$, but the sum of invalid payments decreased. This is in contradiction to the minimality of p''.

Thus, the players in Γ buy their edges in G using their valid payments in p'', and $c_i(p'') \leqslant c_i(p') < c_i(p)$, contradicting p is a SE. □

Appendix A

A.1. Proof of Lemma 2.1

Lemma 2.1 is a direct corollary of the following lemma.

In the lemma we use the following definition of *vertex disjoint* paths. Let (u_1, v_1) and (u_2, v_2) be two pairs of vertices, let P_1 be a path from u_1 to v_1 and let P_2 be a path from u_2 to v_2. We say that the paths P_1 and P_2 are *vertex disjoint*, if $P_1 \setminus \{u_1, v_1\}$ and $P_2 \setminus \{u_2, v_2\}$ have no common vertice.

Lemma A.1. *Let G be an SPG with source s and sink t. For any path, Q, from s to t, and any pair of vertices (s', t'), one of the following holds*:

(1) *Any path from s' to t' and the path Q are vertex disjoint.*
(2) *There exist two vertices $x, y \in Q$, such that for any path Q' from s' to t', $Q'_{s',x}$ and Q are vertex disjoint, and similarly, $Q'_{y,t'}$ and Q are vertex disjoint.* (*We call the vertices x, y the* intersecting vertices with respect to Q.)

Proof. The proof requires the following definition of embedding and lemma which appear in Milchtaich (2006a) for undirected graphs. We start with the definition.

Definition A.2. A graph G' is *embedded* in a graph G if G can be obtained from G' by applying the following operations: (1) *subdivision of an edge*: replaces an edge (u, v) by two edges with a single common new vertex, i.e., (u, w) and (w, v) where w is a new vertex; (2) *addition of a new edge* joining two existing vertices; (3) *extension composition*.

Note that if G' is embedded in a sub-graph of G, it is also embedded in G. The graph in Fig. 5 is called a *Braess graph*. The proof requires the following lemma.

Lemma A.3. *(See (Milchtaich, 2006a).) A graph G is an SPG if and only if the Braess graph is not embedded in G.*

It follows from the lemma that the SPG G does not embed a Braess graph when viewed as an undirected graph, since a (directed) SPG is an SPG when viewed as an undirected graph.

Consider a path Q from s, the source of G, to t, its sink. If every path from s' to t' is vertex disjoint to Q we are done (since condition (1) of the lemma holds). Otherwise, consider a path Q' from s' to t' that is not vertex disjoint

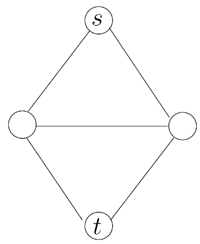

Fig. 5. Braess graph.

to Q. Let x be the first common vertex and y the last common vertex (note that Q' contains both x and y). Then, $Q'_{s',x}$ and Q are vertex disjoint, and similarly, $Q'_{y,t'}$ and Q are vertex disjoint. Consider an additional path Q'' that connects s' and t', and let $\bar{Q}''$ be its extension to a path connecting s and t.

If Q'' is vertex disjoint to Q, and $s' \notin Q$ or $t' \notin Q$, then the undirected graph induced by $H = \bar{Q}'' \cup Q'_{s',x} \cup Q'_{y,t'} \cup Q$ embeds a Braess graph. Otherwise, let z be the first common vertex and w the last common vertex. Then, $Q''_{s',z}$ and Q are vertex disjoint, and similarly, $Q''_{w,t'}$ and Q are vertex disjoint. Assume by contradiction that $\{x, y\} \neq \{z, w\}$ (note that if $\{x, y\} = \{z, w\}$, then condition (2) of the lemma holds). If $x \neq z$ then the undirected graph induced by $H = \bar{Q}''_{s,z} \cup Q'_{s',x} \cup Q$ embeds a Braess graph. Otherwise, $y \neq w$, the undirected graph induced by $H = Q'_{y,t'} \cup \bar{Q}''_{w,t} \cup Q$ embeds a Braess graph. □

A.2. *Proof of Lemma 3.2*

For the proof of Lemma 3.2 we will use the following notation. Given a joint action R, we denote by $Q^i(R)$ the path of player i in the profile R (note that $Q^i(R)$ is essentially R_i). Recall that, given two vertices u, v on the path $Q^i(R)$, $Q^i_{u,v}(R)$ is the subpath of $Q^i(R)$ from u to v and we denote by $c^i_{u,v}(R)$ the cost of player i on the subpath $Q^i_{u,v}(R)$.

Let $\texttt{MIN-COST}_i$ be the set of profiles that minimize the cost of player i among all profiles and let $\texttt{min-cost}_i$ be their cost to player i. That is, $\texttt{min-cost}_i = \min_{T\in\Sigma} c_i(T)$ and $\texttt{MIN-COST}_i = \{T \mid c_i(T) = \texttt{min-cost}_i\}$.

