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On the approximability of Dodgson and Young elections ☆

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ABSTRACT

The voting rules proposed by Dodgson and Young are both designed to find an alternative closest to being a Condorcet winner, according to two different notions of proximity; the score of a given alternative is known to be hard to compute under either rule. In this paper, we put forward two algorithms for approximating the Dodgson score: a combinatorial, greedy algorithm and an LP-based algorithm, both of which yield an approximation ratio of H_{m-1}, where m is the number of alternatives and H_{m-1} is the $(m-1)$st harmonic number. We also prove that our algorithms are optimal within a factor of 2, unless problems in $\mathcal{NP}$ have quasi-polynomial-time algorithms. Despite the intuitive appeal of the greedy algorithm, we argue that the LP-based algorithm has an advantage from a social choice point of view. Further, we demonstrate that computing any reasonable approximation of the ranking produced by Dodgson's rule is $\mathcal{NP}$-hard. This result provides a complexity-theoretic explanation of sharp discrepancies that have been observed in the social choice theory literature when comparing Dodgson elections with simpler voting rules. Finally, we show that the problem of calculating the Young score is $\mathcal{NP}$-hard to approximate by any factor. This leads to an inapproximability result for the Young ranking.

1. Introduction

The discipline of voting theory deals with the following setting: there is a group of n agents and each of them ranks a set of m alternatives; one alternative is to be elected. The big question is: which alternative best reflects the social good?

This question is fundamental to the study of multiagent systems, because the agents of such a system often need to combine their individual objectives into a single output or decision that best reflects the aggregate needs of all the agents in the system. For instance, web meta-search engines [12] and recommender systems [21] have used methods based on voting theory.

Reflecting on this question, the French philosopher and mathematician Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet, suggested the following intuitive criterion: the winner should be an alternative that beats every other alternative

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in a *pairwise election*, i.e., an alternative that a (strict) majority of the agents prefers over any other alternative. Sadly, it is fairly easy to see that the preferences of the majority may be cyclic, hence a *Condorcet winner* does not necessarily exist. This unfortunate phenomenon is known as the *Condorcet paradox* (see Black [5]).

In order to circumvent this result, several researchers have proposed choosing an alternative that is "as close as possible" to a Condorcet winner. Different notions of proximity can be considered, and yield different voting rules. One such suggestion was advocated by Charles Dodgson, better known by his pen name Lewis Carroll, author of "Alice's Adventures in Wonderland." The *Dodgson score* [5] of an alternative, with respect to a given set of agents' preferences, is the minimum number of exchanges between adjacent alternatives in the agents' rankings one has to introduce in order to make the given alternative a Condorcet winner. A *Dodgson winner* is any alternative with a minimum Dodgson score.

Young [45] raised a second option: measuring the distance by agents. Specifically, the *Young score* of an alternative is the size of the largest subset of agents such that, if only these ballots are taken into account, the given alternative becomes a Condorcet winner. A *Young winner* is any alternative with the maximum Young score. Alternatively, one can perceive a Young winner as the alternative that becomes a Condorcet winner by removing the fewest agents.

Though these two voting rules sound appealing and straightforward, they have been criticized because they fail to meet several well-studied classical fairness criteria [18,6]. However, impossibility results tell us that every voting rule likewise fails to satisfy some such criterion. Thus, there is no hope of finding a voting rule that is perfect for all situations. Instead, social choice theory has advanced our understanding of an ever-increasing body of voting rules, each of which has unique features, virtues, and vices. Practitioners can choose from this body whichever rules best apply to their particular situations. Dodgson and Young voting are two such rules, as are the two approximation algorithms introduced later in this article.

A less ambiguous drawback of Dodgson and Young voting is that they are notoriously complicated to resolve. As early as 1989, Bartholdi, Tovey and Trick [2] showed that the Dodgson score decision problem is $\mathcal{NP}$-complete, and that pinpointing a Dodgson winner is $\mathcal{NP}$-hard. This important paper was one of the first to introduce complexity-theoretic considerations to social choice theory. Hemaspaandra et al. [23] refined the aforementioned result by showing that the Dodgson winner decision problem is complete for Θ_2^p, the class of problems that can be solved by $\mathcal{O}(\log n)$ queries to an $\mathcal{NP}$ set. Subsequently, Rothe et al. [41] proved that the Young winner problem is also complete for Θ_2^p.

These complexity-theoretic results give rise to the agenda of *approximately* calculating an alternative's score, under the Dodgson and Young schemes. This is clearly an interesting computational problem, as an application area of algorithmic techniques.

However, from the point of view of social choice theory, it is not immediately apparent that an approximation of a voting rule is satisfactory, since an "incorrect" alternative—in our case, one that is not closest to a Condorcet winner—might be elected. The key insight is that an approximation of a voting rule is a voting rule in its own right, and in some cases one can argue that this new voting rule has desirable properties. We discuss this point at length, and justify our approach, in Section 7.

1.1. Our results

In the context of approximating the Dodgson score, we devise a greedy algorithm for the Dodgson score which has an approximation ratio of H_{m-1}, where m is the number of alternatives and H_{m-1} is the $(m-1)$st harmonic number. We then propose a second algorithm that is based on solving a linear programming relaxation of the Dodgson score and has the same approximation ratio. Although the former algorithm gives us a better intuition into the combinatorial structure of the problem, we show that the latter has the advantage of being score monotonic, which is a desirable property from a social choice point of view. We further observe that it follows from the work of McCabe-Dansted [30] that the Dodgson score cannot be approximated within sublogarithmic factors by polynomial-time algorithms unless $\mathcal{P} = \mathcal{NP}$. We prove a more explicit inapproximability result of $(1/2 - \epsilon)\ln m$, under the assumption that problems in $\mathcal{NP}$ do not have algorithms running in quasi-polynomial time; this implies that the approximation ratio achieved by our algorithms is optimal up to a factor of 2.

A number of recent papers [38,39,27–29] have established that there are sharp discrepancies between the Dodgson ranking and the rankings produced by other rank aggregation rules. Some of these rules (e.g., Borda and Copeland) are polynomial-time computable, so the corresponding results can be viewed as negative results regarding the approximability of the Dodgson ranking by polynomial-time algorithms. We show that the problem of distinguishing between whether a given alternative is the unique Dodgson winner or in the last $O(\sqrt{m})$ positions in any Dodgson ranking is $\mathcal{NP}$-hard. This theorem provides a complexity-theoretic explanation for some of the observed discrepancies, but in fact is much wider in scope as it applies to any efficiently computable rank aggregation rule.

At first glance, the problem of calculating the Young score seems simple compared with the Dodgson score (we discuss in Section 6 why this seems so). Therefore, we found the following result quite surprising: it is $\mathcal{NP}$-hard to approximate the Young score within any factor. Specifically, we show that it is $\mathcal{NP}$-hard to distinguish between the case where the Young score of a given alternative is 0, and the case where the score is greater than 0. As a corollary we obtain an inapproximability result for the Young ranking. We also show that it is $\mathcal{NP}$-hard to approximate the dual Young score within $O(n^{1-\epsilon})$, for any constant $\epsilon > 0$. We define the dual Young score below in the preliminaries section.

1.2. Related work

The agenda of approximating voting rules was recently pursued by Ailon et al. [1], Coppersmith et al. [10], and Kenyon-Mathieu and Schudy [26]. These papers deal, directly or indirectly, with the Kemeny rank aggregation rule, which chooses a ranking of the alternatives instead of a single winning alternative. The Kemeny rule picks the ranking that has the maximum number of agreements with the agents' individual rankings regarding the correct order of pairs of alternatives. Ailon et al. improve the trivial 2-approximation algorithm to an involved, randomized algorithm that gives an 11/7-approximation; Kenyon-Mathieu and Schudy further improve the approximation, and obtain a polynomial-time approximation scheme (PTAS).

Two recent papers study the approximability of Dodgson elections; both papers appeared after the conference version of the current paper. Faliszewski, Hemaspaandra, and Hemaspaandra show that minimax as a scoring rule is an m^2 approximation of the Dodgson score [16]. Caragiannis et al. [7] give a number of algorithms that approximate to Dodgson score and also have nice fairness properties, such as homogeneity and monotonicity, but whose approximations are either asymptotically worse than ours or not polynomial-time computable. Moreover, their approximation algorithms are much more complex, both descriptionally and in running time. They also provide upper bounds such that any score-based voting rule whose scoring rule approximates Dodgson scoring to within the bounds fails to meet certain fairness criteria. In Section 4 we discuss further the latter results.

Two recent papers have directly put forward algorithms for the Dodgson winner problem [24,32]. Both papers independently build upon the same basic idea: if the number of agents is significantly larger than the number of alternatives, and one looks at a uniform distribution over the preferences of the agents, with high probability one obtains an instance on which it is trivial to compute the Dodgson score of a given alternative. This directly gives rise to an algorithm that can usually compute the Dodgson score (under the assumption on the number of agents and alternatives). However, this is not an approximation algorithm in the usual sense, since the algorithm *a priori* gives up on certain instances,[1] whereas an approximation algorithm is judged by its worst-case guarantees. In addition, this algorithm would be useless if the number of alternatives is not small compared with the number of agents.[2]

In a similar vein, McCabe-Dansted [31] suggested several new variations on Dodgson's rule. These rules are shown to give an *additive* approximation to the Dodgson score. Specifically, they can underestimate the score by an additive term of at most $(m-1)!(m-1)e$, where m is the number of alternatives. We note that this result would only be meaningful if there are very few alternatives, and in addition it does not provide the tight worst-case *multiplicative* guarantees that we achieve in this paper. McCabe-Dansted further shows that, similarly to the rules discussed above, the new rules usually select the Dodgson winner under certain distributions.

Betzler et al. [4] have investigated the parameterized computational complexity of the Dodgson and Young rules. The authors have devised a fixed parameter algorithm for exact computation of the Dodgson score, where the fixed parameter is the "edit distance", i.e., the number of exchanges. Specifically, if k is an upper bound on the Dodgson score of a given alternative, n is the number of agents, and m the number of alternatives, the algorithm runs in time $\mathcal{O}(2^k \cdot nk + nm)$. Notice that in general it may hold that $k = \Omega(nm)$. In contrast, the Young score decision problem is $W[2]$-complete; this implies that there is no algorithm that computes the Young score exactly, and whose running time is polynomial in nm and only exponential in k, where the parameter k is the number of remaining votes. These results complement ours nicely, as we shall also demonstrate that computing the Dodgson score is in a sense easier than computing the Young score, albeit in the context of approximation.

Putting computational complexity aside, some research by social choice theorists has considered comparing the ranking produced by Dodgson, i.e., the ordering of the alternatives by nondecreasing Dodgson score, with elections based on simpler voting rules. Such comparisons have always revealed sharp discrepancies. For example, the Dodgson winner can appear in any position in the Kemeny ranking [38] and in the ranking of any positional scoring rule [39] (e.g., Borda or Plurality), Dodgson rankings can be exactly the opposite of Borda [29] and Copeland rankings [27], while the winner of Kemeny or Slater elections can appear in any position of the Dodgson ranking [28].

More distantly related to our work is research that is concerned with exactly resolving hard-to-compute voting rules by heuristic methods. Typical examples include papers regarding the Kemeny rule [9] and the Slater rule [8]. Another more remotely related field of research is concerned with finding approximate, efficient representations of voting rules, by eliciting as little information as possible; this line of research employs techniques from learning theory [36].

[1] Technically speaking, this algorithm correctly computes the Dodgson score in worst-case polynomial time, but only when the domain is restricted to those instances on which the algorithm does not give up, and there does not seem to be a characterization of this domain restriction that does not refer in a fairly direct way back to the algorithm itself. Thus, in many natural settings one cannot before an election is held guarantee that the algorithm will work.

[2] This would normally not happen in political elections, but can certainly be the case in many other settings. For instance, consider a group of agents trying to reach an agreement on a joint plan, when multiple alternative plans are available. Specifically, think of a group of investors deciding which company to invest in.

1.3. Structure of the paper

In Section 2, we introduce some notations and definitions. In Section 3, we present our upper bounds for approximating the Dodgson score. We study the monotonicity properties of our algorithms in Section 4. In Section 5, we present our lower bounds for approximating the Dodgson score and ranking. In Section 6, we prove that the Young score, dual Young score, and Young ranking are inapproximable. Finally, we discuss our approach in Section 7.

2. Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of agents, and let A be a set of *alternatives*. We denote $|A|$ by m, and denote the alternatives themselves by letters, such as $a \in A$. Indices referring to agents appear in superscript. Each agent $i \in N$ holds a binary relation R^i over A that satisfies antisymmetry, transitivity and totality. Informally, R^i is a ranking of the alternatives. Let $L = L(A)$ be the set of all rankings over A; we have that each $R^i \in L$. We denote $R^N = \langle R^1, \ldots, R^n\rangle \in L^N$, and refer to this vector as a *preference profile*. We may also use Q^i to denote the preferences of agent i, in cases where we want to distinguish between two different rankings R^i and Q^i. For sets of alternatives $B_1, B_2 \subseteq A$, we write $B_1 R^i B_2$ if for all $a \in B_1$ and $b \in B_2$, aR^ib. If $B_1 = \{a\}$ (respectively, $B_2 = \{a\}$) for some a, we sometimes write aR^iB_2 (respectively, B_1R^ia) instead of $\{a\}R^iB_2$ (respectively, $B_1R^i\{a\}$).

Let $a, b \in A$. Denote $P(a,b) = \{i \in N\colon\ aR^ib\}$. We say that *a beats b in a pairwise election* if $|P(a,b)| > n/2$, that is, a is preferred to b by a majority of agents. A *Condorcet winner* is an alternative that beats every other alternative in a pairwise election.

The *Dodgson score* of a given alternative a^*, with respect to a given preference profile R^N, is the least number of exchanges between adjacent alternatives in R^N needed to make a^* a Condorcet winner. For instance, let $N = \{1, 2, 3\}$, $A = \{a, b, c\}$, and let R^N be given by:

R^1	R^2	R^3
a	b	a
b	a	c
c	c	b

where the top alternative in each column is the most preferred one.

In this example, the Dodgson score of a is 0 (a is a Condorcet winner), the Dodgson score of b is 1, and the Dodgson score of c is 3. Bartholdi et al. [2] have shown that the Dodgson score decision problem—the problem of determining, for a given preference profile R^N, alternative a, and natural number k, whether the Dodgson score of a in R^N is at most k—is $\mathcal{NP}$-complete.

The *Young score* of a^* with respect to R^N is the size of a largest subset of agents for whom a^* is a Condorcet winner. This is the definition given by Young himself [45], and used in subsequent articles [41]. If, for every nonempty subset of agents, a^* is not a Condorcet winner, its Young score is 0. In the above example, the Young score of a is 3, the Young score of b is 1, and the Young score of c is 0.

Equivalently, a Young winner is an alternative such that one has to remove the minimum number of agents in order to make it a Condorcet winner. We call this number the *dual* Young score. Note that, in the context of approximation, these two definitions are not equivalent; we employ the former (original, prevalent) definition, but touch on the latter as well.

As the decision problem version of the Young winner problem (the decision problem is to determine, given a preference profile and an alternative a, whether a is the Young winner in that profile) is known to be Θ_2^P-complete [41], and thus $\mathcal{NP}$-hard, the Young score problem must also be hard; otherwise, we would be able to calculate the scores of all the alternatives efficiently, and identify the alternatives with minimum score.

Linear and integer programs are fundamental tools for solving optimization problems. See Cormen et al. [11] for a nice introduction to the subject from a computer science perspective, which we summarize here. A *linear program* in its canonical form consists of, for some $p, q \in \mathbb{N}$, a $p \times q$ matrix M, a p-vector A and a q-vector B, and seeks to find a q-vector that maximizes BX (called the *objective function*) subject to the constraints $MX \leqslant A$, where any X satisfying the constraints (though which may not necessarily be a maximum) is called *feasible*. An *integer linear program* is a linear program with the additional restriction that X may only take integral values. As is commonly done, we will often write the linear programs we use as seeking to minimize rather than maximize the objective function, and express some of the constraints as lower rather than upper bounds. Through simple algebraic manipulation, these expressions can always be translated into equivalent ones that are in the canonical form defined above.

For a linear program in the form given above, its *dual* is the linear program defined as the problem of finding a p-vector $Y \geqslant 0$ that minimizes AY subject to the constraints $M^{\mathrm{T}}Y \geqslant B$ (where M^{T} is the transpose of M). It is easy to see that BX of any feasible solution X to the original problem (known as the *primal* linear program) is a lower bound on AY of any feasible solution Y to its dual (or vice versa if the primal is expressed as a minimization problem), so X and Y are optimal solutions to their respective problems whenever $AY = BX$. The converse is also true, though not as easy to see, and plays a fundamental role in the analysis of algorithms for solving linear programs. We will use this fact in the paper.

R^1	R^2	R^3
b	b	c
d_1	d_4	b
d_2	d_5	d_6
c	a^*	d_7
d_3	c	d_8
a^*	d_1	a^*
d_4	d_2	d_1
d_5	d_3	d_2
d_6	d_6	d_3
d_7	d_7	d_4
d_8	d_8	d_5

(a) Initial profile.

R^1	R^2	R^3
b	b	c
d_1	d_4	b
d_2	d_5	d_6
a^*	a^*	d_7
c	c	d_8
d_3	d_1	a^*
d_4	d_2	d_1
d_5	d_3	d_2
d_6	d_6	d_3
d_7	d_7	d_4
d_8	d_8	d_5

(b) After step 1.

R^1	R^2	R^3
b	a^*	c
d_1	b	b
d_2	d_4	d_6
a^*	d_5	d_7
c	c	d_8
d_3	d_1	a^*
d_4	d_2	d_1
d_5	d_3	d_2
d_6	d_6	d_3
d_7	d_7	d_4
d_8	d_8	d_5

(c) After step 2.

R^1	R^2	R^3
b	a^*	c
d_1	b	a^*
d_2	d_4	b
a^*	d_5	d_6
c	c	d_7
d_3	d_1	d_8
d_4	d_2	d_1
d_5	d_3	d_2
d_6	d_6	d_3
d_7	d_7	d_4
d_8	d_8	d_5

(d) After step 3.

Fig. 1. An example of the execution of the greedy algorithm.

Linear programs are widely used by engineers and computer scientists to solve a wide range of problems. They can be solved in polynomial time, though the degrees of these polynomials are so high that in practice worst-case superpolynomial-time algorithms are used instead.

3. Approximability of Dodgson scores

We begin by presenting our approximation algorithms for the Dodgson score. Let us first introduce some notation.

Let $a^* \in A$ be a distinguished alternative, whose Dodgson score we wish to compute. Define the *deficit* of a^* with respect to $a \in A$, simply denoted $\operatorname{def}(a)$ when the identity of a^* is clear, as the number of additional agents i whose ranking relation R^i must be changed from to aR^ia^* to a^*R^ia in order for a^* to beat a in a pairwise election. Equivalently,

$$\operatorname{def}(a) = \max\{0, 1 + \lfloor n/2 \rfloor - P(a^*, a)\}.$$

For instance, if four agents prefer a to a^* and only one agent prefers a^* to a, then $\operatorname{def}(a) = 2$. If a^* beats a in a pairwise election (namely a^* is preferred to a by a majority of agents) then $\operatorname{def}(a) = 0$. We say that alternatives $a \in A$ with $\operatorname{def}(a) > 0$ are *alive*. Alternatives that are not alive, i.e., those with $\operatorname{def}(a) = 0$, are *dead*.

3.1. A greedy algorithm

In this section we present a combinatorial, greedy algorithm for approximating the Dodgson score of a given alternative. Consider, once again, a special alternative a^*, and recall that a live alternative is one with a positive deficit. In each step, the algorithm selects an optimally cost-effective push of alternative a^* in the preference of some agent. The *cost-effectiveness* of pushing a^* in the preference of an agent $i \in N$ is the ratio between the total number of positions a^* is moved upwards in the preference of i *compared with the original profile* R^N, and the number of *currently* live alternatives that a^* overtakes as a result of this push.

After selecting an optimally cost-effective push, i.e., the push with the lowest cost-effectiveness, the algorithm decreases $\operatorname{def}(a)$ by one for each live alternative a that a^* overtakes. Alternatives $a \in A$ with $\operatorname{def}(a) = 0$ become dead. The algorithm terminates when no live alternatives remain. The output of the algorithm is the total number of positions that alternative a^* is pushed upwards in the preferences of all agents.

Greedy algorithm:

1. Let A' be the set of live alternatives, namely those alternatives $a \in A$ with $\operatorname{def}(a) > 0$.
2. While $A' \neq \emptyset$:
   - Perform an optimally cost-effective push, namely push a^* in the preferences of agent $i \in N$ in a way that minimizes the ratio between the *total* number of positions that a^* moves upwards in the preferences of i and the number of currently live alternatives that a^* overtakes as a result of this push.
   - Recalculate A'.
3. Return the number of exchanges performed.

An example of the execution of the algorithm is depicted in Fig. 1 (see also Fig. 2 and the related discussion in Section 4). In the initial profile of this example, alternative a^* has deficits $\operatorname{def}(b) = 2$, $\operatorname{def}(c) = 1$, and $\operatorname{def}(d_i) = 0$. Hence, alternatives b and c are alive and alternatives $d_1, \ldots, d_8$ are dead. At the first step of the algorithm, there are several different ways of pushing alternative a^* upwards in order to overtake one of the live alternatives b and c or both. Among them, the one with the smallest cost-effectiveness is to push a^* upwards in the preference R^1. In this way, a^* moves two positions upwards and overtakes the live alternative c for a cost-effectiveness of 2. Any other push of a^* in the initial profile has cost-effectiveness at least 2.5 since a^* has to be pushed at least three positions upwards in order to overtake one live alternative and at least five positions upwards in order to overtake both b and c. After step 1, alternative c is dead. Then, in step 2, there are three ways to push alternative a^* upwards so that it overtakes the live alternative b: either pushing it at the top of R^1 (this has cost effectiveness 5 because a^* would have moved five positions in total compared to the initial profile in R^1), or pushing it at the top of R^2 (with cost-effectiveness 3), or pushing it four positions upwards in R^3 (with cost-effectiveness 4). The algorithm picks the second option. Then, in step 3, the algorithm can either push alternative a^* at the top position of R^1 or push it four positions upwards in R^3. The former has a cost-effectiveness of 5 (recall that cost-effectiveness is defined using the total number of positions a^* would move compared to its position at the initial profile) while the latter has a cost-effectiveness of 4 and is the push the algorithm picks. After step 3, all alternatives are dead and the algorithm terminates by returning the total number of positions a^* is pushed upwards, i.e., 9.

By the definition of the algorithm, it is clear that it produces a profile where a^* is a Condorcet winner. It is important to notice that if a^* is initially a Condorcet winner then the algorithm calculates a score of zero, so as a voting rule the algorithm satisfies the Condorcet criterion. Also, during each iteration of line 2 an optimally cost-effective push may not be unique, in which case the algorithm chooses, in a manner that does not affect our approximation results, exactly one of these optimally cost-effective pushes.

Theorem 3.1. *For any input a^* and R^N with m alternatives, the greedy algorithm returns an H_{m-1}-approximation of the Dodgson score of a^*, where for all natural numbers k, $H_k = \sum_{i=1}^{k} \frac{1}{k}$ is the k-th harmonic number.*

Proof. We base our proof on the connection between our problem and the Constrained Set Multicover problem, for which Rajagopalan and Vazirani [37] give an approximation algorithm and use the dual fitting technique to prove its approximation ratio (see also [44, pp. 112–116]).

Constrained Set Multicover

Instance:A ground set A, a set of integers $\{r_a\}_{a\in A}$, one for each element of $a \in A$, representing the *covering requirement* for a, an indexed collection $\mathcal{S} = \{S_j \mid S_j \subseteq A\}$ of subsets of A (crucially, the same subset may occur more than once in this collection, as long as each copy has a distinct index), and a set of integers $\{c_{S_j}\}$, one for each member of $\mathcal{S}$, representing the *cost* of that member.

Question: What is the smallest number $\bar{c}$ for which there is a subcollection $\mathcal{C}$ of $\mathcal{S}$ such that

1. $\bar{c} = \sum_{S_j \in \mathcal{C}} c_{S_j}$,
2. each member of $\mathcal{S}$ appears at most once in $\mathcal{C}$, and
3. each element $a \in A$ appears in at least r_a members of $\mathcal{C}$?

We may view the problem of approximating the Dodgson score as a variation of Constrained Set Multicover. The ground set is the set of live alternatives. For each live alternative $a \in A \setminus \{a^*\}$, its deficit def($a$) is in fact its covering requirement, i.e., the number of different sets it has to belong to in the final cover. For each agent $i \in N$ that ranks a^* in place r^i, we have a group $\mathcal{S}^i$ consisting of the sets S_k^i for $k = 1, \dots, r^i - 1$, where the set S_k^i contains the (initially) live alternatives that appear in positions $r^i - k$ to $r^i - 1$ in the preference of agent i. The set S_k^i has cost k. Now, the covering problem to be solved is the following. We wish to select at most one set from each of the different groups so that each alternative $a \in A \setminus \{a^*\}$ appears in at least def(a) sets and the total cost of the selected sets is minimized. The optimal cost is the Dodgson score of a^* and, hence, the cost of any approximate cover that satisfies the covering requirements and the constraints is an upper bound on the Dodgson score.

