Approximation algorithms for MAX 4-SAT and rounding procedures for semidefinite programs *

Eran Halperin *   *Uri Zwick* †

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Abstract

Karloff and Zwick obtained recently an optimal 7/8-approximation algorithm for MAX 3-SAT. In an attempt to see whether similar methods can be used to obtain a 7/8-approximation algorithm for MAX SAT, we consider the most natural generalization of MAX 3-SAT, namely MAX 4-SAT. We present a semidefinite programming relaxation of MAX 4-SAT and a new family of rounding procedures that try to cope well with clauses of various sizes. We study the potential, and the limitations, of the relaxation and of the proposed family of rounding procedures using a combination of theoretical and experimental means. We select two rounding procedures from the proposed family of rounding procedures. Using the first rounding procedure we seem to obtain an *almost optimal* 0.8721-approximation algorithm for MAX 4-SAT. Using the second rounding procedure we seem to obtain an *optimal* 7/8-approximation algorithm for *satisfiable* instances of MAX 4-SAT. On the other hand, we show that no rounding procedure from the family considered can be shown, using the current techniques, to yield an approximation algorithm for MAX 4-SAT whose performance guarantee on all instances of the problem is greater than 0.8724. We also show that the the integrality ratio of the proposed relaxation, as a relaxation of MAX $\{1,4\}$-SAT is at most 0.8754.

The 0.8721-approximation for MAX 4-SAT that we seem to obtain substantially improves the performance guarantees of all previous algorithms suggested for the problem. It is extremely close to being optimal as a $(7/8+\epsilon)$-approximation algorithm for MAX 4-SAT, for any fixed $\epsilon > 0$, would imply that P=NP. Our investigation also indicates, however, that additional ideas are required in order to obtain optimal 7/8-approximation algorithms for MAX 4-SAT and MAX SAT.

Although most of this paper deals specifically with the MAX 4-SAT problem, we believe that the new family of rounding procedures introduced, and the methodology used in the design and in the analysis of the various rounding procedures considered would have a much wider range of applicability.

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†Department of Computer Science, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: {heran,zwick}@math.tau.ac.il.

1 Introduction

MAX SAT is one of the most natural optimization problems. An instance of MAX SAT in the Boolean variables $x_1, \ldots, x_n$ is composed of a collection of clauses. Each clause is the disjunction of an arbitrary number of literals. Each literal is a variable, x_i, or a negation, $\bar{x}_i$, of a variable. Each clause has a non-negative weight w associated with it. The goal is to find a 0-1 assignment of values to the Boolean variables $x_1, \ldots, x_n$ so that the sum of the weights of the satisfied clauses is maximized.

Following breakthrough developments by many authors that established close connections between *Probabilistically Checkable Proofs (PCPs)* and the hardness of approximation, we now know that MAX SAT is APX-hard (or MAX SNP-hard) [PY91, FGL+96, AS92, ALM+92, BGLR93, BGS95, KMSV94]. This means that there is a constant $\epsilon > 0$ such that, assuming P$\neq$NP, there is no polynomial time approximation algorithm for MAX SAT with a performance guarantee of at least $1 - \epsilon$ on all instances of the problem. Approximation algorithms for MAX SAT were designed by many authors, including [Joh74, Yan94, GW94, GW95, AOH96, Asa97, Zwi99, AW00]. The best *proven* performance ratio known for the problem is currently about 0.7846 [AW00].

In another breakthrough, Håstad [Hås97] showed recently that no polynomial time approximation algorithm for MAX SAT can have a performance guarantee of more than 7/8, unless P=NP. Håstad's shows, in fact, that no polynomial time approximation algorithm for satisfiable instances of MAX {3}-SAT can have a performance guarantee of more than 7/8. MAX {3}-SAT is the subproblem of MAX SAT in which each clause is of size *exactly* three. An instance is satisfiable if there is an assignment that satisfies all its clauses.