Recall that S is a strong equilibrium that minimizes the cost of player i, i.e., $S = \arg\min_{T\in SE} c_i(T)$. Let $\bar{S}$ be a profile in $\texttt{MIN-COST}_i$ that is obtained from S by a deviation of a coalition with a minimal number of players, and if one of the minimal coalitions contains player i, we choose that profile. Let Γ be the set of players deviating. (Note that given $T \in \texttt{MIN-COST}_i$ and S, the coalition to deviate from S to T is $\Gamma_{S,T} = \{j\colon S_j \neq T_j\}$.)

For any $j \in \Gamma$, let x_j, y_j denote the two intersecting vertices of s_j and t_j with respect to $Q^i(\bar{S})$ (whose existence is guaranteed by Lemma A.1). Note that such intersecting vertices exist for any player $j \in \Gamma$ (otherwise, we can remove from the coalition Γ players that do not have intersecting vertices, as they do not affect player i's cost, contradicting the minimality of the coalition Γ).

Let S^* be the profile obtained from S by the coalition Γ, when each player $j \in \Gamma$ deviates to $Q^j(S^*) = Q^j_{s_j,x_j}(S) \cup Q^i_{x_j,y_j}(\bar{S}) \cup Q^j_{y_j,t_j}(S)$. First we argue that S^* is optimal for player i.

Claim A.4. *Profile S^* minimizes the cost of player i, i.e., $S^* \in \texttt{MIN-COST}_i$.*

Proof. For every player $j \in \Gamma$ we have $Q^j_{x_j,y_j}(S^*) = Q^i_{x_j,y_j}(\bar{S}) \supseteq Q^j_{x_j,y_j}(\bar{S}) \cap Q^i_{x_j,y_j}(\bar{S})$. Therefore $Q^j(S^*) \cap Q^i(S^*) = Q^i_{x_j,y_j}(\bar{S}) \supseteq Q^j(\bar{S}) \cap Q^i(S^*)$. This implies that $c_i(S^*) \leqslant c_i(\bar{S})$, and since $c_i(\bar{S}) = \texttt{min-cost}_i$ the claim follows. □

Now we assume by way of contradiction that $c_i(S^*) = \texttt{min-cost}_i < c_i(S)$. There are two cases.

Case a: If for every player $j \in \Gamma$, $c_j(S^*) < c_j(S)$, it contradicts that S is a SE.

Case b: Otherwise, there exists a player j such that $c_j(S^*) \geqslant c_j(S)$. Our main goal is to show that one can construct a new coalition of a smaller size or a minimal coalition containing player i, where player $i \notin \Gamma$, whose deviation will lead to a minimal cost path for player i. In both cases we reach a contradiction to our choice of the coalition Γ.

Let $Q' = Q^i_{s,x_j}(S^*) \cup Q^j_{x_j,y_j}(S) \cup Q^i_{y_j,t}(S^*)$. Note that Q' is a path connecting the source s to the sink t and utilizing $Q^j_{x_j,y_j}(S)$. Let Γ' be the set of players $k \in \Gamma$ for which there exists a path that edge intersects $Q^j_{x_j,y_j}(S)$, and let x'_k, y'_k denote the two intersecting vertices w.r.t. Q' (whose existence is guaranteed by Lemma A.1). For any player $k \in \Gamma'$, we define x''_k (respectively, y''_k) to be the latter (former) of the two vertices x_j and x'_k (y_j and y'_k) on Q'.

Consider the profile S', obtained as follows. For any player $k \in \Gamma'$, $Q^k(S') = Q^k_{s_k,x''_k}(S^*) \cup Q^j_{x''_k,y''_k}(S) \cup Q^k_{y''_k,t_k}(S^*)$, and for any player $k \notin \Gamma'$, $Q^k(S') = Q^k(S^*)$. Note that we are guarantee that there is at least one edge in $Q^j_{x''_k,y''_k}(S)$. Note that $Q^j(S') = Q^j(S)$ and therefore player j is not part of the coalition required to deviate from S to S'. If we can show that $S' \in \texttt{MIN-COST}_i$ we will reach a contradiction to the minimality of the coalition Γ that deviated from S to some profile in $\texttt{MIN-COST}_i$. The following two lemmas would be required to complete our proof.

Lemma A.5. *It holds that* $c^j_{x_j,y_j}(S') \leqslant c^j_{x_j,y_j}(S)$.