We can thus define this covering problem as:

Set Multicover with Group Constraints

Instance:A ground set A, a set of integers $\{r_a\}_{a\in A}$, one for each element of $a \in A$, a collection $\mathcal{S} = \{S_j \mid S_j \subseteq A\}$ of subsets of A, a set of integers $\{c_{S_j}\}$, and a partitioning of $\mathcal{S}$ into groups $\mathcal{S}^i$ for $i \in N$.

Question: What is the smallest number $\bar{c}$ for which there is a subcollection $\mathcal{C}$ of $\mathcal{S}$ such that

1. $\bar{c} = \sum_{S_j \in \mathcal{C}} c_{S_j}$,
2. each member of $\mathcal{S}$ appears at most once in $\mathcal{C}$,
3. each element $a \in A$ appears in at least r_a members of $\mathcal{C}$, and
4. at most one member from each group $\mathcal{S}^i$ appears in $\mathcal{C}$?

In terms of this covering problem, the greedy algorithm mentioned above can be thought of as follows. In each step, it selects an optimally cost-effective set where the cost-effectiveness of a set is defined as the ratio between the cost of the set and the number of live alternatives it covers that have not been previously covered by sets belonging to the same group. For these live alternatives, the algorithm decreases their covering requirements at the end of the step. The algorithm terminates when all alternatives have died (i.e., their covering requirement has become zero). The output of the algorithm consists of the maximum-cost sets that were picked from each group.

We find it convenient to formulate the Dodgson score problem as the following integer linear program.

$$\begin{aligned}
\text{minimize} \quad & \sum_{i\in N}\sum_{k=1}^{r^i-1} k\cdot x_{S_k^i} \\
\text{subject to} \quad & \forall a\in A\setminus\{a^*\}, \quad \sum_{i\in N}\sum_{S\in\mathcal{S}^i:\, a\in S} x_S \geqslant \operatorname{def}(a) \\
& \forall i\in N, \quad \sum_{S\in\mathcal{S}^i} x_S \leqslant 1 \\
& x\in\{0,1\}.
\end{aligned}$$

The variable x_S associated with a set S denotes whether S is included in the solution ($x_S = 1$) or not ($x_S = 0$). We relax the integrality constraint in order to obtain a linear programming relaxation and we compute its dual linear program.

$$\begin{aligned}
\text{maximize} \quad & \sum_{a\in A\setminus\{a^*\}} \operatorname{def}(a)\cdot y_a - \sum_{i\in N} z^i \\
\text{subject to} \quad & \forall i\in N, k=1,\ldots,r^i-1, \quad \sum_{a\in S_k^i} y_a - z^i \leqslant k \\
& \forall i\in N, \quad z^i \geqslant 0 \\
& \forall a\in A\setminus\{a^*\}, \quad y_a \geqslant 0.
\end{aligned}$$

For a set S that is picked by the algorithm to cover alternative $a\in A\setminus\{a^*\}$ for the j-th time (the j-th copy of a), we set $p(a, j)$ to be equal to the cost-effectiveness of S when it is picked. Informally, p distributes equally the cost of S among the copies of the live alternatives it covers. When the algorithm covers a live alternative a by picking a set S that belongs to group $\mathcal{S}^i$, we use the notation $j_i(a)$ to denote the index of the copy of a the algorithm covers by picking this set. Denote by T^i the set of live alternatives covered by the sets of group $\mathcal{S}^i$ that are picked by the algorithm throughout its execution.

Now, we shall show that by setting

$$y_a = \frac{p(a, \operatorname{def}(a))}{H_{m-1}}$$

for each alternative $a\in A\setminus\{a^*\}$ and

$$z^i = \frac{1}{H_{m-1}}\sum_{a\in T^i}\big(p\big(a,\operatorname{def}(a)\big) - p\big(a, j_i(a)\big)\big)$$

for each agent $i\in N$, the constraints of the dual linear program are satisfied. The variables y_a are clearly non-negative. Since the algorithm picks a set of optimal cost-effectiveness at each step, the cost-effectiveness of the set picked does not decrease with time. Hence, $p(a, \operatorname{def}(a)) \geqslant p(a, j)$ for every alternative a with $\operatorname{def}(a) > 0$ and $j \leqslant \operatorname{def}(a)$. This implies that z^i is non-negative as well.

In order to show that the first constraint of the dual linear program is also satisfied, consider an agent $i\in N$ and integer k such that $1\leqslant k\leqslant r^i-1$. We have

$$\begin{aligned}
\sum_{a\in S_k^i} y_a - z^i &= \frac{1}{H_{m-1}}\Bigg[\sum_{a\in S_k^i} p\big(a,\operatorname{def}(a)\big) - \sum_{a\in T^i}\big(p\big(a,\operatorname{def}(a)\big) - p\big(a, j_i(a)\big)\big)\Bigg] \\
&\leqslant \frac{1}{H_{m-1}}\Bigg[\sum_{a\in S_k^i} p\big(a,\operatorname{def}(a)\big) - \sum_{a\in S_k^i\cap T^i}\big(p\big(a,\operatorname{def}(a)\big) - p\big(a, j_i(a)\big)\big)\Bigg] \\
&= \frac{1}{H_{m-1}}\Bigg[\sum_{a\in S_k^i\setminus T^i} p\big(a,\operatorname{def}(a)\big) + \sum_{a\in S_k^i\cap T^i} p\big(a, j_i(a)\big)\Bigg]. \qquad (1)
\end{aligned}$$

Now, for each alternative $a\in S_k^i$, we define $\nu(a)$ as follows. If $a\in S_k^i\cap T^i$, $\nu(a)$ is the time step in which the algorithm covered alternative a by picking a set of group $\mathcal{S}^i$. Otherwise, if $a\in S_k^i\setminus T^i$, $\nu(a)$ is the time step in which alternative a died. Now, number the alternatives in S_k^i in nondecreasing order of $\nu(\cdot)$, breaking ties arbitrarily. Let $a_1, a_2, \ldots, a_{|S_k^i|}$ be this order. Consider alternative a_t with $1\leqslant t\leqslant |S_k^i|$. Observe that, due to the definition of the order of alternatives in S_k^i, after step $\nu(a_t)$ is performed, the alternatives $a_t, a_{t+1}, \ldots, a_{|S_k^i|}$ have not died yet and the sets of group $\mathcal{S}^i$ that have been picked by the algorithm so far (if any) do not contain any of them. Hence, at step $\nu(a_t)$, the algorithm has the option to pick set S_k^i of cost k in order to cover at least these $|S_k^i| - t + 1$ alternatives. So, the cost-effectiveness of the set that is actually picked by the algorithm at step $\nu(a_t)$ is at most $\frac{k}{|S_k^i|-t+1}$. This argument implies that

$$p\big(a_t, j_i(a_t)\big) \leqslant \frac{k}{|S_k^i| - t + 1} \tag{2}$$

if $a_t \in S_k^i \cap T^i$, and

$$p\big(a_t, \mathrm{def}(a_t)\big) \leqslant \frac{k}{|S_k^i| - t + 1} \tag{3}$$

otherwise (if $a_t \in S_k^i \setminus T^i$).

Using (2) and (3) together with (1), we obtain that

$$\sum_{a \in S_k^i} y_a - z^i \leqslant \frac{1}{H_{m-1}} \sum_{t=1}^{|S_k^i|} \frac{k}{|S_k^i| - t + 1} = \frac{kH_{|S_k^i|}}{H_{m-1}} \leqslant k,$$

implying that the constraints of the dual linear program are always satisfied. The last inequality follows since, obviously, $|S_k^i| \leqslant m - 1$.

Now, denote by OPT the optimal objective value of the integer linear program. By duality, we have that any feasible solution to the dual of its linear programming relaxation has objective value at most OPT. Hence,

$$\begin{aligned} H_{m-1} \cdot \mathrm{OPT} &\geqslant H_{m-1}\Bigg(\sum_{a \in A \setminus \{a^*\}} \mathrm{def}(a) \cdot y_a - \sum_{i \in N} z^i\Bigg) \\ &= \sum_{a \in A \setminus \{a^*\}} \mathrm{def}(a) \cdot p\big(a, \mathrm{def}(a)\big) - \sum_{i \in N} \sum_{a \in T^i} \Big(p\big(a, \mathrm{def}(a)\big) - p\big(a, j_i(a)\big)\Big) \\ &= \sum_{i \in N} \sum_{a \in T^i} p\big(a, j_i(a)\big). \end{aligned}$$

The theorem follows since the last expression is equal to the total cost of the sets picked at all steps of the algorithm and clearly upper-bounds the cost of the final solution. □

3.2. An LP-based algorithm

The analysis of the greedy algorithm suggests an LP-based algorithm for approximating the Dodgson score of an alternative a^* without explicitly providing a way to push a^* upwards in the preference of some agents so that a^* becomes the Condorcet winner. This algorithm uses the same LP relaxation of the Dodgson score that was used in the analysis of the greedy algorithm. The algorithm computes the optimal objective value, and returns this value multiplied by H_{m-1} as a score of the alternative a^*. The idea that the relaxation of the ILP for the Dodgson score induces a rule that is similar to Dodgson is not new (see, e.g., [30]).

For completeness, we reformulate the LP in a more detailed form that takes the preference profile as a parameter as well; this shall be useful in the following section, where we discuss the monotonicity properties of the algorithm. Given a profile $R = R^N$ with a set of agents N and a set of m alternatives A, we denote by $r^i(R)$ the rank of alternative a^* in the preference of agent i. We use the notation $\mathrm{def}(a, R)$ for the deficit of a^* against an alternative a in the profile R. Recall that alternatives $a \in A \setminus \{a^*\}$ such that $\mathrm{def}(a, R) > 0$ are said to be alive. For every agent $i \in N$ that ranks a^* in place $r^i(R)$, we denote by $\mathcal{S}^i(R)$ the subcollection that consists of the sets $S_k^i(R)$ for $k = 1, \ldots, r^i(R) - 1$, where the set $S_k^i(R)$ contains the live alternatives that appear in positions $r^i(R) - k$ to $r^i(R) - 1$ in the preference of agent i. We denote by $\mathcal{S}(R)$ the union of the subcollections $\mathcal{S}^i(R)$ for $i \in N$.

The LP-based algorithm uses the following LP relaxation of the Dodgson score of alternative a^* in the profile R:

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in N} \sum_{k=1}^{r^i(R)-1} k \cdot x_{S_k^i(R)} \\ \text{subject to} \quad & \forall a \in A \setminus \{a^*\}, \quad \sum_{i \in N} \sum_{S \in \mathcal{S}^i(R):\, a \in S} x_S \geqslant \mathrm{def}(a, R) \\ & \forall i \in N, \quad \sum_{S \in \mathcal{S}^i(R)} x_S \leqslant 1 \\ & \forall S \in \mathcal{S}(R), \quad 0 \leqslant x_S \leqslant 1. \end{aligned}$$

We remark that, as it is the case with the greedy algorithm, if a^* is initially a Condorcet winner then the algorithm calculates a score of zero. In any other case, we can easily show that the score returned by the algorithm is between the Dodgson score of alternative a^* and its Dodgson score multiplied by H_{m-1}. Indeed, by the analysis in the proof of Theorem 3.1 (see the last derivation), we know that the score returned by the greedy algorithm and, consequently, the Dodgson score of a^* is not higher than the optimal objective value of the LP relaxation multiplied by H_{m-1}. Furthermore, since the optimal objective value of the LP relaxation is a lower bound on the Dodgson score of a^*, the LP-based algorithm returns a score that is an H_{m-1}-approximation of the Dodgson score. This is formalized in the following theorem.

Theorem 3.2. *For any input a^* and R^N with m alternatives, the LP-based algorithm returns an H_{m-1}-approximation of the Dodgson score of a^*.*

4. Interlude: on the desirability of approximation algorithms as voting rules

So far we have looked at Dodgson approximations through the algorithmic lens. We now wish to briefly explore the social choice point of view. We argue that a Dodgson approximation is equivalent to a new voting rule, which is guaranteed to elect an alternative that is not far from being a Condorcet winner. In other words, a perfectly sensible definition of a "socially good" winner, given the circumstances, is simply the alternative chosen by the approximation algorithm. Note that the approximation algorithm can be designed to satisfy the Condorcet criterion, i.e., always elect a Condorcet winner if one exists. Since the Dodgson score of a Condorcet winner is zero, choosing such a winner when one exists has no impact on the approximation ratio.

