Truth, Envy, and Truthful Market Clearing Bundle Pricing

Edith Cohen[1], Michal Feldman[2], Amos Fiat[3], Haim Kaplan[3], and Svetlana Olonetsky[3]

[1] AT&T Labs-Research, 180 Park Avenue, Florham Park, NJ
[2] School of Business Administration, The Hebrew University of Jerusalem
[3] The Blavatnik School of Computer Science, Tel Aviv University

Abstract. We give a non-trivial class of valuation functions for which we give auctions that are efficient, truthful and envy-free.
We give interesting classes of valuations for which one can design such auctions. Surprisingly, we also show that minor modifications to these valuations lead to impossibility results, the most surprising of which is that for a natural class of valuations, one cannot achieve efficiency, truthfulness, envy freeness, individual rationality, and no positive transfers.
We also show that such auctions also imply a truthful mechanism for computing bundle prices ("shrink wrapped" bundles of items), that clear the market. This extends the class of valuations for which truthful market clearing prices mechanisms exist.

1 Introduction

In this paper we consider auctions that are

1. Efficient — the mechanism maximizes the sum of the valuations of the agents. Alternately, efficient mechanisms are said to maximize social welfare.
2. Incentive compatible (IC) — it is a dominant strategy for agents to report their private information [11].
3. Envy-free (EF) - no agent wishes to exchange her outcome with that of another [6, 7, 22, 15, 16, 24].
4. Make no positive transfers (NPT)— the payments of all agents are non-negative.
5. Individually rational (IR) — no agent gets negative utility.

We argue that such auctions are natural and interesting for a variety of reasons:

- It is not clear how to obtain efficiency without truthfullness.
- An auction that is not envy-free discriminates between bidders, moreover — it is relatively easy for bidders to realize that they are being discriminated against. Experiments suggest that human subjects prefer degraded performance over discrimination, e.g., ([21]).
- Posted prices that clear the market are inherently envy-free, but computing such Walrasian pricing, even if it exists, is itself not necessarily truthful. Auctions that are both truthful and envy-free can be interpreted as a mechanism for computing

market clearing prices where the posted price is associated with bundles of items rather than individual items. We present a natural subset of gross substitute valuations for which we show that
  - There is no incentive compatible mechanism to computes Walrasian prices.
  - We give a truthful mechanism for computing market clearing *bundle* prices.
- Auctions that are not individually rational or make positive transfers represent situations when one forces the bidders to participate against their will or subsidizes the auction. That said, we show a class of valuation functions (capacitated valuations with unequal capacities), for which Walrasian prices exist, but no auction exists that is incentive compatible, envy-free, makes no positive transfers, and is individually rational. Moreover, for a subset of this class, we give an efficient, incentive compatible, envy-free and individually rational auction (albeit — with positive transfers).

We consider a specific class of additive valuations where agents have a limit on the number of goods they may receive. We refer to such valuations as *capacitated* valuations and seek mechanisms that maximize social welfare and are simultaneously incentive compatible, envy-free, individually rational, and have no positive transfers. Capacitated valuations are a special case of gross substitute valuations (Kelso and Crawford [12]). Capacitated valuations are a natural generalization of the unit demand valuation. One may view the capacity of an agent as the size of the market basket, an agent with capacity c may carry no more than c items. If such a bidder gets more than c items, (say, in a shrink wrapped bundle containing $2c$ goods, of various types), she can discard any excess items.

If capacities are infinite, then sequentially repeating the 2nd price Vickrey auction meets these requirements. In 1983, Leonard showed that for unit capacities, VCG with Clarke pivot payments is also envy-free[4]. In this paper we consider generalizations of the setting considered by Leonard. For homogeneous capacities (all capacities equal) we show that VCG with Clarke pivot payments is envy-free (VCG with Clarke pivot payments is always efficient, incentive compatible, individually rational, and has no positive transfers). Also, we show that there is no incentive compatible mechanism to compute Walrasian prices for capacities > 1. For heterogeneous capacities, we show that there is no mechanism with all 5 properties, but at least in some cases, one can achieve both incentive compatibility and envy freeness.