Proof. It is sufficient to prove that for every player $k \in \Gamma'$

$$Q^k(S) \cap Q^j_{x_j,y_j}(S) \subseteq Q^k(S') \cap Q^j_{x_j,y_j}(S). \tag{A.1}$$

There are four cases:

Case a: $x_j \preccurlyeq x'_k$, and $y'_k \preccurlyeq y_j$. Then, the following hold:

$$Q^k_{s_k,x'_k}(S') \cap Q^j(S) = \emptyset,$$
$$Q^k_{y'_k,t_k}(S') \cap Q^j(S) = \emptyset,$$
$$Q^k_{x'_k,y'_k}(S') = Q^j_{x'_k,y'_k}(S),$$

and Eq. (A.1) follows.

Case b: $x'_k \preccurlyeq x_j$ and $y_j \preccurlyeq y'_k$. Then, it holds that $Q^k_{x_j,y_j}(S') = Q^j_{x_j,y_j}(S)$, and Eq. (A.1) follows.

Case c: $x'_k \preccurlyeq x_j$ and $y'_k \preccurlyeq y_j$. Then, it holds that:

$$Q^k_{x_j,y'_k}(S') = Q^j_{x_j,y'_k}(S),$$
$$Q^k_{y'_k,t_k}(S') \cap Q^j(S) = \emptyset,$$

and Eq. (A.1) follows.

Case d: $x_j \preccurlyeq x'_k$, and $y_j \preccurlyeq y'_k$. This case is symmetric to case c. Hence, Eq. (A.1) follows. □

Lemma A.6. *It holds that*

$$c^j_{x_j,y_j}(S) \leqslant c^j_{x_j,y_j}(S^*). \tag{A.2}$$

Proof. Since $c_j(S) \leqslant c_j(S^*)$, it is sufficient to show that

$$c^j_{s_j,x_j}(S) = c^j_{s_j,x_j}(S^*),$$
$$c^j_{y_j,t_j}(S) = c^j_{y_j,t_j}(S^*).$$

For every player $k \in \Gamma$, if there exists a vertex $v_{k,j}$ which is reachable from both s_j and s_k, and x_j is reachable from $v_{k,j}$, then $x_k = x_j$, and from the construction of the profile S^* it follows that $Q^k_{s_k,x_k}(S^*) = Q^k_{s_k,x_k}(S)$. Otherwise, no path of k can intersect a path from s_j to x_j, and therefore k's action has no influence on the cost of this subpath. Therefore $c^j_{s_j,x_j}(S) = c^j_{s_j,x_j}(S^*)$. Similarly we obtain $c^j_{y_j,t_j}(S) = c^j_{y_j,t_j}(S^*)$. □

To complete the proof we have two cases, depending on whether player i is in Γ.

Case b1: Assume $Q^i_{x_j,y_j}(S') = Q^j_{x_j,y_j}(S')$. (This holds if either player $i \in \Gamma$ or $Q^i_{x_j,y_j}(S) = Q^j_{x_j,y_j}(S)$ in S.) In this case we have

$$c^i_{x_j,y_j}(S') = c^j_{x_j,y_j}(S') \leqslant c^j_{x_j,y_j}(S) \leqslant c^j_{x_j,y_j}(S^*) = c^i_{x_j,y_j}(S^*),$$

where the first equality follows from the construction of S', the first inequality follows from Lemma A.5, the second inequality follows from Lemma A.6 and the second equality follows from the construction of S^*. Thus, $c^i(S') \leqslant c^i(S^*)$, and hence $S' \in \texttt{MIN-COST}_i$. This is in contradiction to the minimality of the coalition Γ size that deviates from S to some profile in $\texttt{MIN-COST}_i$.

Case b2: Assume that $Q^i_{x_j,y_j}(S') \neq Q^j_{x_j,y_j}(S')$. Thus, player $i \notin \Gamma$. Here we will reach a contradiction to the fact that player $i \notin \Gamma$.

Consider the profile S'' that is obtained from S' by the following deviation of player i: $Q^i(S'') = Q^i_{s,x_j}(S') \cup Q^j_{x_j,y_j}(S') \cup Q^i_{y_j,t}(S')$, i.e., player i deviates to the subpath $Q^j_{x_j,y_j}(S')$ from x_j to y_j. Then, we get

$$c^i_{x_j,y_j}(S'') = c^j_{x_j,y_j}(S'') \leqslant c^j_{x_j,y_j}(S') \leqslant c^j_{x_j,y_j}(S) \leqslant c^j_{x_j,y_j}(S^*) = c^i_{x_j,y_j}(S^*),$$

where the first equality follows from the construction of S'', the first inequality follows from the fact that player i joins the path $Q^j_{x_j,y_j}(S')$, the second inequality follows from Lemma A.5, the third inequality follows from Lemma A.6 and the last equality follows from the construction of S^*. Thus, we get that $c^i(S'') \leqslant c^i(S^*)$, where the profile S'' is obtained by deviation of the minimal coalition $\Gamma \setminus \{j\} \cup \{i\}$. Contradicting the fact that player i is not in the minimal coalition Γ.

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