Our approximation algorithms should therefore be compared according to two conceptually different, but not orthogonal, dimensions: their algorithmic properties and their social choice properties. From an algorithmic point of view, the greedy algorithm gives us a better sense of the combinatorial structure of the problem. In the sequel we suggest, however, that the LP-based algorithm has some desirable properties from the social choice point of view.

In most algorithmic mechanism design settings [34], such as combinatorial auctions or scheduling, one usually seeks approximation algorithms that are truthful, i.e., the agents cannot benefit by lying. However, the well-known Gibbard–Satterthwaite Theorem [22,42] precludes voting rules that are both truthful and reasonable, in a sense. Therefore, other desiderata are looked for in voting rules. (Of course, other social choice properties are interesting to look at in their own right, independent of the Gibbard–Satterthwaite Theorem).

Let us reiterate that both the greedy algorithm and the LP-based algorithm satisfy the Condorcet property. Let us now consider the *monotonicity* property, one of the major desiderata on the basis of which voting rules are compared. Many different notions of monotonicity can be found in the literature; for our purposes, a (score-based) voting rule is *score monotonic* if and only if making an alternative more preferable in the rankings of some agent cannot worsen the score of the alternative, that is, increase it when a lower score is desirable (as in Dodgson), or decrease it when a higher score is desirable. All prominent score-based voting rules (e.g., positional scoring rules, Copeland, Maximin) are score monotonic; it is straightforward to see that the Dodgson and Young rules are score monotonic as well.

We first claim that our LP-based algorithm is score monotonic.

Theorem 4.1. *The LP-based algorithm is score monotonic.*

Proof. We will consider two different inputs to the LP-based algorithm for computing the score of an alternative a^*: one with a profile $R = R^N$ and another with a profile $\bar{R}$ that is obtained from R by pushing alternative a^* upwards in the preferences of some of the agents (abusing notation somewhat, we will sometimes let R, respectively $\bar{R}$, denote the input having profile R, respectively $\bar{R}$). Given an optimal solution x for R, we will construct a feasible solution $\bar{x}$ for $\bar{R}$ that does not exceed x. This is a sufficient condition for the assertion of the theorem.

By the definition of profile $\bar{R}$, it holds that $r^i(R) \geqslant r^i(\bar{R})$ for every $i \in N$. We partition the subcollection $\mathcal{S}^i(R)$ into the following two disjoint subcollections:

$$\mathcal{S}^{i,1}(R) = \left\{S^i_k(R)\colon k = r^i(R) - r^i(\bar{R}) + 1, \ldots, r^i(R) - 1\right\}$$

and

$$\mathcal{S}^{i,2}(R) = \left\{S^i_k(R)\colon k = 1, \ldots, r^i(R) - r^i(\bar{R})\right\}.$$

For every $i \in N$, there is a one-to-one and onto correspondence between the sets in $\mathcal{S}^i(\bar{R})$ and the sets in $\mathcal{S}^{i,1}(R)$, where for $k \in \{1, \ldots, r^i(\bar{R})\}$ the set $S^i_k(\bar{R})$ of $\mathcal{S}^i(\bar{R})$ corresponds to the set $S^i_{k+r^i(R)-r^i(\bar{R})}(R)$ of $\mathcal{S}^{i,1}(R)$ and vice versa. The solution $\bar{x}$ for the second input is constructed by simply setting

$$\bar{x}_{S^i_k(\bar{R})} = x_{S^i_{k+r^i(R)-r^i(\bar{R})}(R)}$$

for $i \in N$ and $k \in \{1, \ldots, r^i(\bar{R}) - 1\}$.

R^1	R^2	R^3	R^4	R^5	R^6
a_4	a_4	a_4	a_3	a_2	a_1
a_3	a_3	b_4	b_9	b_{13}	b_{16}
a_2	a_2	b_5	b_{10}	b_{14}	b_{17}
a_1	a_1	b_6	b_{11}	b_{15}	a^*
.	b_1	b_7	b_{12}	a^*	.
.	b_2	b_8	a^*	.	.
.	b_3	a^*	.	.	.
.	a^*	.	.	.	.
.	.	.	.	.	.
a^*	.	.	.	.	.

(a) Original profile.

R^1	R^2	Q^3	Q^4	Q^5	Q^6
a_4	a_4	a_4	a_3	a_2	a_1
a_3	a_3	b_4	b_9	b_{13}	a^*
a_2	a_2	b_5	b_{10}	a^*	b_{16}
a_1	a_1	b_6	a^*	b_{14}	b_{17}
.	b_1	a^*	b_{11}	b_{15}	.
.	b_2	b_7	b_{12}	.	.
.	b_3	b_8	.	.	.
.	a^*	.	.	.	.
.	.	.	.	.	.
a^*	.	.	.	.	.

(b) Improvement of a^*.

Fig. 2. The voting rule corresponding to the greedy algorithm is not score monotonic: an example.

We will first prove that the solution $\bar{x}$ is a feasible solution for $\bar{R}$. The definition of the $\bar{x}$-variables clearly implies that the second and third sets of constraints are satisfied (since the solution x is feasible). Also, the first set of constraints is trivially satisfied for each alternative a with $\operatorname{def}(a,\bar{R})=0$. Assume now that alternative a has $\operatorname{def}(a,\bar{R})>0$. Let e_a^i be 1 if agent i ranks alternative a above a^* in R and below it in $\bar{R}$; otherwise let e_a^i be 0. Then, it can be easily seen that $\operatorname{def}(a,R)=\operatorname{def}(a,\bar{R})+\sum_{i\in N} e_a^i>0$. Hence, by the correspondence between the sets in $\mathcal{S}^i(\bar{R})$ and the sets in $\mathcal{S}^{i,1}(R)$, it follows that for every set $S\in\mathcal{S}^i(\bar{R})$ that contains alternative a, its corresponding set in $\mathcal{S}^{i,1}(R)$ also contains a. Using this observation and the definition of the solution $\bar{x}$, we obtain that

$$\sum_{i\in N}\sum_{S\in\mathcal{S}^i(\bar{R}):\, a\in S}\bar{x}_S=\sum_{i\in N}\sum_{S\in\mathcal{S}^{i,1}(R):\, a\in S}x_S
=\sum_{i\in N}\left(\sum_{S\in\mathcal{S}^i(R):\, a\in S}x_S-\sum_{S\in\mathcal{S}^{i,2}(R):\, a\in S}x_S\right).$$

Let $\alpha=\sum_{S\in\mathcal{S}^{i,2}(R):\, a\in S}x_S$. Observe that if $e_a^i=0$, then no set $S\in\mathcal{S}^{i,2}(R)$ contains a, thus $\alpha=0$. Otherwise, if $e_a^i=1$, then the second constraint in the LP implies that $\alpha\leqslant 1$. In other words, in any case α is upper-bounded by e_a^i. Using this observation and, additionally, the fact that $\operatorname{def}(a,R)=\operatorname{def}(a,\bar{R})+\sum_{i\in N}e_a^i$, we conclude that

$$\sum_{i\in N}\sum_{S\in\mathcal{S}^i(\bar{R}):\, a\in S}\bar{x}_S\geqslant\sum_{i\in N}\sum_{S\in\mathcal{S}^i(R):\, a\in S}x_S-\sum_{i\in N}e_a^i\geqslant\operatorname{def}(a,R)-\sum_{i\in N}e_a^i=\operatorname{def}(a,\bar{R}),$$

as desired.

It is not hard to see that the objective of $\bar{R}$ is upper bounded by the objective of R. Indeed, the coefficient of each $\bar{x}$-variable in the objective of $\bar{R}$ is at most equal to the coefficient of the x-variable of the corresponding set in $\mathcal{S}^{i,1}(R)$ in R, i.e., the variable $\bar{x}_{S_k^i(\bar{R})}$ is multiplied by k in the objective of $\bar{R}$ while the variable $x_{S^i_{k+r^i(R)-r^i(\bar{R})}(R)}$ is multiplied by $k+r^i(R)-r^i(\bar{R})\geqslant k$ in R. □

In contrast, let us now consider the greedy algorithm. We design a preference profile and a push of a^* that demonstrate that the algorithm is not score monotonic. Agents 1 through 6 vote according to the profile R^N given in Fig. 2(a). The positions marked by "." are placeholders for the rest of the alternatives, in some arbitrary order. Let $A'=\{a_1,\dots,a_4\}$ and $A''=\{b_1,\dots,b_{17}\}$. Notice that $\operatorname{def}(a)=1$ for all $a\in A'$ and $\operatorname{def}(b)=0$ for all $b\in A''$. The optimal sequence of exchanges moves a^* all the way to the top of the preferences of agent 2, with a cost of seven. The greedy algorithm, given this preference profile, indeed chooses this sequence.

On the other hand, consider the profile $(R^1,R^2,Q^3,Q^4,Q^5,Q^6)$ given in Fig. 2(b) (where the position of a^* was improved by two positions in the preferences of agents 3 through 6). First notice that the deficits have not changed compared to the profile R^N. The greedy algorithm would in fact push a^* to the top of the preferences of agents 6, 5, 4, and 3 (in this order), with a total cost of ten. Note that the optimal solution still has a cost of seven. In conclusion, the greedy algorithm is not score monotonic while the LP-based algorithm is score monotonic.

It should be mentioned that the following stronger notion of monotonicity is often considered in the literature: pushing a winning alternative in the preferences of the agents cannot harm it, that is, cannot make it lose the election. We say that a voting rule that satisfies this property is *monotonic*. Interestingly, Dodgson itself is not monotonic [6,19], a fact that is considered by many to be a serious flaw. However, this does not preclude the existence of an approximation algorithm for the Dodgson score that is monotonic as a voting rule. Additionally, there are other prominent social choice properties that are often considered, e.g., homogeneity: a voting rule is said to be homogeneous if duplicating the electorate does not change the outcome of the election.

The existence of algorithms that approximate the Dodgson score well and also satisfy additional social choice properties is addressed by Caragiannis et al. [7]. Among other results, Caragiannis et al. show that a monotonic approximation algorithm for the Dodgson score cannot have an approximation ratio smaller than 2, and complement this result by designing a monotonic exponential-time 2-approximation algorithm. Building on the results in this paper, they are able to construct a monotonic polynomial-time $\mathcal{O}(\log m)$-approximation algorithm. We nevertheless feel that our more preliminary discussion of score monotonicity is worthwhile: in this setting approximation algorithms should also be compared by their social choice properties.

With respect to our approximations, Caragiannis et al. provide the following results (see [7,43] for definitions of the properties discussed below).

Theorem 4.2. *(See [7].) Any homogeneous Dodgson approximation has approximation ratio at least $\Omega(m \log m)$.*

Theorem 4.3. *(See [7].) Let V be a Dodgson approximation. If V satisfies combinativity or Smith consistency, then its approximation ratio is at least $\Omega(nm)$. If V satisfies mutual majority consistency, invariant loss consistency, or independence of closes, then its approximation ratio is at least $\Omega(n)$.*

As a corollary, we get the following result for our Dodgson approximations

Theorem 4.4. *Neither the greedy Dodgson approximation nor the linear programming Dodgson approximation rule satisfies homogeneity, combinativity, Smith consistency, mutual majority consistency, invariant loss consistency, or independence of clones.*

Caragiannis et al. [7] do propose a homogeneous Dodgson approximation (that is also monotonic), but its approximation ratio of $\mathcal{O}(m \log m)$ is inevitably worse than the ratio provided by the Dodgson approximations considered above.

5. Lower bounds for the Dodgson rule

McCabe-Dansted [30] gives a polynomial-time reduction from the Minimum Dominating Set problem to the Dodgson score problem with the following property: given a graph G with k vertices, the reduction creates a preference profile with $n = \Theta(k)$ agents and $m = \Theta(k^4)$ alternatives, such that the size of the minimum dominating set of G is $\lfloor k^{-2}\,\mathrm{sc}_D(a^*)\rfloor$, where $\mathrm{sc}_D(a^*)$ is the Dodgson score of a distinguished alternative $a^* \in A$. We observe that since the Minimum Dominating Set problem is known to be $\mathcal{NP}$-hard to approximate to within logarithmic factors [40], it follows that the Dodgson score problem is also hard to approximate to a factor of $\Omega(\log m)$. Due to the relation of Minimum Dominating Set to Minimum Set Cover, using an inapproximability result due to Feige [17], the explicit inapproximability bound can become $(\frac{1}{4} - \epsilon) \ln m$ under the assumption that problems in $\mathcal{NP}$ do not have quasi-polynomial-time algorithms. This means that our algorithms are asymptotically optimal.

5.1. Inapproximability of the Dodgson score

In the following, we present an alternative and more natural reduction directly from Minimum Set Cover that allows us to obtain a better explicit inapproximability bound. This bound implies that our greedy algorithm is optimal up to a factor of 2.