Let $[s] = \{1, \dots, s\}$ of s be the set of goods to be allocated amongst n agents with private valuations. An agent's valuation function is a mapping from every subset of the goods into the non negative reals. A *mechanism* receives the valuations of the agents as input, and determines an allocation a_i and a payment p_i for every agent. We assume that agents have quasi-linear utilities; that is, the utility of agent i is the difference between her valuation for the bundle allocated to her and her payment.

Given an efficient, incentive compatible, envy-free, auction, with no positive transfers and individually rational, we can convert the allocations a_i and associated prices p_i into market clearing bundle prices. As the auction is efficient, we can assume that all items are allocated. (In capacitated valuations, there may be later discarded simply because the agents do not have the capacity to accept them). For every agent i we create a

[4] Lehmann, Lehmann, and Nisam, [13], show that computing VCG in the case of gross substitutes is poly time.

bundle of all items in a_i, and attach the price p_i to this shrink-wrapped bundle of goods. Of all these bundles, the bundle a_i, and it's associated price, p_i, maximize the utility for agent z, the bundle a_j and it's associated price, p_j, maximize the utility for agent j, etc.

Most of our results concern the class of capacitated valuations: every agent i has an associated capacity c_i, and her value is additive up to the capacity, *i.e.*, for every set $S \subseteq [s]$,

$$v_i(S) = \max \left\{ \sum_{j \in T} v_i(j) \middle| T \subseteq S, |T| = c_i \right\},$$

where $v_i(j)$ denotes the agent i's valuation for good j.

Consider the following classes of valuation functions:

1. Gross substitutes: good x is said to be a gross substitute of good y if the demand for x is monotonically non-decreasing with the price of y, *i.e.*,

$$\partial(\text{demand } x)/\partial(\text{price } y) \geq 0 .$$

   A valuation function is said to obey the gross substitutes condition if for every pair of goods x and y, good x is a gross substitute of good y.
2. Subadditive valuations: A valuation $v : 2^{[s]} \to \mathbb{R}_{\geq 0}$ is said to be subadditive if for every two disjoint subsets $S, T \subseteq [s]$, $v(S) + v(T) \geq v(S \cup T)$.
3. Superadditive valuations: A valuation $v : 2^{[s]} \to \mathbb{R}_{\geq 0}$ is said to be superadditive if for every two disjoint subsets $S, T \subseteq [s]$, $v(S) + v(T) \leq v(S \cup T)$.

Capacitated valuations are a subset of gross substitutes, which are themselves a subset of subadditive valuations.

In a Walrasian equilibrium (See [12, 10]), prices are *item prices*, that is, prices are assigned to *individual goods* so that every agent chooses a bundle that maximizes her utility and the market clears. Thus, Walrasian prices automatically lead to an envy-free allocation. Every Walrasian pricing gives a mechanism that is efficient and envy-free, has no positive transfers, and is individually rational [13].

We remark that while Walrasian pricing ⇒ EF, NPT, IR, the converse is not true. Even a mechanism that is EF, NPT, IR, *and* IC does not imply Walrasian prices. Note that envy-free prices may be assigned to bundles of goods which cannot necessarily be interpreted as item prices. It is well known that in many economic settings, bundle prices are more powerful than item prices [1, 19]. [12] showed that gross substitutes imply the existence of Walrasian equilibrium, Gul and Stacchetti [10] show that this is necessarily the case.

As capacitated valuations are also gross substitutes (see Theorem 2.4 in Section 2.2), it follows that capacitated valuations always have a Walrasian equilibrium. Walrasian prices, however, may not be incentive compatible. In fact, we show (Proposition 3.1) that even with 2 agents with capacities 2 and 3 goods, there is *no incentive compatible* mechanism that produces Walrasian prices.

For superadditive valuations it is known that Walrasian equilibrium may not exist. Pápai [18] has characterized the family of mechanisms that are simultaneously EF and IC under superadditive valuations. In particular, VCG with Clarke pivot payments satisfies these conditions. However, Pápai's result for superadditive valuations does not hold for subadditive valuations. Moreover, Clarke pivot payments do not satisfy envy freeness even for the more restricted family of capacitated valuations, as demonstrated in the following example:

Example 1.1. Consider an allocation problem with two agents, $\{1, 2\}$, and two goods, $\{a, b\}$. Agent 1 has capacity $c_1 = 1$ and valuation $v_1(a) = v_1(b) = 2$, and agent 2 has capacity $c_2 = 2$ and valuation $v_2(a) = 1$, $v_2(b) = 2$. According to VCG with Clarke pivot payments, agent 1 is given a and pays 1, while agent 2 is given b and pays nothing (as he imposes no externality on agent 1). Agent 1 would rather switch with agent 2's allocation and payment (in which case, her utility grows by 1), therefore, the mechanism is not envy-free.