Karloff and Zwick [KZ97] obtained recently an optimal 7/8-approximation algorithm for MAX 3-SAT, the version of MAX SAT in which each clause is of size *at most* three. (This claim appears in [KZ97] as a conjecture. It has since been proved.) Their algorithm uses semidefinite programming. A much simpler approximation algorithm has a performance guarantee of 7/8 if all clauses are of size *at least* three. If all clauses are of size at least three then a *random* assignment satisfies, on the average, at least 7/8 of the clauses.

We thus have a performance guarantee of 7/8 for instances in which all clauses are of size at most three, and for instances in which all clauses are of size at least three. Can we get a performance guarantee of 7/8 for *all* instances of MAX SAT? In an attempt to answer this question, we check the prospects of obtaining a 7/8-approximation algorithm for MAX 4-SAT, the subproblem of MAX SAT in which each clause is of size *at most* four. As it turns out, this is already a challenging problem.

The 7/8-approximation algorithm for MAX 3-SAT begins by solving a semidefinite programming relaxation of the problem. It then *rounds* the solution of this program using a random hyperplane passing through the origin. It is natural to try to obtain a similar approximation algorithm for MAX 4-SAT. It is not difficult, see Section 2, to obtain a semidefinite programming relaxation of MAX 4-SAT. It is again natural to try to round this solution using a random hyperplane. It turns out, however, that the performance guarantee of this algorithm is only 0.845173. Although this is much better than all previous performance guarantees for MAX 4-SAT, this performance guarantee is, unfortunately, below 7/8.

As the semidefinite programming relaxation of MAX 4-SAT is the strongest relaxation of its kind (see again Section 2), it seems that a different rounding procedure should be used. We describe, in Section 3, a new family of rounding procedures. This family extends all the families of rounding procedures previously suggested for maximum satisfiability problems. The difficulty in developing good rounding procedures for MAX 4-SAT is that rounding procedures that work well for the short clauses, do not work so well for the longer clauses, and vice versa. Rounding procedures from the new family try to work well on all clause sizes simultaneously. We initially hoped that an appropriate rounding procedure from this family could be used to obtain 7/8-approximation algorithms for MAX 4-SAT and perhaps even MAX SAT. It turns out, however, that the new family falls just short of this mission. The checks that we have made suggest that a rounding procedure from the family, which we explicitly describe, can be used to obtain a 0.8721-approximation algorithm for MAX 4-SAT. Unfortunately, no rounding procedure from the family yields an approximation algorithm for MAX 4-SAT with a performance guarantee larger than 0.8724, or more precisely, no such rounding procedure can be shown to have a performance ratio larger than 0.8724 using the type of local analysis used here and in all previous related papers.

We have more success with MAX $\{2,3,4\}$-SAT, the version of MAX SAT in which the clauses are of size two, three or four. We present a second rounding procedure from the family that seems to yield an optimal 7/8-approximation algorithm for MAX $\{2,3,4\}$-SAT. A 7/8-approximation algorithm for MAX $\{2,3,4\}$-SAT yields immediately an optimal 7/8-approximation algorithm for *satisfiable* instances of MAX 4-SAT, as clauses of size one can be easily eliminated from satisfiable instances.

To determine the performance guarantee obtained using a given rounding procedure R, or at least a lower bound on this ratio, we have to find the *global* minimum of a function $ratio_R(v_0, v_1, v_2, v_3, v_4)$, given a set of constraints on the unit vectors $v_0, v_1, v_2, v_3, v_4 \in \mathbb{R}^5$. The function $ratio_R$ is a fairly complicated function determined by the rounding procedure R. As five unit vectors are determined, up to rotations, by the $\binom{5}{2} = 10$ angles between them, the function $ratio_R$ is actually a function of 10 real variables. Finding the global minimum of $ratio_R$ analytically is a formidable task. In the course of our investigation we experimented with hundreds of rounding procedures. Finding these minima 'by hand' was not really an option. We have implemented a set of Matlab functions that use numerical techniques to find these minima.