Theorem 5.1. *There exists a $\beta > 0$ such that it is $\mathcal{NP}$-hard to approximate the Dodgson score of a given alternative in an election with m alternatives to within a factor of $\beta \ln m$. Furthermore, for any $\epsilon > 0$, there is no polynomial-time $(\frac{1}{2} - \epsilon) \ln m$-approximation for the Dodgson score of a given alternative unless all problems in $\mathcal{NP}$ have algorithms running in time $k^{\mathcal{O}(\log\log k)}$, where k is the input size.*

Proof. We use a reduction from Minimum Set Cover (defined formally below, when we present our reduction) and the following well-known statements of its inapproximability.

Theorem 5.2. *(See Raz and Safra [40].) There exists a constant $\alpha > 0$ such that, given an instance $(U, \mathcal{S})$ of Minimum Set Cover with $|U| = n$ and an integer $K \leqslant n$, it is $\mathcal{NP}$-hard to distinguish between the following two cases:*

- *$(U, \mathcal{S})$ has a cover of size at most K.*
- *Any cover of $(U, \mathcal{S})$ has size at least $\alpha K \ln n$.*

Theorem 5.3. *(See Feige [17].) For any constant $\epsilon > 0$, given an instance $(U, \mathcal{S})$ of Minimum Set Cover with $|U| = n$ and an integer $K \leqslant n$, there is no polynomial-time algorithm that distinguishes between the following two cases:*

- *$(U, \mathcal{S})$ has a cover of size at most K, and*
- *Any cover of $(U, \mathcal{S})$ has size at least $(1 - \epsilon) K \ln n$,*

unless $\mathcal{NP} \subseteq DTIME(n^{O(\log\log n)})$.

The known inapproximability results for the problems that we use in our proofs are better understood by considering their relation to a generic NP-hard problem such as Satisfiability (see [44], Chapter 29). For example, what Theorem 5.2 essentially states is that there exists a polynomial-time reduction which, on input an instance ϕ of Satisfiability, constructs an instance $(U, \mathcal{S})$ of Minimum Set Cover with the following properties: if ϕ is satisfiable, $(U, \mathcal{S})$ has a cover of size at most K, and if ϕ is not satisfiable, any cover of $(U, \mathcal{S})$ has size at least $\alpha K \ln n$. This reduction implies that it is NP-hard to approximate Minimum Set Cover within a factor of $\alpha \ln n$. The interpretation of the remaining inapproximability results that are used or proved in the paper is similar.

Given an instance of Minimum Set Cover consisting of a set of n elements, a collection of sets over these elements and an integer $K \leqslant n$, we construct a preference profile with $m = (1+\zeta)n + \lceil \alpha\zeta K n \ln n \rceil + 1$ alternatives and a specific alternative a^* in which we show that if we could distinguish in polynomial time between the following two cases:

- a^* has Dodgson score at most $(1+\zeta)Kn$, and
- a^* has Dodgson score at least $\alpha\zeta K n \ln n$,

then we could have distinguished between the two cases of Theorems 5.2 and 5.3 for the original Minimum Set Cover instance, contradicting the above inapproximability statements. Here, α is the inapproximability constant in Theorem 5.2 or 5.3 (in the latter $\alpha = 1 - \epsilon$), and ζ is an arbitrarily large positive constant. In this way, we obtain an inapproximability bound of $\frac{\alpha\zeta}{1+\zeta} \ln n$. Since $m = (1+\zeta)n + \lceil \alpha\zeta K n \ln n \rceil + 1$, it holds that $\ln n \geqslant \frac{1}{2} \ln m - \mathcal{O}(\ln \ln m)$, and hence the inapproximability bound for Dodgson score can be expressed in terms of the number of alternatives m as stated in Theorem 5.1.

We now present our reduction. Given an instance $(U, \mathcal{S})$ of Minimum Set Cover consisting of a set U of n elements, a collection $\mathcal{S}$ of sets $S_1, S_2, \ldots, S_{|\mathcal{S}|}$ and an integer $K \leqslant n$, we construct the following preference profile. There are the following alternatives:

- A set of n basic alternatives, each corresponding to an element of U. Abusing notation, we also call this set U.
- A set Z of ζn alternatives where ζ is a positive constant.
- A set F of $\lceil \alpha\zeta K n \ln n \rceil$ alternatives, where α is the constant from Theorem 5.2.
- A specific alternative a^*.

There are the following $2|\mathcal{S}| + 1$ agents:

- A critical agent ℓ^i for each set $S_i \in \mathcal{S}$.
- An indifferent agent r^i for each set $S_i \in \mathcal{S}$.
- A special agent v^*.

The preferences of the agents are defined as follows:

- The special agent v^* ranks a^* in the first position of its preferences and the rest of the alternatives occupy the remaining positions in arbitrary order (i.e., $a^* R^{v^*} (U \cup Z \cup F)$).
- The critical agent ℓ^i ranks the basic alternatives corresponding to the elements of S_i in the first positions of its preference (in arbitrary order), next the alternatives of Z, next a^*, next the alternatives of F, and, in the last positions of its preference, the basic alternatives corresponding to the elements in $U \backslash S_i$ (i.e., $S_i R^{\ell^i} Z R^{\ell^i} a^* R^{\ell^i} F R^{\ell^i} (U \backslash S_i)$).
- We construct the ranking of the indifferent agents as follows. Initialize $S'_1, S'_2, \ldots, S'_{|\mathcal{S}|}$ as $S'_1 \leftarrow S_1, S'_2 \leftarrow S_2, \ldots, S'_{|\mathcal{S}|} \leftarrow S_{|\mathcal{S}|}$. For each element u of U, choose arbitrarily $j \in \{1, 2, \ldots, |\mathcal{S}|\}$ such that $u \in S'_j$ and set $S'_j \leftarrow S'_j \setminus \{u\}$. Denote by $\mathcal{S}'$ the collection $S'_1, S'_2, \ldots, S'_{|\mathcal{S}|}$ resulting after each $u \in U$ has been processed in this way. The indifferent agent r^i ranks the basic alternatives corresponding to the elements in $U \setminus S'_i$ in the first positions of its preference, followed the alternatives of F, then a^*, then the alternatives of Z, and then in the last positions of its preference the basic alternatives corresponding to elements in S'_i (if any)—i.e., $(U \setminus S'_i) R^{r^i} F R^{r^i} a^* R^{r^i} Z R^{r^i} S'_i$.

Clearly, a^* is preferred to any alternative in Z by the special agent and by the $|\mathcal{S}|$ indifferent agents, i.e., by a majority of agents. Similarly, a^* is preferred to any alternative in F by the special agent and by the $|\mathcal{S}|$ critical agents. Now, for each element of U, denote by f_u the number of sets in $\mathcal{S}$ that contain u. Then, a^* is preferred to u by the special agent, by the $|\mathcal{S}| - f_u$ critical agents corresponding to sets in $\mathcal{S}$ that do not contain u, and by the $f_u - 1$ indifferent agents corresponding to sets in $\mathcal{S}'$ that contain u (i.e., by $|\mathcal{S}|$ agents in total). Hence, a^* has a deficit of exactly 1 with respect to each of the alternatives in U.

Theorem 5.1 follows by the next two lemmas that give bounds on the Dodgson score of alternative a^* in the two cases of interest: when $(U, \mathcal{S})$ has a cover of size at most K (Lemma 5.4) and when any cover of $(U, \mathcal{S})$ has size at least $\alpha K \ln n$ (Lemma 5.5).

Lemma 5.4. *If $(U, \mathcal{S})$ has a cover of size K, then a^* has Dodgson score at most $(1+\zeta)Kn$.*

Proof. Let $H \subseteq \mathcal{S}$ be a cover for $(U, \mathcal{S})$ with $|H| = K$. By the definition of a cover, H covers all elements of U. Hence, by pushing a^* to the first position in the preference of the critical agent ℓ^i such that $S_i \in H$, a^* will decrease its deficit with respect to each of the basic alternatives by 1, and hence it will become a Condorcet winner. The total number of positions a^* rises is at most $|H| \cdot (|Z| + n) = (1 + \zeta)nK$. □

Lemma 5.5. *If every cover of $(U, \mathcal{S})$ has size at least $\alpha K \ln n$, then a^* has Dodgson score at least $\alpha\zeta K n \ln n$.*

Proof. We first assume that the minimum number of positions a^* has to rise in order to beat the basic alternatives and become a Condorcet winner includes raising a^* by at least $|F|$ positions in the ranking of some indifferent agent r^i. Hence, a^* rises $|F|$ positions in the preference of r^i in order to reach position $|U \setminus S'_i| + 1$ and at least n additional positions in order to beat the basic alternatives. Its Dodgson score is thus at least $|F| + n \geqslant \alpha\zeta K n \ln n$.

Now, assume that the minimum number of positions a^* has to rise in order to beat the basic alternatives does not include raising a^* by at least $|F|$ positions in the ranking of some indifferent agent. We will show that if the Dodgson score of a^* is less than $\alpha\zeta K n \ln n$, then there exists a cover of $(U, \mathcal{S})$ of size less than $\alpha K \ln n$, contradicting the assumption of the lemma.

Let H be the set of critical agents in whose preferences a^* is pushed at least $|Z|$ positions higher. Over all the preference lists of all the agents in H, a^* rises a total of $|H| \cdot |Z|$ positions in order to reach position $|S_i| + 1$ in each list, plus at least n additional positions in order to decrease by 1 its deficit with respect to each of the alternatives in U. So, recalling $|Z| = \zeta n$, a^* rises at least $\zeta |H| n + n$ positions. Denoting the Dodgson score of a^* by $\mathrm{sc}_D(a^*)$, we thus have $|H| \leqslant \frac{1}{\zeta n} \mathrm{sc}_D(a^*) - \frac{1}{\zeta} < \alpha K \ln n$. The proof is completed by observing that the union of the sets S_i for each critical agent ℓ^i belonging to H contains all the basic alternatives, i.e., H corresponds to a cover for $(U, \mathcal{S})$ of size less than $\alpha K \ln n$. □

This completes the proof of Theorem 5.1. □

5.2. Inapproximability of Dodgson rankings

A question related to the approximability of Dodgson scores is the approximability of the Dodgson ranking, that is, the ranking of alternatives given by ordering them by nondecreasing Dodgson score. To the best of our knowledge, no rank aggregation function, which maps preference profiles to rankings of the alternatives, is known to provably produce rankings that are close to the Dodgson ranking [38,39,27–29] (see the survey of related work in Section 1).

Our next result establishes that efficient approximation algorithms for Dodgson ranking are unlikely to exist unless $\mathcal{P} = \mathcal{NP}$. It does so by proving that the problem of distinguishing between whether a given alternative is the unique Dodgson winner or in the last $O(\sqrt{m})$ positions is $\mathcal{NP}$-hard. This result provides a complexity-theoretic explanation for the sharp discrepancies observed in the Social Choice Theory literature when comparing Dodgson elections with simpler, efficiently computable, voting rules.

Theorem 5.6. *Given a preference profile with m alternatives and an alternative a^*, it is $\mathcal{NP}$-hard to decide whether a^* is a Dodgson winner or has rank at least $m - 6\sqrt{m}$ in any Dodgson ranking.*

Proof. We use a reduction from Minimum Vertex Cover in 3-regular graphs, and exploit a result concerning its inapproximability that follows from the work of Berman and Karpinski [3]. Our approach is similar to the proof of Theorem 5.1, albeit considerably more involved. We use the following result.

Theorem 5.7. *(See Berman and Karpinski [3], see also [25].) Given a 3-regular graph G with $n = 22t$ nodes for some integer $t > 0$ and an integer K in $[n/2, n-6]$, it is $\mathcal{NP}$-hard to distinguish between the following two cases:*

- *G has a vertex cover of size at most K.*
- *Any vertex cover of G has size at least $K + 6$.*

Given an instance of Minimum Vertex Cover consisting of a 3-regular graph G with $n = 22t$ nodes $v_0, v_1, \ldots, v_{n-1}$ and an integer $K \in [n/2, n-6]$, we construct in polynomial time a preference profile in which if we could distinguish whether a particular alternative is a Dodgson winner or not very far from the last position in any Dodgson ranking, then we could also distinguish between the two cases mentioned in Theorem 5.7 for the original Minimum Vertex Cover instance. See page 46 for an example of the construction. The Dodgson election has the following sets of alternatives:

- A special alternative a^*.
- A set F of $4Kn/11 + 3n/2$ alternatives. These alternatives are partitioned into n disjoint blocks $F_0, F_1, \ldots, F_{n-1}$ so that each block contains either $\lceil 4K/11 + 3/2 \rceil$ or $\lfloor 4K/11 + 3/2 \rfloor$ alternatives.
- A set A of n alternatives $a_0, a_1, \ldots, a_{n-1}$.