Two extremal cases of capacitated valuations are "no capacity constraints", or, all capacities are equal to one. If capacities are infinite, running a Vickrey 2nd price auction [23] for every good, independently, meets all requirements (IC + Walrasian $\Rightarrow$ efficient, IC, EF, NPT, IR). If all agent capacities are one, [14] shows that VCG with Clarke pivot payments is envy-free, and it is easy to see that it also meets the stronger notion of an incentive compatible Walrasian equilibrium. For arbitrary capacities (not only all ∞ or all ones), we distinguish between *homogeneous* capacities, where all agent capacities are equal, and *heterogeneous* capacities, where agent capacities are arbitrary.

When considering incentive compatible and heterogeneous capacities, we distinguish between capacitated valuations with *public* or *private* capacities: being incentive compatible with respect to private capacities and valuation is a more difficult task than incentive compatible with respect to valuation, where capacities are public. In this paper, we primarily consider public capacities.

The main technical results of this paper (which are also summarized in Figure 1) are as follows:

- For arbitrary homogeneous capacities c, such that $(c \equiv c_1 = c_2 = \cdots = c_n)$:
  - VCG with Clarke pivot payments is efficient, IC, NPT, IR, and EF.
  - However, there is no incentive compatible mechanism that produces Walrasian prices, even for $c = 2$.
- For arbitrary heterogeneous capacities $c = (c_1, c_2, \ldots, c_n)$:
  - Under the VCG mechanism with Clarke pivot payments (public capacities), a higher capacity agent will never envy a lower capacity agent.
  - In the full version we also show that
    * There is no mechanism that is IC, NPT, and EF (for public and hence also for private capacities).
    * 2 agents, public capacities - there exist mechanisms that are IC, IR, and EF.
    * 2 agents, 2 goods - there exist mechanisms that are IC, IR, and EF for every subadditive valuation.

	Subadditive	Gross substitutes	capacitated - heterogeneous	capacitated - homogeneous
Walras.	NO [10]	YES [10]	(→) YES	(→) YES
Walras.+IC	NO (←)	NO (←)	NO (←)	**NO** (Proposition 3.1)
EF + IC	? **YES*** for $m = 2, n = 2$	? (→) **YES** for $m = 2, n = 2$	? **YES*** for $m = 2$	YES (↑)
EF + IC + NPT	NO (←)	NO (←)	**NO***	**YES***

Fig. 1. This table specifies the existence of a particular type of mechanism (rows) for various families of valuation functions (columns). Efficiency is required in all entries. The valuation families satisfy capacitated homogeneous ⊂ capacitated heterogeneous ⊂ gross substitutes ⊂ subadditive. Wherever results are implied from other table entries, this is specified with corresponding arrows. We note that for the family of additive valuations (no capacities), all entries are positive, as the Clarke pivot mechanism satisfies all properties. * Appears in full version only.

2 Model and Preliminaries

Let $[s] = \{1, \ldots, s\}$ be a set of goods to be allocated to a set $[n] = \{1, \ldots, n\}$ of agents.

An allocation $a = (a_1, a_2, \ldots, a_n)$ assigns agent i the bundle $a_i \subseteq [s]$ and is such that $\bigcup_i a_i \subseteq [s]$ and $a_i \cap a_j = \emptyset$ for $i \neq j$.[5] We use $\mathcal{L}$ to denote the set of all possible allocations.

For $S \subseteq [s]$, let $v_i(S)$ be the valuation of agent i for set S. Let $v = (v_1, v_2, \ldots, v_n)$, where v_i is the valuation function for agent i.

Let V_i be the domain of all valuation functions for agent $i \in [n]$, and let $V = V_1 \times V_2 \times \cdots \times V_n$.