The discussion so far centered on the quality of the rounding procedures considered. We also consider the quality of the suggested semidefinite programming relaxation itself. The integrality ratio of the MAX 4-SAT relaxation cannot be more than 7/8, as it is also a relaxation of MAX 3-SAT. We also show that the integrality ratio of the relaxation, considered as a relaxation of the problem MAX $\{1,4\}$-SAT, is at most 0.8754. (MAX $\{1,4\}$-SAT is the subproblem of MAX 4-SAT in which all clauses are of size one or four.) The fact that this ratio is, at best, just above 7/8 is another indication of the difficulty of obtaining optimal 7/8-approximation algorithm for MAX 4-SAT and MAX SAT. It may also indicate that a stronger semidefinite programming relaxation would be needed to accomplish this goal.

The fact that numerical optimization techniques were used to compute the performance guarantees of the algorithms means that we cannot claim the existence of a 0.8721-approximation algorithm for MAX 4-SAT, and of a 7/8-approximation algorithm for MAX $\{2,3,4\}$-SAT as theorems. We believe, however, that it is possible to prove these claims analytically and promote them to the status of theorems, as was eventually done with the optimal 7/8-approximation algorithm for MAX 3-SAT. This would require, however, considerable effort. It may make more sense, therefore, to look for an approximation algorithm that seems to be a 7/8-approximation algorithm for MAX 4-SAT before proceeding to this stage.

In addition to implementing a set of Matlab functions that try to find the performance guarantee of a given rounding procedure from the family considered, we have also implemented a set of functions that *search* for good rounding procedures. The whole project required about 3000 lines of code. The two rounding procedures mentioned above, and several other interesting rounding procedures mentioned is Section 5, were found automatically using this system, with some manual help from the authors. The total running time used in the search for good rounding procedures is measured by months.

We end this section with a short survey of related results. The 7/8-approximation algorithm for MAX 3-SAT is based on the MAX CUT algorithm of Goemans and Williamson [GW95]. A 0.931-approximation algorithm for MAX 2-SAT was obtained by Feige and Goemans [FG95]. Asano and Williamson [AW00] obtained a 0.7846-approximation algorithm for MAX SAT. Trevisan [Tre97] obtained a 0.8-approximation algorithm for *satisfiable* MAX SAT instances. The last two results are also the best previously published results for MAX 4-SAT. The second author [Zwi99] recently described a new rounding technique, called *outward rotations*, which was also suggested by Nesterov [Nes98] and Ye [Ye99]. Using this method he describes an approximation algorithm for MAX NAE-SAT, a generalization of the MAX SAT problem, with a *conjectured* performance ratio of 0.7977. Using this algorithm as a building block, Asano and Williamson [AW00] obtain an approximation algorithm for MAX SAT whose performance ratio, if the 0.7977 conjecture is true, is 0.8331.

2 Semidefinite programming relaxation of MAX 4-SAT

Karloff and Zwick [KZ97] describe a canonical way of obtaining semidefinite programming relaxations for any constraint satisfaction problem. Obtaining the semidefinite relaxation of MAX 4-SAT using this approach is fairly straightforward.

Assume that $x_1, \ldots, x_n$ are the variables of the MAX 4-SAT instance. We let $x_0 = 0$ and $x_{n+i} = \bar{x}_i$, for $1 \le i \le n$. The semidefinite program corresponding to the instance has a variable unit vector v_i, corresponding to each literal x_i, and scalar variables z_i, z_{ij}, z_{ijk} or z_{ijkl} corresponding to the clauses x_i, $x_i \vee x_j$, $x_i \vee x_j \vee x_k$ and $x_i \vee x_j \vee x_k \vee x_l$ of the instance, where $1 \le i, j, k, l \le 2n$. Note that all clauses, including those that contain negated literals can be expressed in this form. Clearly, we require $v_{n+i} = -v_i$, or equivalently $v_i \cdot v_{n+i} = -1$, for $1 \le i \le n$.