- An alternative u_j for each edge e_j of G. Let U be the set of these $3n/2$ alternatives. We denote by S_i the set of the three alternatives of U that correspond to the edges of G which are incident to node v_i.

For each node v_i of G, there are two agents: one left agent ℓ^i and one right agent r^i. The preferences of the left agent ℓ^i are as follows:

- The three alternatives of S_i are ranked by agent ℓ^i in the first three positions of its preference (in arbitrary order).
- From position 4 to position $4n/11+3$, agent ℓ^i ranks the alternatives $a_i, a_{(i+1) \bmod n}, \dots, a_{(i+4n/11-1) \bmod n}$ in this order.
- In position $4n/11+4$, agent ℓ^i ranks a^*.
- From position $4n/11+5$ to position $4Kn/11+41n/22+4$, agent ℓ^i ranks the alternatives of F in the following order. The alternatives of F_i are ranked in positions from $4n/11+5$ to $4n/11+4+|F_i|$ (in arbitrary order). Next, agent ℓ^i ranks the alternatives of sets $F_0, \dots, F_{i-1}, F_{i+1}, \dots, F_{n-1}$ in this order (the relative order of the alternatives of the same block is arbitrary).
- From position $4Kn/11+41n/22+5$ to position $4Kn/11+5n/2+4$, agent ℓ^i ranks the alternatives $a_{(i+4n/11) \bmod n}, a_{(i+4n/11+1) \bmod n}, \dots, a_{(i-1) \bmod n}$ in this order.
- In the last $3n/2-3$ positions, agent ℓ^i ranks the alternatives of $U \setminus S_i$ (in arbitrary order).

The preferences of the right agent r^i are as follows:

- In the first $3n/2-3$ positions, agent r^i ranks the alternatives of $U \setminus S_i$ in reverse relative order to the order ℓ^i ranks them.
- From position $3n/2-2$ to position $4Kn/11+3n-|F_i|-3$, agent r^i ranks the alternatives of the blocks $F_{n-1}, F_{n-2}, \dots, F_{i+1}, F_{i-1}, \dots, F_0$ in this order so that the alternatives of block F_j are ranked in reverse relative order to the order ℓ^i ranks them.
- From position $4Kn/11+3n-|F_i|-2$ to position $4Kn/11+40n/11-|F_i|-5$, agent r^i ranks the alternatives $a_{(n-i-1) \bmod n}, a_{(n-i-2) \bmod n}, \dots, a_{(4n/11-i+2) \bmod n}$ in this order.
- In position $4Kn/11+40n/11-|F_i|-4$, agent r^i ranks a^*.
- From position $4Kn/11+40n/11-|F_i|-3$ to position $4Kn/11+40n/11-4$, agent r^i ranks the alternatives of F_i in reverse relative order to the order ℓ^i ranks them.
- From position $4Kn/11+40n/11-3$ to position $4Kn/11+4n-2$, agent r^i ranks the alternatives $a_{(4n/11-i+1) \bmod n}, a_{(4n/11-i) \bmod n}, \dots, a_{(n-i) \bmod n}$ in this order.
- The three alternatives of S_i are ranked in the last three positions in the preference of agent r^i, in reverse relative order to the order ℓ^i ranks them.

We observe that a^* beats all alternatives but the alternatives of U. In particular, a^* is preferred to each alternative of F by $n+1$ agents. Specifically, a^* is ranked above an alternative belonging to the block F_i by the n left agents and by the right agent r^i. Also, the alternative a_i is ranked below a^* by the $7n/11$ left agents $\ell^{(i+1) \bmod n}, \ell^{(i+2) \bmod n}, \dots, \ell^{(i+7n/11-1) \bmod n}$ and by the $4n/11+2$ right agents $r^{(i+7n/11-1) \bmod n}, r^{(i+7n/11) \bmod n}, \dots, r^i$. Hence, a^* beats all alternatives in set A as well since it is ranked above each of them by $n+2$ agents. Also, a^* is ranked above the alternative u_j corresponding to the edge e_j of G by the left agents ℓ^i and $\ell^{i'}$ and by all right agents besides r^i and $r^{i'}$ so that nodes v_i and $v_{i'}$ are the endpoints of edge e_j in G. Hence, a^* has a deficit of 1 with respect to each of the alternatives in U.

We also observe that the alternatives in F beat each alternative in A. Note that each agent other than r^i who prefers a^* to an alternative in A also prefers an alternative in block F_i to the alternative in A. Hence, each alternative of F beats each alternative of A since it is ranked above it by $n+1$ agents. Furthermore, similarly to a^*, each alternative in F is preferred to each alternative of U by n agents. Also, when an alternative f of F is ranked above another alternative f' of F by agent ℓ^i, f' is ranked above f by agent r^i. Hence, an alternative of F has a deficit of 1 with respect to U and each other alternative in F, and a deficit of 2 with respect to a^*.

Furthermore, observe that each alternative in A is ranked above the alternative u_j corresponding to the edge e_j of G by the left agents ℓ^i and $\ell^{i'}$ and by all right agents besides r^i and $r^{i'}$ so that nodes v_i and $v_{i'}$ are the endpoints of edge e_j in G, i.e., by n agents. Also, when an alternative a of A is preferred to another alternative a' of A by agent ℓ^i, a' is preferred to a by agent r^i. Hence, an alternative in A has a deficit of 1 with respect to each alternative in U and each remaining alternative in A, a deficit of 2 with respect to each alternative in F, and a deficit of 3 with respect to a^*. This immediately yields that the Dodgson score of each alternative in A is at least $8Kn/11+11n/2+2$.

Similarly, when an alternative u of U is preferred to another alternative u' of U by agent ℓ^i, u' is preferred to u by agent r^i. Hence, an alternative in U has a deficit of 1 with respect to each of the alternatives in A and F, each other alternative in U, and a^*. This immediately yields that the Dodgson score of each alternative in U is at least $4Kn/11+4n$. A summary of the deficit of each alternative with respect to any other alternative is presented in Table 1.

The next lemma gives upper and lower bounds on the Dodgson score of the alternatives in F.

Lemma 5.8. *Each alternative in F has Dodgson score between $4Kn/11+3n+1$ and $4Kn/11+37n/11+2K/11+3/4$.*

Table 1
The deficit of each alternative (rows) against any other alternative (columns).

	a^*	Any alt. in F	Any alt. in A	Any alt. in U
a^*	–	0	0	1
Any alt. in F	2	1	0	1
Any alt. in A	3	2	1	1
Any alt. in U	1	1	1	1

Proof. Since each alternative in F has a deficit of 1 with respect to each alternative in U and each other alternative in F, and a deficit of 2 with respect to a^*, its Dodgson score is at least $|U| + |F| - 1 + 2 = 4Kn/11 + 3n + 1$.

Now, consider an alternative f belonging to block F_i. f is at distance at most

$$\left\lfloor \frac{|F_i| - 1}{2} + 1 \right\rfloor \leqslant \frac{|F_i| + 1}{2} \leqslant \frac{\lceil 4K/11 + 3/2 \rceil + 1}{2} \leqslant 2K/11 + 7/4$$

from a^* in the preferences of either the left agent ℓ^i or the right agent r^i. Hence, by raising f at most $2K/11 + 7/4$ positions in the preferences of either ℓ^i or r^i, its deficit with respect to a^* decreases by 1. Consider a left agent $\ell^{i'}$ with $i' \neq i$ and let F' be the subset of alternatives in F that are higher than f in the preferences of $\ell^{i'}$. By pushing f to the first position in the preferences of agent $\ell^{i'}$ (i.e., $4n/11 + 3 + |F'|$ positions in addition to the $2K/11 + 7/4$ positions mentioned above), f decreases its deficit by 1 with respect to each alternative of F' and a^*, as well as with respect to the three alternatives of $S_{i'}$ in the first three positions in the preferences of $\ell^{i'}$. Now, consider the right agent $r^{i'}$. In the preferences of $r^{i'}$, f is ranked higher than the alternatives in F' and lower than the alternatives in $F \backslash F' - \{f\}$. Hence, by pushing f to the first position in the preferences of agent $r^{i'}$ (i.e., $|F \backslash F' - \{f\}| + 3n/2 - 3 = 4Kn/11 - |F'| + 3n - 4$ additional positions), f decreases its deficit by 1 with respect to each alternative of $F \backslash F' - \{f\}$ as well the alternatives of $U \backslash S_{i'}$ in the first $3n/2 - 3$ positions in the preferences of r^i. Hence, by pushing $4Kn/11 + 37n/11 + 2K/11 + 3/4$ positions, f becomes a Condorcet winner. $\square$

The next two lemmas give bounds on the Dodgson score of alternative a^* in the two cases of interest: when G has a vertex cover of size at most K (Lemma 5.9), and when any vertex cover of G has size at least $K + 6$ (Lemma 5.10).

Lemma 5.9. *If G has a vertex cover of size at most K, then the Dodgson score of a^* is less than $4Kn/11 + 3n$.*

Proof. Let $H \subseteq V$ be a vertex cover of G with $|H| = K$. By the definition of the vertex cover, H covers all edges of G and this implies that $\bigcup_{i: v_i \in H} S_i = U$. Hence, by pushing a^* to the first position in the preferences of each of the K left agents ℓ^i such that $v_i \in H$, a^* decreases it deficit with respect to each of the alternatives in U by 1, and becomes a Condorcet winner. The total number of positions a^* rises is $K(4n/11 + 3) < 4Kn/11 + 3n$. The last inequality is true since $K < n$. $\square$

Lemma 5.10. *If any vertex cover of G has size at least $K + 6$, then the Dodgson score of a^* is larger than* $4Kn/11 + 37n/11 + 2K/11 + 3/4$.

Proof. First assume that the minimum sequence of exchanges that makes a^* beat the alternatives of U and become a Condorcet winner includes pushing a^* to one of the first $3n/2 - 3$ positions in the preferences of some right agent r^i. Certainly, not all alternatives of U are beaten in this way since the three alternatives of S_i are ranked below a^* by agent r^i. So, in order to beat the remaining 3 alternatives of S_i, a^* has to either be pushed to one of the first three positions of a left agent or to one of the first $3n/2 - 3$ positions of another right agent $r^{i'}$ with $i' \neq i$. Hence, a^* must be first pushed to position $3n/2 - 2$ of agent r^i (i.e., $|F \backslash F_i| + 7n/11 - 2$ positions), to position 4 of a left agent ($4n/11$ additional positions) or to position $3n/2 - 2$ of agent $r^{i'}$ ($|F \backslash F_{i'}| + 7n/11 - 2$ additional positions), and then rise at least $3n/2$ additional positions in order to beat all alternatives of U. In total, a^* rises at least

$$\begin{aligned}
&|F \backslash F_i| + 7n/11 - 2 + \min\{4n/11, |F \backslash F_{i'}| + 7n/11 - 2\} + 3n/2 \\
&\quad \geqslant |F| - |F_i| + 5n/2 - 2 \\
&\quad \geqslant 4Kn/11 + 4n - \lceil 4K/11 + 3/2 \rceil - 2 \\
&\quad \geqslant 4Kn/11 + 4n - 4K/11 - 9/2 \\
&\quad = 4Kn/11 + 37n/11 + n/11 + 6n/11 - 4K/11 - 9/2 \\
&\quad \geqslant 4Kn/11 + 37n/11 + 22/11 + 6(K+6)/11 - 4K/11 - 9/2 \\
&\quad > 4Kn/11 + 37n/11 + 2K/11 + 3/4
\end{aligned}$$

positions. The fourth inequality holds since $n \geqslant 22$ and $n \geqslant K + 6$.

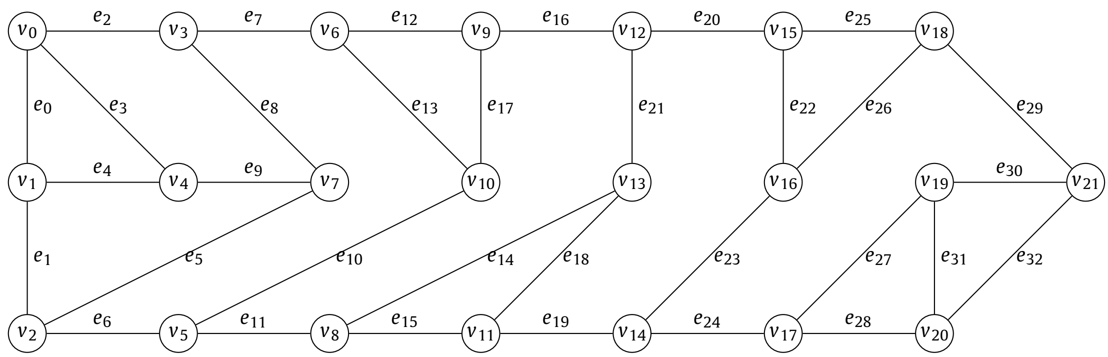

Fig. 3. A 3-regular graph with 22 nodes.