An allocation function $a : V$ maps $v \in V$ into an allocation

$$a(v) = (a_1(v), a_2(v), \ldots, a_n(v)) .$$

A payment function $p : V$ maps $v \in V$ to $\mathbb{R}^n_{\geq 0}$: $p(v) = (p_1(v), p_2(v), \ldots, p_n(v))$, where $p_i(v) \in \mathbb{R}_{\geq 0}$ is the payment of agent i. Payments are from the agent to the mechanism (if the payment is negative then this means that the transfer is from the mechanism to the agent).

A mechanism is a pair of functions, $\langle a, p\rangle$, where a is an allocation function, and p is a payment function. For a valuation v, the utility to agent i in a mechanism $\langle a, p\rangle$ is defined as $v_i(a_i(v)) - p_i(v)$. Such a utility function is known as quasi-linear.

For a valuation v, we define (v'_i, v_{-i}) to be the valuation obtained by substituting v'_i for v_i, i.e.,

$$(v'_i, v_{-i}) = (v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_n).$$

[5] Here we deal with indivisible goods, although our results also extend to divisible goods with appropriate modifications.

A mechanism is incentive compatible (IC) if for all i, v, and v'_i:

$$v_i(a_i(v)) - p_i(v) \geq v_i(a_i(v'_i, v_{-i})) - p_i(v'_i, v_{-i});$$

this holds if and only if

$$p_i(v) \leq p_i(v', v_{-i}) + \Big(v_i(a_i(v)) - v_i(a_i(v'_i, v_{-i}))\Big). \tag{1}$$

A mechanism is envy-free (EF) if for all $i, j \in [n]$ and all v:

$$v_i(a_i(v)) - p_i(v) \geq v_i(a_j(v)) - p_j(v);$$

this holds if and only if

$$p_i(v) \leq p_j(v) + \Big(v_i(a_i(v)) - v_i(a_j(v))\Big). \tag{2}$$

Given valuation functions $v = (v_1, v_2, \ldots, v_n)$, a social optimum Opt is an allocation that maximizes the sum of valuations

$$\mathrm{Opt} \in \arg\max_{a \in \mathcal{L}} \sum_{i=1}^{n} v_i(a_i) \ .$$

Likewise, the social optimum when agent i is missing, Opt^{-i}, is the allocation

$$\mathrm{Opt^{-i}} \in \arg\max_{a \in \mathcal{L}} \sum_{j \in [n] \setminus \{i\}} v_j(a_j) \ .$$

2.1 VCG mechanisms

A mechanism $\langle a, p \rangle$ is called a VCG mechanism [4, 23] if:

- $a(v) = \mathrm{Opt}$, and
- $p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(a_j(v))$, where h_i does not depend on v_i, $i \in [n]$.

For connected domains, the only efficient incentive compatible mechanism is VCG (See Theorem 9.37 in [17]). Since capacitated valuations induce a connected domain, we get the following proposition.

Proposition 2.1. *With capacitated valuations, a mechanism is efficient and IC if and only if it is VCG.*

VCG with *Clarke pivot payments* has

$$h_i(v_{-i}) = \max_{a \in \mathcal{L}} \sum_{j \neq i} v_j(a) \qquad (= \sum_{j \neq i} v_j(\mathrm{Opt}_j^{-i})).$$

Agent valuations for bundles of goods are non negative. The only mechanism that is efficient, incentive compatible, individually rational, and with no positive transfers is VCG with Clarke pivot payments.

The following proposition, which appears in [18], provides a criterion for the envy freeness of a VCG mechanism.

Proposition 2.2. *[18] Given a VCG mechanism, specified by functions $\{h_i\}_{i \in [n]}$, agent i does not envy agent j iff for every v,*

$$h_i(v_{-i}) - h_j(v_{-j}) \leq v_j(\mathrm{Opt}_j) - v_i(\mathrm{Opt}_j).$$

2.2 Gross substitutes and capacitated valuations

We define the notion of gross substitute valuations and show that every capacitated valuation (i.e., additive up to the capacity) has the gross substitutes property. As this discussion refers to a valuation function of a single agent, we omit the index of the agent.

Fix an agent and let $D(p)$ be the collection of all sets of goods that maximize utility for the agent under price vector p, $D(p) = \arg\max_{S \subseteq [s]}\{v(S) - \sum_{j \in S} p_j\}$.