The objective of the semidefinite program is to maximize the function

$$\sum_i w_i z_i + \sum_{i,j} w_{ij} z_{ij} + \sum_{i,j,k} w_{ijk} z_{ijk} + \sum_{i,j,k,l} w_{ijkl} z_{ijkl} \ ,$$

where the w_i's, w_{ij}'s, w_{ijk}'s and w_{ijkl}'s are the non-negative weights of the different clauses, subject to the following collection of constraints. For ease of notation, we write down the constraints that correspond to the clauses x_1, $x_1 \vee x_2$, $x_1 \vee x_2 \vee x_3$ and $x_1 \vee x_2 \vee x_3 \vee x_4$. The constraints corresponding to the other clauses are easily obtained by plugging in the corresponding indices. The constraints corresponding to x_1 and $x_1 \vee x_2$ are quite simple:

$$z_1 = \tfrac{1 - v_0 \cdot v_1}{2} \quad , \quad z_{12} \le \tfrac{3 - v_0 \cdot v_1 - v_0 \cdot v_2 - v_1 \cdot v_2}{4} \quad , \quad z_{12} \le 1 \ .$$

The constraints corresponding to $x_1 \vee x_2 \vee x_3$ are slightly more complicated:

$$\begin{array}{ll} z_{123} \le \tfrac{4 - (v_0 + v_1) \cdot (v_2 + v_3)}{4} \quad , & z_{123} \le \tfrac{4 - (v_0 + v_2) \cdot (v_1 + v_3)}{4} \\ z_{123} \le \tfrac{4 - (v_0 + v_3) \cdot (v_1 + v_2)}{4} \quad , & z_{123} \le 1 \end{array}$$

It is not difficult to check that the first three constraints are equivalent to the requirement that $z_{123} \leq (4-(v_{i_0}\cdot v_{i_1}+v_{i_1}\cdot v_{i_2}+v_{i_2}\cdot v_{i_3}+v_{i_3}\cdot v_{i_0}))/4$, for any permutation i_0,i_1,i_2,i_3 on $0,1,2,3$. We will encounter similar constraints for the 4-clauses.

The constraints corresponding to $x_1 \vee x_2 \vee x_3 \vee x_4$ are even more complicated. For any permutation i_0,i_1,i_2,i_3,i_4 on $0,1,2,3,4$ we require:

$$z_{1234} \leq \frac{5-(v_{i_0}\cdot v_{i_1}+v_{i_1}\cdot v_{i_2}+v_{i_2}\cdot v_{i_3}+v_{i_3}\cdot v_{i_4}+v_{i_4}\cdot v_{i_0})}{4}\,,$$

$$z_{1234} \leq \frac{5-(v_{i_0}+v_{i_4})\cdot(v_{i_1}+v_{i_2}+v_{i_3})+v_{i_0}\cdot v_{i_4}}{4}\quad,\quad z_{1234}\leq 1\,.$$

The first line above contributes 12 different constraints, the second line contributes 10 different constraints. Together with the constraint $z_{1234} \leq 1$ we get a total of 23 constraints per 4-clause. In addition, for every distinct $0 \leq i_1,i_2,i_3,i_4,i_5 \leq 2n$, we require

$$\sum_{1\leq j<k\leq 3} v_{i_j}\cdot v_{i_k} \geq -1 \quad\text{and}\quad \sum_{1\leq j<k\leq 5} v_{i_j}\cdot v_{i_k} \geq -2\,.$$

Although we do not consider clauses of size larger than 4 in this paper, we remark that for any k, and for any permutation $i_0,i_1,\ldots,i_k$ on $0,1,\ldots,k$, $z_{12\ldots k} \leq ((k+1)-\sum_{j=0}^{k} v_{i_j}\cdot v_{i_{j+1}})/4$, where the index $j+1$ is interpreted modulo $k+1$, is a valid constraint in the semidefinite programming relaxation of MAX k-SAT.