Now, assume that the minimum sequence of exchanges for making a^* a Condorcet winner does not include raising a^* to any of the first $3n/2-3$ positions of any right agent. We will show that if a^* has Dodgson score at most $4Kn/11+37n/11+2K/11+3/4$, then G has a vertex cover of size less than $K+6$, contradicting the assumption of the lemma.

Let H be the set of left agents where a^* rises to one of the first three positions in order to beat all the alternatives of U. In total, a^* rises $4|H|n/11$ positions in order to reach position 4 in the preferences of each of the agents in H plus at least $3n/2$ additional positions in order to decrease its deficit with respect to the alternatives in U by at least 1, i.e., at least $4|H|n/11+3n/2$ positions in total. Hence, by denoting the Dodgson score of a^* by $\mathrm{sc}_D(a^*)$, we have $|H| \leqslant \frac{11}{4n}(\mathrm{sc}_D(a^*)-3n/2)$.

Since $\bigcup_{i:\ell^i\in H} S_i = U$, the set of nodes of G consisting of nodes v_i such that agent ℓ^i belongs to H is a vertex cover of G of size $|H|$. Assuming that the Dodgson score of a^* is at most $4Kn/11+37n/11+2K/11+3/4$, we have

$$\begin{aligned}|H| &\leqslant \frac{11}{4n}\big(\mathrm{sc}_D\big(a^*\big)-3n/2\big)\\ &\leqslant \frac{11}{4n}(4Kn/11+41n/22+2K/11+3/4)\\ &< K+6,\end{aligned}$$

where the last inequality follows since $K \leqslant n-6$. □

By Lemmas 5.8, 5.9, and 5.10, we obtain that if G has a vertex cover of size at most K, then a^* is the unique Dodgson winner, while if every vertex cover of G has size at least $K+6$, then a^* is below all alternatives in F in any Dodgson ranking. Denote by m the total number of alternatives and recall that $m=|F|+|A|+|U|+1=4Kn/11+4n+1$. Then, the rank of a^* in the second case is at least

$$\begin{aligned}|F|+1 &= 4Kn/11+3n/2+1 = m-5n/2 = m-\sqrt{25n^2/4}\\ &\geqslant m-\sqrt{25nK/2} \geqslant m-6\sqrt{4Kn/11+4n+1} = m-6\sqrt{m},\end{aligned}$$

where the first inequality follows since $K \geqslant n/2$. By Theorem 5.7, we obtain the desired result. □

An example. We present an example of the construction in the proof of Theorem 5.6. Consider an instance of Minimum Vertex Cover with the 22-node 3-regular graph of Fig. 3, and $K=12$.

The corresponding preference profile has 185 alternatives and 44 agents. In particular, the set F has 129 alternatives $f_0, f_1, \dots, f_{128}$, which are partitioned into 22 blocks as follows. Block F_0 contains the six alternatives $f_0, f_1, \dots, f_5$, block F_1 contains the six alternatives $f_6, \dots, f_{11}, \dots,$ block F_{18} contains the six alternatives $f_{108}, \dots, f_{113}$, block F_{19} contains the five alternatives $f_{114}, \dots, f_{118}, \dots,$ and block F_{21} contains the alternatives $f_{124}, \dots, f_{128}$. The set A has 22 alternatives $a_0, \dots, a_{21}$. The set U has 33 alternatives $u_0, \dots, u_{32}$, one alternative for each edge of the graph. The agents are partitioned into 22 left agents and 22 right agents. In order to compute the preferences of an agent, say agent ℓ^{17}, we first compute the set S_{17}, which contains the alternatives corresponding to the edges incident to node v_{17} of the graph, i.e., $S_{17}=\{u_{24}, u_{27}, u_{28}\}$. Now, the preferences of agent ℓ^{17} are:

$$\begin{aligned}&S_{17}R^{\ell^{17}}a_{17}R^{\ell^{17}}a_{18}R^{\ell^{17}}\cdots R^{\ell^{17}}a_1R^{\ell^{17}}a_2R^{\ell^{17}}a^*R^{\ell^{17}}F_{17}R^{\ell^{17}}F_0R^{\ell^{17}}\cdots F_{16}\\ &R^{\ell^{17}}F_{18}R^{\ell^{17}}\cdots R^{\ell^{17}}F_{21}R^{\ell^{17}}a_3R^{\ell^{17}}\cdots R^{\ell^{17}}a_{16}R^{\ell^{17}}(U\setminus S_{17}).\end{aligned}$$

Similarly, the preferences of agent r^{17} are:

$$(U \setminus \overleftarrow{S_{17}}) R^{\ell^{17}} \overleftarrow{F_{21}} R^{\ell^{17}} \overleftarrow{F_{20}} R^{\ell^{17}} \cdots R^{\ell^{17}} \overleftarrow{F_{18}} R^{\ell^{17}} \overleftarrow{F_{16}} R^{\ell^{17}} \cdots R^{\ell^{17}} \overleftarrow{F_0} R^{\ell^{17}}$$
$$a_4 R^{\ell^{17}} a_3 R^{\ell^{17}} \cdots R^{\ell^{17}} a_{15} R^{\ell^{17}} a^* R^{\ell^{17}} \overleftarrow{F_{17}} a_{16} R^{\ell^{17}} a_5 R^{\ell^{17}} \overleftarrow{S_{17}},$$

where the symbol $\leftarrow$ on top of a set of alternatives is used to denote that their order in the preferences of r^{17} is the reverse of the order ℓ^{17} ranks them.

6. Inapproximability of Young scores and rankings

Recall that the Young score of a given alternative $a^* \in A$ is the size of the largest subset of agents for which a^* is a Condorcet winner.

It is straightforward to obtain a simple ILP for the Young score problem. As before, let $a^* \in A$ be the alternative whose Young score we wish to compute. Let the variables of the program be $x^i \in \{0,1\}$ for all $i \in N$; $x^i = 1$ if and only if agent i is included in the subset of agents for a^*. Define constants $e_a^i \in \{-1, 1\}$ for all $i \in N$ and $a \in A \setminus \{a^*\}$, which depend on the given preference profile; $e_a^i = 1$ if and only if agent i ranks a^* higher than a. The ILP that computes the Young score of a^* is given by:

$$\begin{aligned} &\text{maximize} \quad \sum_{i \in N} x^i \\ &\text{subject to} \quad \forall a \in A \setminus \{a^*\}, \quad \sum_{i \in N} x^i e_a^i \geqslant 1 \\ &\qquad\qquad\quad \forall i \in N, \quad x^i \in \{0,1\}. \end{aligned} \tag{4}$$

The ILP (4) for the Young score is seemingly simpler than the one for the Dodgson score. This might seem to indicate that the problem can be easily approximated by similar techniques. Therefore, the following result is quite surprising.

Theorem 6.1. *It is $\mathcal{NP}$-hard to approximate the Young score by any factor.*

Proof. This result becomes more self-evident when we notice that the Young score has the rare property of being non-monotonic as an optimization problem, in the following sense: given a subset of agents that make a^* a Condorcet winner, it is not necessarily the case that a smaller subset of the agents would satisfy the same property. This stands in contrast to many approximable optimization problems, in which a solution that is worse than an optimal solution is also a valid solution. Consider the Set Cover problem, for instance: if one adds more subsets to a valid cover, one obtains a valid cover. The same goes for the Dodgson score problem: if a sequence of exchanges makes a^* a Condorcet winner, introducing more exchanges on top of the existing ones would not undo this fact.

In order to prove the inapproximability of the Young score, we define the following problem.

NONEMPTY SUBSET
Instance: An alternative a^*, and a preference profile $R^N \in L^N$.
Question: Is there a nonempty subset of agents $C \subseteq N$, $C \neq \emptyset$, for which a^* is a Condorcet winner?

To prove Theorem 6.1, it is sufficient to prove that NONEMPTY SUBSET is $\mathcal{NP}$-hard. Indeed, this implies that it is $\mathcal{NP}$-hard to distinguish whether the Young score of a given alternative is zero or greater than zero, which directly entails that the score cannot be approximated.

Lemma 6.2. NONEMPTY SUBSET *is $\mathcal{NP}$-complete.*

Proof. The problem is clearly in $\mathcal{NP}$; a witness is given by a nonempty set of agents for which a^* is a Condorcet winner.

In order to show $\mathcal{NP}$-hardness, we present a polynomial-time reduction from the $\mathcal{NP}$-hard Exact Cover by 3-Sets (X3C) problem [20] to our problem. An instance of the X3C problem includes a finite set of elements U, $|U| = n$ (where n is divisible by 3), and a collection $\mathcal{S}$ of 3-element subsets of U, $\mathcal{S} = \{S_1, \ldots, S_k\}$, such that for every i, $1 \leqslant i \leqslant k$, $S_i \subseteq U$ and $|S_i| = 3$. The question is whether the collection $\mathcal{S}$ contains an *exact cover* for U, i.e., a subcollection $\mathcal{S}^* \subseteq \mathcal{S}$ of size $n/3$ such that every element of U occurs in exactly one subset in $\mathcal{S}$.

We next give the details of the reduction from X3C to NONEMPTY SUBSET. Given an instance of X3C, defined by the set U and a collection of 3-element sets $\mathcal{S}$, we construct the following instance of NONEMPTY SUBSET.

Define the set of alternatives as $A = U \cup \{a\} \cup \{a^*\}$. Let the set of agents be $N = N' \cup N''$, where N' and N'' are defined as follows. The set N' is composed of k agents, corresponding to the k subsets in $\mathcal{S}$, such that for all $i \in N'$, agent i prefers the alternatives in $U \setminus S_i$ to a^*, and prefers a^* to all the alternatives in $S_i \cup \{a\}$ (i.e., $(U \setminus S_i)\ R^i\ a^* R^i\ (S_i \cup \{a\})$). Subset N'' is composed of $\frac{n}{3} - 1$ agents who prefer a to a^* and a^* to U (i.e., for all $i \in N''$, $aR^i a^* R^i U$).

We next show that there is an exact cover in the given instance if and only if there is nonempty subset of agents for which a^* is a Condorcet winner in the constructed instance.

Sufficiency: Let $\mathcal{S}^*$ be an exact cover by 3-sets of U, and let $N^* \subseteq N'$ be the subset of agents corresponding to the $\frac{n}{3}$ subsets $S_i \in \mathcal{S}^*$. We show that a^* is a Condorcet winner for $C = N^* \cup N''$. Since $\mathcal{S}^*$ is an exact cover, for all $b \in U$ there exists exactly one agent in N^* that prefers a^* to b and $\frac{n}{3} - 1$ agents in N^* that prefer b to a^*. In addition, all $\frac{n}{3} - 1$ agents in N'' prefer a^* to b. Therefore, a^* beats b in a pairwise election.

It remains to show that a^* beats a in a pairwise election. This is true since all $\frac{n}{3}$ agents in N^* prefer a^* to a, and there are only $\frac{n}{3} - 1$ agents in N'' who prefer a to a^*. It follows that a^* is a Condorcet winner for $N^* \cup N''$.

Necessity: Assume the given instance of X3C has no exact cover. We have to show that there is no subset of agents for which a^* is a Condorcet winner. Let $C \subseteq N$, $C \neq \emptyset$, and let $N^* = C \cap N'$. We distinguish among three cases.

Case 1: $|N^*| = 0$. It must hold that $C \cap N'' \neq \emptyset$. In this case, a^* loses to a in a pairwise election, since all the agents in N'' prefer a to a^*.

Case 2: $0 < |N^*| \leqslant \frac{n}{3}$. Since there is no exact cover, the corresponding sets S_i cannot cover U. Thus there exists $b \in U$ that is ranked higher than a^* by all agents in N^*. In order for a^* to beat b in a pairwise election, C must include at least $|N^*| + 1$ agents from N''. However, this means that a beats a^* in a pairwise election (since a is ranked lower than a^* by $|N^*|$ agents, and higher than a^* by at least $|N^*| + 1$ agents). It follows that a^* is not a Condorcet winner for C.

Case 3: $|N^*| > \frac{n}{3}$. Let us award each alternative $b \in A \setminus \{a^*\}$ a point for each agent that ranks it above a^*, and subtract a point for each agent that ranks it below a^*. a^* is a Condorcet winner if and only if the score of every $b \in A \setminus \{a^*\}$, counted this way, is negative. This implies that a^* is a Condorcet winner only if for every subset $B \subseteq A$ of alternatives, the total score of the alternatives in B is at most $-|B|$.