Definition 2.3. *[10] A valuation function $v : 2^{[s]} \to \mathbb{R}_{\geq 0}$ satisfies the gross substitutes condition if the following holds: Let $p = (p_1, \ldots, p_s)$ and $q = (q_1, \ldots, q_s)$ be two price vectors such that the price for good j is no less under q than under p: i.e., $q_j \geq p_j$, for all j. Consider the set of all items whose price is the same under p and q, $E(p, q) = \{1 \leq j \leq s | p_j = q_j\}$, then for any $S^p \in D(p)$ there exists some $S^q \in D(q)$ such that $S^p \cap E(p, q) \subseteq S^q \cap E(p, q)$.*

Theorem 2.4. *Every capacitated valuation function (additive up to the capacity) obeys the gross substitutes condition.*

As a corollary, we get that capacitated valuations admit a Walrasian equilibrium. However, not necessarily within an IC mechanism.

3 Envy-Free and Incentive Compatible Assignments with Capacities

The main result of this section is that Clarke pivot payments are envy-free when capacities are homogeneous. This follows from a stronger result, which we establish for heterogeneous capacities, showing that with Clarke pivot payments, no agent envies a lower-capacity agent.

We first observe that one cannot aim for an incentive compatible mechanism with Walrasian prices (if this was possible then envy freeness would follow immediately).

3.1 No Incentive Compatible Walrasian pricing

	a	b	c
Agent 1	$1+\epsilon$	$1+\epsilon$	$1-\epsilon$
Agent 2	$1-\epsilon/2$	1	$1+\epsilon$

(a) Matrix v

	a	b	c
Agent 1	$1-\epsilon$	0	0
Agent 2	$1-\epsilon/2$	1	$1+\epsilon$

(b) Matrix v'

Fig. 2. IC Walrasian prices in v implies no IR in v'.

Proposition 3.1. *Capacitated valuations with homogeneous capacities $c \geq 2$ have no incentive compatible mechanism which produces Walrasian prices (See Figure 2). Details appear in full version.*

3.2 Truthful and Envy-Free Capacitated Allocations

The following theorem establishes a general result for capacitated valuations: in a VCG mechanism with Clarke pivot payments, no agent will ever envy a lower-capacity agent.

Theorem 3.2. *If we apply the VCG mechanism with Clarke pivot payments on the assignment problem with capacitated valuations, then*

- *The mechanism is incentive compatible, individually rational, and makes no positive transfers (follows from VCG with CPP).*
- *No agent of higher capacity envies an agent of lower or equal capacity.*

The input to the VCG mechanism consists of capacities and valuations. The agent capacity, $c_i \geq 0$ (the capacity of agent i), is publicly known. The number of units of good j, $q_j \geq 0$ is also public knowledge. The valuations $v_i(j)$ are private.

The b-Matching Graph Given capacities c_i, q_j, and a valuation matrix v, we construct an edge-weighted bipartite graph G as follows:

- We associate a vertex with every agent $i \in [n]$ on the left, let $\mathcal{A}$ be the set of these vertices.
- We associate a vertex with every good $j \in [s]$ on the right, let $\mathcal{I}$ be the set of these vertices.
- Edge (i, j), $i \in \mathcal{A}$, $j \in \mathcal{I}$, has weight $v_i(j)$.
- Vertex $i \in \mathcal{A}$ (associated with agent i) has *degree constraint* c_i.
- Vertex $j \in \mathcal{I}$ (associated with good j) has degree constraint q_j.

We seek an allocation a ($= a(v)$) where a_{ij} is the number of units of good j allocated to agent i. The value of the allocation is $v(a) = \sum_{ij} a_{ij} v_i(j)$. We seek an allocation of maximal value that meets the degree constraints: $\sum_j a_{ij} \leq c_i$, $\sum_i a_{ij} \leq q_j$, this is known as a b-matching problem and has an integral solution if all constraints are integral, see [20]. Let $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ denote the i'th row of a, which corresponds to the bundle allocated to agent i.

Let $v_k(a_i) = \sum_{j \in [s]} a_{ij} v_k(j)$ denote the value to agent k of bundle a_i. Let M denote some allocation that attains the maximal social value, $M \in \arg\max_a v(a)$. Finally, let G^{-i} be the graph derived from G by removing the vertex associated with agent i and all its incident edges, and let M^{-i} be a matching of maximal social value with agent i removed.