We shall calculate the total score of the alternatives in U from the agents in N^*. Every agent in N^* prefers a^* to 3 alternatives in U and prefers $n-3$ alternatives in U to a^*. Thus, every agent in N^* contributes $(n-3) - 3 = n - 6$ points to the total score of U. Summing over all the agents in N^*, we have that the total score of U from N^* is $|N^*|(n-6)$. By $|N^*| > \frac{n}{3}$, we have that

$$|N^*|(n-6) \geqslant \left(\left(\frac{n}{3} - 1\right) + 2\right)(n-6) = \left(\frac{n}{3} - 1\right)n - 6.$$

Recall that every agent in N'' prefers a^* to all alternatives in U. However, since $|N''| = \frac{n}{3} - 1$, agents from N'' can only subtract $(\frac{n}{3} - 1)n$ from the total score of U. We conclude that the total score of U is at least -6. Since we can assume that $|U| = n > 6$,[3] a^* cannot beat all the alternatives in U in pairwise elections. □

This concludes the proof of Theorem 6.1. □

A short discussion is in order. Theorem 6.1 states that the Young score cannot be efficiently approximated to any factor. The proof shows that, in fact, unless $\mathcal{P} = \mathcal{NP}$ it is impossible to efficiently distinguish between a zero and a nonzero score. However, the proof actually shows more: it constructs a family of instances, where it is hard to distinguish between a score of zero and almost $2m/3$. Now, if one looks at an alternative formulation of the Young score problem where all the scores are scaled by an additive constant, it is no longer true that it is hard to approximate the score to *any* factor; however, the proof still shows that it is hard to approximate the Young score, even under this alternative formulation, to a factor of $\mathcal{O}(m)$.

The strong inapproximability result for the Young score intuitively implies that the Young ranking cannot be approximated. The following corollary, whose proof is a straightforward variation on the proof of Lemma 6.2, shows that this is indeed the case. It can be viewed as an analog of Theorem 5.6 for Young.

Corollary 6.3. *For any constant $\epsilon > 0$, given a preference profile with m alternatives and an alternative a^*, it is $\mathcal{NP}$-hard to decide whether a^* has rank $\mathcal{O}(m^\epsilon)$ or is ranked in place m (that is, ranked last) in any Young ranking.*

Proof. Let $\epsilon > 0$ be a constant. We perform the same reduction as before, with the following differences. Let A' be the set of alternatives constructed in the reduction of Lemma 6.2, and $m' = |A'|$; we add a set B of $(m')^{1/\epsilon}$ additional alternatives, i.e., $A = A' \cup B$, $m = |A| = m' + (m')^{1/\epsilon}$. The set of agents is $N' \cup N'' \cup N^*$, the preferences of N' and N'' restricted to A' are as before, and all these agents rank B at the bottom. All the agents in N^* rank a^* last; for each $b \in A' \setminus \{a^*\}$, there is $i \in N^*$ that ranks b first and B just above a^*, i.e.,

$$bR^i\left(A' \setminus \{a^*, b\}\right)R^iBR^ia^*.$$

For each $c \in B$, there is $i \in N^*$ that ranks c first and the rest of B just above a^*, namely

$$cR^i\left(A' \setminus \{a^*\}\right)R^i\left(B \setminus \{c\}\right)R^ia^*.$$

[3] X3C is obviously tractable for a constant n, as one can examine all the families $\mathcal{S}' \subseteq \mathcal{S}$ of constant size in polynomial time.

Notice that the Young score of the alternatives in $A' \setminus \{a^*\}$ is at least one. The Young score of any alternative $c \in B$ is exactly one, since exactly one agent (in N^*) does not rank $A' \setminus \{a^*\}$ above c. Now, if there is an exact 3-cover, then the Young score of a^* is at least $2n/3 - 1$ (according to the proof of Lemma 6.2), so a^* is ranked above all the alternatives in B, that is, in the top $m' + 1 = \mathcal{O}(m^\epsilon)$ places. On the other hand, if there is no exact 3-cover, then the Young score of a^* is zero by the same arguments as in Lemma 6.2, since the agents in N^* all rank a^* last. Hence a^* is placed last in any Young ranking. □

As noted in Section 2, one can imagine another alternative formulation of the Young score. Indeed, one might ask: given a preference profile, what is the *smallest* number of agents that must be *removed* in order to make a^* a Condorcet winner? This minimization problem, where the score is the number of agents that are removed, is referred to as the *Dual Young score* by Betzler et al. [4]. Of course, a Young winner according to the primal formulation is always a winner according to the dual formulation, and vice versa. Notice that it is easy to obtain an ϵn-approximation under the dual formulation for any constant $\epsilon > 0$ by enumerating all subsets of agents of size at least $n - 1/\epsilon$ and checking whether a^* is the Condorcet winner in the preferences of these agents. Our next result states that the dual Young score is hard to approximate significantly better.

Theorem 6.4. *For any constant $\epsilon > 0$, the dual Young score is $\mathcal{NP}$-hard to approximate within $O(n^{1-\epsilon})$.*

Proof. We rely on a statement regarding the inapproximability of Vertex Cover that is weaker than Theorem 5.7; the one we used in the proof for the inapproximability of the Dodgson ranking.

Theorem 6.5. *(See Berman and Karpinski [3], see also [25].) Given a 3-regular graph G and an integer $K \geqslant 1$, it is $\mathcal{NP}$-hard to distinguish between the following two cases*:

- *G has a vertex cover of size at most K.*
- *Any vertex cover of G has size at least $K + 2$.*

Our reduction extends the one in the proof of Lemma 6.2. Consider a 3-regular graph $G = (V_1, E)$ with p nodes and an integer $K \geqslant 1$. Also, let $\epsilon \in (0, 1)$ be a constant and let $n = \lceil p^{1/\epsilon} \rceil$. Denote by $H = (V_2, F)$ the complete graph with $n - p$ nodes.

Define the set of alternatives as $A = E \cup F \cup \{a\} \cup \{a^*\}$. Let the set of agents be $N = N' \cup N'' \cup N'''$, where N', N'', and N''' are defined as follows. The set N' consists of p agents corresponding to the p nodes of G, such that for all $i \in N'$, agent i prefers the alternatives in $F \cup E \setminus E_i$ to a^* (where the set E_i consists of the edges of E which are incident to node i), and prefers a^* to all the alternatives in $E_i \cup \{a\}$ (i.e., $((F \cup E) \setminus E_i) R^i a^* R^i (E_i \cup \{a\})$). The set N'' contains $n - p$ agents corresponding to the $n - p$ nodes of H, such that for all $i \in N''$, agent i prefers the alternatives in $E \cup F \setminus F_i$ to a^* (where the set F_i consists of the edges of F which are incident to node i), and prefers a^* to all the alternatives in $F_i \cup \{a\}$ (i.e., $((E \cup F) \setminus F_i) R^i a^* R^i (F_i \cup \{a\})$). Subset N''' consists of $n - p + K - 2$ agents who prefer a to a^* and a^* to $E \cup F$ (i.e., $a R^i a^* R^i (E \cup F)$).

Theorem 6.4 now follows by Theorem 6.5 and the next two lemmas.

Lemma 6.6. *If G has a vertex cover of size at most K, then the dual Young score of alternative a^* is at most n^ϵ.*

Proof. Let $C \subseteq V_1$ be a vertex cover of G of size at most K. Consider the following sets of agents: a set $N^* \subseteq N'$ that contains the agents that correspond to nodes in the vertex cover C, a set N^+ of all the agents of N'' besides one, and the set N'''.

Recall that $N^* \cup N^+$ has size at most $n - p + K - 1$ while N''' has size $n - p + K - 2$. Since C is a vertex cover in G, each alternative in E is ranked lower than a^* by at least one agent of N^*. Also, the nodes corresponding to the agents in N^+ form a vertex cover of H. So, each alternative in F is ranked lower than a^* by at least one agent of N^+. Hence, by considering the agents of $N^* \cup N^+ \cup N'''$, a^* beats any other alternative in their pairwise comparison and its dual Young score is at most $p - |C| + 1 \leqslant p \leqslant n^\epsilon$. □

Lemma 6.7. *If G has no vertex cover of size less than $K + 2$, then the dual Young score of alternative a^* is n.*

Proof. We will show that there is no nonempty subset of agents that make a^* a Condorcet winner. Indeed, assume for contradiction that there exists such a subset that contains the sets of agents $N^* \subseteq N'$, $N^+ \subseteq N''$, and $N^- \subseteq N'''$.

If $|N^+| < n - p - 1$ or $|N^*| < K + 2$ then there exists an alternative of E or F which is not ranked lower than a^* by any agent of $N^* \cup N^+$. In both cases, N^- must have size at least $|N^*| + |N^+|$ in order for a^* to beat every alternative in $E \cup F$ in their pairwise comparison. However, a^* does not beat a and cannot be a Condorcet winner.

Therefore, it holds that $|N^+| \in \{n - p - 1, n - p\}$ and $|N^*| \geqslant K + 2$. If $|N^+| = n - p - 1$, then some alternative of F is ranked below a^* by at most one agent of N^+. It is also ranked above a^* by the agents of N^* and below it by the agents of N^-. In total, it is ranked above a^* by at least $n-p+K$ agents while it is ranked below a^* by at most $n-p+K-1$ agents. Hence, a^* cannot be a Condorcet winner in this case.

If $|N^+| = n-p$ then each alternative in F is ranked below a^* by two agents of N^+. In total, it is ranked above a^* by at least $n-p+K$ agents while it is ranked below a^* by at most $n-p+K$ agents. Again, a^* cannot be a Condorcet winner. □

This concludes the proof of Theorem 6.4. □

The proof of Theorem 6.4 provides an alternative proof of Theorem 6.1. In terms of the Young score, it implies that, for every constant $\epsilon > 0$, there are instances for which it is hard to distinguish between a score of zero and a score of at least $n - n^{\epsilon}$. So, for the formulation of the Young score where all the scores are scaled by an additive constant, it provides additional information which is complementary to the one provided by the proof of Theorem 6.1: it implies that it is hard to approximate the Young score, even under this alternative formulation, within a factor of $\mathcal{O}(n)$.

7. Discussion

Generally speaking, we have taken the following approach: winner determination under the Dodgson and Young voting rules is intractable, therefore we aim to approximate the Dodgson or Young *score*. Other goals may seem more natural. For example, one can ask for a randomized algorithm that selects the winner with high probability, or an algorithm that selects an alternative that is ranked high by the voting rule in question. Note that our Theorem 5.6 rules out the latter goal. Nevertheless, a social choice justification for approximating a voting rule's score is called for. Below we concentrate on the Dodgson score because our positive results concern this rule.

Dodgson's reasoning in designing his voting rule is a special case of a more general framework called *distance rationalizability*, which was proposed by Meskanen and Nurmi [33], and recently received some attention in the AI literature [13,15,14]. The reasoning behind this framework is that a voting rule should elect an alternative that is closest to being a consensus winner, according to a natural notion of consensus and a natural notion of distance.

Dodgson's rule employs a very natural notion of consensus (Condorcet winner) and arguably a natural notion of distance (number of swaps between adjacent alternatives). These are normative statements, as is common in social choice theory. However, viewed through the distance rationalizability lens, an approximation of the Dodgson score is simply an approximation of a natural distance function, much like approximations for other hard problems that involve distances, e.g., facility location problems. In facility location problems there is a direct connection between distances and the quality of the solution (e.g., the larger the distances, the more costly it would be to build an appropriate infrastructure). Work in progress by Boutilier and Procaccia suggests that, similarly, in the distance rationalizability framework the distance function can be proportional to a direct quantitative measure of an alternative's quality: the closer the alternative is to consensus according to the distance function, the faster it leads to consensus in a social choice model that involves dynamic preferences, as put forward by Parkes and Procaccia [35].

Therefore, we can argue that an alternative is increasingly more socially desirable the smaller its Dodgson score, that is, the score itself is meaningful and not just the Dodgson ranking, and therefore a good approximation of the Dodgson score may also single out socially desirable winners. Moreover, as argued in Section 4, whenever approximation algorithms satisfy additional social choice desiderata, they may ultimately be adopted as socially sensible voting rules in their own right.

Interestingly, Dodgson's rule is considered to be especially flawed from a social choice point of view, and this may be one of the reasons why it was never employed in real-world decision making. Some well-known voting rules like Copeland and Maximin are Condorcet-consistent, and in addition avoid the main drawbacks from which Dodgson suffers (e.g., they are monotonic). Nevertheless, our thesis is that Dodgson approximations can in fact be superior to the original rule, from both the computational and the social choice points of view, and ultimately may serve as realistic choices for preference aggregation in human societies and in multiagent systems. The results given above are the starting point of this line of inquiry; some of us make the point more forcefully in follow-up work [7], which directly builds on the results of this paper.

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