Specializing the Clarke pivot rule to our setting, the payment of agent k is

$$p_k = v(M^{-k}) - v(M) + v_k(M_k) \ . \tag{3}$$

In the special case of permutation games (the number of agents and goods is equal, and every agent can receive at most one good), the social optimum corresponds to a maximum weighted matching in G. Such "permutation games" were first studied by [14] who showed that Clarke pivot payments are envy-free. However, the shadow variables technique used in this proof does not seem to generalize for larger capacities.

Proof sketch of Theorem 3.2: Let agent 1 and agent 2 be two arbitrary agents such that $c_1 \geq c_2$. Agent 1 does not envy agent 2 if and only if

$$v_1(M_1) - p_1 \geq v_1(M_2) - p_2$$

By substituting the Clarke pivot payments (3) and rearranging, this is true if and only if

$$v(M^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2). \qquad (4)$$

Thus in order to prove the theorem we need to establish (4).

We construct a new allocation D^{-2} on G^{-2} (from the allocations M and M^{-1}) such that

$$v(D^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2)\ . \qquad (5)$$

From the optimality of M^{-2}, it must hold that $v(M^{-2}) \geq v(D^{-2})$. Combining this with (5) shall establish (4), as required.

In what follows we make several preparations for the construction of the allocation D^{-2}. Given M and M^{-1}, we construct a directed bipartite graph G_f on $\mathcal{A} \cup \mathcal{I}$ coupled with a flow f as follows. For every pair of vertices $i \in \mathcal{A}$ and $j \in \mathcal{I}$,

- If $M_{ij} - M^{-1}_{ij} > 0$, then G_f includes arc $i \to j$ with flow $f_{i \to j} = M_{ij} - M^{-1}_{ij}$.
- If $M_{ij} - M^{-1}_{ij} < 0$, then G_f includes arc $j \to i$ with flow $f_{j \to i} = M^{-1}_{ij} - M_{ij}$.
- If $M_{ij} = M^{-1}_{ij}$, then G_f contains neither arc $i \to j$ nor arc $j \to i$.

We define the *excess* of a vertex i in G_f, χ_i, to be the difference between the amount of flow flowing out of the vertex and the amount of flow flowing into the vertex. Clearly the sum of all excesses is zero. A vertex is said to be a *source* if its excess is positive, and said to be a *target* if its excess is negative.

Using the flow decomposition theorem, we can decompose the flow f into simple paths and cycles, where each path connects a source to a target. Associated with each path and cycle T is a positive flow value $f(T) > 0$. Given an arc $x \to y$, $f_{x \to y}$ is obtained by summing up the values $f(T)$ of all paths and cycles T that contain $x \to y$. Notice that $M^{-1}_{1j} = 0$ for all j and therefore $f_{1 \to j} \geq 0$ for all j. It follows that there are no arcs of the form $j \to 1$ in G_f. The following observation can be easily verified.

Observation 3.3 *For each path $P = u_1, u_2, \ldots, u_t$ in a flow decomposition of G_f, where u_1 is a source and u_t is a target, it holds that $f(P) \leq \min\{\chi_{u_1}, |\chi_{u_t}|\}$.*

We define the *value* of a path or a cycle $T = u_1, u_2, \ldots, u_t$ in G_f, to be

$$v(P) = \sum_{\substack{u_i \in \mathcal{A},\\ u_{i+1} \in \mathcal{I}}} v_{u_i}(u_{i+1}) - \sum_{\substack{u_i \in \mathcal{I},\\ u_{i+1} \in \mathcal{A}}} v_{u_{i+1}}(u_i).$$

It is easy to verify that $\sum_T f(T) \cdot v(T) = v(M) - v(M^{-1})$, where we sum over all paths and cycles T in our decomposition.

We will repeatedly do the following procedure: Let M, M^{-1}, f and G_f be as above.

The proofs of the following Lemmata are omitted.

Lemma 3.4. *Let $T = u_1, u_2, \ldots, u_t$ be a cycle in G_f or a path in the flow decomposition of G_f, and let ϵ be the minimal flow along any arc of T. We construct an allocation $\widehat{M}$ (= $\widehat{M}(T)$) from M by canceling the flow along T, start with $\widehat{M} = M$ and then for each $(u_i, u_{i+1}) \in T$ set:*

$$\widehat{M}_{u_i u_{i+1}} = M_{u_i u_{i+1}} - \epsilon \qquad u_i \in \mathcal{A}, u_{i+1} \in \mathcal{I}$$
$$\widehat{M}_{u_{i+1} u_i} = M_{u_{i+1} u_i} + \epsilon \qquad u_i \in \mathcal{I}, u_{i+1} \in \mathcal{A}\,.$$

Alternatively, we construct $\widehat{M}^{-1}$ (= $\widehat{M}^{-1}(T)$) from M^{-1}, starting from $\widehat{M}^{-1} = M^{-1}$ and then for each $(u_i, u_{i+1}) \in T$ set

$$\widehat{M}^{-1}_{u_i u_{i+1}} = M^{-1}_{u_i u_{i+1}} + \epsilon \qquad u_i \in \mathcal{A}, u_{i+1} \in \mathcal{I},$$
$$\widehat{M}^{-1}_{u_{i+1} u_i} = M^{-1}_{u_{i+1} u_i} - \epsilon \qquad u_i \in \mathcal{I}, u_{i+1} \in \mathcal{A}.$$

The allocations $\widehat{M}$, $\widehat{M}^{-1}$ are valid (do not violate capacity constraints).

The remainder of the proof requires several additional lemmata.

Lemma 3.5. *It is without loss of generality to assume that M^{-1} is such that*

1. *There are no cycles of zero value in G_f.*
2. *There is no path $P = u_1, u_2, \ldots, u_t$ of zero value such that $u_1 \neq 1$ is a source and u_t is a target.*

Thus, in the sequel we assume that M^{-1} satisfies conditions (1),(2) of Lemma 3.5[6].

Lemma 3.6. *The graph G_f does not contain a cycle.*

In particular, Lemma 3.6 implies that there are no cycles in our flow decomposition. We next show that the only source vertex in G_f is the vertex corresponding to agent 1.

Lemma 3.7. *The vertex that corresponds to agent 1 is the unique source vertex.*

Lemma 3.7 implies that all the paths in our flow decomposition originate at agent 1. We are now ready to describe the construction of the allocation D^{-2}:

1. Stage I: initially, $D^{-2} := M^{-1}$.
2. Stage II: for every good j, let $x = \min\{M_{2j}, M^{-1}_{2j}\}$, and set $D^{-2}_{2j} := M^{-1}_{2j} - x$ and $D^{-2}_{1j} := x$.
3. Stage III: for every flow path P in the flow decomposition of G_f that contains agent 2, let $\hat{P}$ be the prefix of P up to agent 2. For every agent to good arc $(i \to j) \in \hat{P}$ set $D^{-2}_{ij} := D^{-2}_{ij} + f(P)$, and for every good to agent arc $(j \to i) \in \hat{P}$ set $D^{-2}_{ij} := D^{-2}_{ij} - f(P)$.

[6] Since Inequality (5) depends only on the value of M^{-1} it does not matter which M^{-1} we work with.

It is easy to verify that D^{-2} indeed does not allocate any good to agent 2. Also, the allocation to agent 1 in D^{-2} is of the same size as the allocation to agent 2 in M^{-1}. Since $c_1 \geq c_2$, D^{-2} is a valid allocation.

To conclude the proof of Theorem 3.2 we now show that:

Lemma 3.8. *Allocation D^{-2} satisfies (5).*

The following is a direct corollary of Theorem 3.2.

Corollary 3.9. *If all agent capacities are equal, then the VCG allocation with Clarke pivot payments is EF.*

4 Discussion and open problems

This work initiates the study of efficient, incentive compatible, and envy-free mechanisms for capacitated valuations.

Our work suggests a host of problems for future research on heterogeneous capacitated valuations and generalizations thereof.

We know that, generally, there may be no mechanism that is both IC and EF even if we allow positive transfers [7].

We conclude by posing the following open problems, from the very concrete to the more general:

- Is there a k agent, $k > 2$, mechanism for heterogenous capacitated allocations, that is is efficient, IC, and EF ? We conjecture that such mechanisms do for any combinatorial auction with subadditive valuations. We know they exist for two agents and public capacities, and for subadditive valuations with two agents and two goods.
- We have focused on efficient mechanisms; *i.e.*, that maximize social welfare. A natural question is how well the optimal social welfare can be *approximated* by a mechanism that is IC, EF, and NPT.
- In [8], Fleischer and Wang consider lower bounds for envy-free and truthful mechanisms for makespan minimization in related machines. Ergo, one can ask these questions not only in the context of efficiency but also in other contexts. This is yet another step in the most general problem of all (see below).
- And, the most general problem of all: can one characterize the set of truthful and envy-free mechanisms? There have been some attempts, including a characterization due to the authors that generalizes Rochet's cyclic monotonicity characterization for truthfulness to a full characterization for truthfulness and envy freeness (See [5]). However, like cyclic monotonicity itself, this characterization is hardly satisfactory.

[7] As an example, consider a setting with two goods, a, b and three agents, where $v_1(a) = v_1(b) = v_1(\{a,b\})$, $v_2(a) = v_2(b) = v_2(\{a,b\})$, and $v_3(a) = v_3(b) = 0$, while $v_3(\{a,b\}) > 0$. One can easily verify that this setting has no mechanism that is simultaneously incentive compatible and envy-free. This example is due to Noam Nisan.

References

1. M. P. R. Ausubel, L. M. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1:1–42, 2002.
2. L. Blumrosen and N. Nisan. Combinatorial auctions. In E. Tardos V. Vazirani N. Nisan, T. Roughgarden, editor, *Algorithmic Game Theory*. Cambridge University Press, 2007.
3. L. Blumrosen and N. Nisan. Informational limitations of ascending combinatorial auctions. *Journal of Economic Theory*, 145:1203–1223, 2001.
4. E. Clarke. Multipart Pricing of Public Goods. *Public Choice*, 1:17–33, 1971.
5. E. Cohen, M. Feldman, A. Fiat, H. Kaplan and S. Olonetsky, On the Interplay between Incentive Compatibility and Envy Freeness. http://arxiv.org/abs/1003.5328
6. L.E. Dubins and E.H. Spanier. How to cut a cake fairly. *American Mathematical Monthly*, 68:1–17, 1961.
7. D. Foley. Resource allocation and the public sector. *Yale Economic Essays*, 7:45–98, 1967.
8. L. Fleischer and Z. Wang. Lower Bound for Envy-Free and Truthful Makespan Approximation on Related Machines. SAGT, 2011.
9. G. Demange, D. Gale and M. Sotomayor. Multi-Item Auctions. Journal of Political Economy,1986
10. F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87:95–124, 1999.
11. L. Hurwicz. Optimality and informational efficiency in resource allocation processes. In K.J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*, 27–46, 1960.
12. A. Kelso and V. Crawford. Job Matching, Coalition Formation, and Gross Substitutes. Econometrica, 1982.
13. Benny Lehmann, Daniel J. Lehmann and Noam Nisan. Combinatorial Auctions with Decreasing Marginal Utilities. ACM Conference on Electronic Commerce, 2001
14. H. B. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *The Journal of Political Economy*, 91:3:461–479, 1983.
15. E. S. Maskin. On the fair allocation of indivisible goods. In *G. Feiwel (ed.), Arrow and the Foundations of the Theory of Economic Policy (essays in honor of Kenneth Arrow)*, volume 59:4, 341–349, 1987.
16. H. Moulin. *Fair Division and Collective Welfare*. MIT Press, 2004.
17. N. Nisan. Introduction to mechanism design. In N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, editors, *Algorithmic Game Theory*. Cambridge University Press, 2007.
18. S. Pápai. Groves sealed bid auctions of heterogeneous objects with fair prices. *Social choice and Welfare*, 20:371–385, 2003.
19. D. Parkes. Iterative combinatorial auctions: Achieving economic and computational efficiency. *Ph.D. Thesis, Department of Computer and Information Science, University of Pennsylvania*, 2001.
20. W. Pulleyblank. Dual integrality in b-matching problems. In R. W. et. al. Cottle, editor, *Combinatorial Optimization*, volume 12 of *Mathematical Programming Studies*, 176–196, 1980. 10.1007/BFb0120895.
21. D. Raz, H. Levy, and B. Avi-Itzhak. A resource-allocation queueing fairness measure. SIGMETRICS, 2004.
22. L. G. Svensson. On the existence of fair allocations. *Journal of Economics*, 43:301–308, 1983.
23. W. Vickrey. Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance*, 1961.
24. H. P. Young. *Equity: In Theory and Practice*. Princeton University Press, 1995.