It is not difficult to verify that all these constraints are satisfied by any valid 'integral' assignment to the vectors v_i and the scalars z_i, z_{ij}, z_{ijk} and z_{ijkl}, i.e., an assignment in which $v_i=(1,0,\ldots,0)$ if $x_i=1$, and $v_i=(-1,0,\ldots,0)$ if $x_i=0$, and in which every z variable is set to 1 if its corresponding clause is satisfied by the assignment $x_1,x_2,\ldots,x_n$, and to 0 otherwise. Thus, the presented semidefinite program is indeed a relaxation of the corresponding MAX 4-SAT instance.

The constraints of the above semidefinite program correspond to the *facets* of a polyhedron corresponding to the Boolean function x_1, $x_1\vee x_2$, $x_1\vee x_2\vee x_3$ and $x_1\vee x_2\vee x_3\vee x_4$. As explained in [KZ97], it is therefore the strongest semidefinite relaxation that considers the clauses of the instance one at a time. Stronger relaxations may be obtained by considering several clauses at once.

The semidefinite programming relaxation of a MAX 4-SAT instance has $n+1$ unknown unit vectors $v_0,v_1,\ldots,v_n$ (the vectors v_{n+i} are just used as shorthands for $-v_i$), $O(n^4)$ scalar variables and $O(n^5)$ constraints. An almost optimal solution in which all unit vectors $v_0,v_1,\ldots,v_n$ lie in $\mathbb{R}^{n+1}$ can be found in polynomial time ([Ali95],[GLS93],[NN94]).

3 Rounding procedures

In this section we consider various procedures that can be used to round solutions of semidefinite programming relaxations and examine the performance guarantees that we get for MAX 4-SAT using them. We start with simple rounding procedures and then move on to more complicated ones. The new family of rounding procedures is then presented in Section 3.5.

3.1 Rounding using a random hyperplane

Following Goemans and Williamson [GW95], many semidefinite programming based approximation algorithms round the solution to the semidefinite program using a *random hyperplane* passing through the origin. A random hyperplane that passes through the origin is chosen by choosing its normal vector r as a uniformly distributed vector on the unit sphere (in $\mathbb{R}^{n+1}$). A vector v_i is then rounded to 0 if v_i and v_0 fall on the same side of the random hyperplane, i.e., if $sgn(r \cdot v_i) = sgn(r \cdot v_0)$, and to 1, otherwise. Note that the rounded values of the variables are usually not independent. More specifically, if v_i and v_j are not perpendicular, i.e., if $v_i \cdot v_j \neq 0$, then the rounded values of x_i and x_j are dependent.

Given a set V of unit vector, we let $prob_H(V)$ denote the probability V is separated by a random hyperplane that passes through the origin, or in other words, that not all the vectors of V fall on the same side of this hyperplane. It is not difficult to see (see, e.g., [GW95]) that

$$prob_H(v_0, v_1) = \frac{\theta_{01}}{\pi} \quad , \quad prob_H(v_0, v_1, v_2) = \frac{\theta_{01} + \theta_{02} + \theta_{12}}{2\pi} ,$$

where $\theta_{ij} = \arccos(v_i \cdot v_j)$ is the angle between v_i and v_j. Evaluating $prob_H(v_0, v_1, v_2, v_3)$ is more difficult. As noted in [KZ97],

$$prob_H(v_0, v_1, v_2, v_3) = 1 - \frac{\mathrm{Vol}(\lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{03}, \lambda_{13}, \lambda_{23})}{\pi^2} ,$$

where $(\lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{03}, \lambda_{13}, \lambda_{23}) = \pi - (\theta_{23}, \theta_{13}, \theta_{03}, \theta_{12}, \theta_{02}, \theta_{01})$ and $\mathrm{Vol}(\lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{03}, \lambda_{13}, \lambda_{23})$ is the volume of a *spherical tetrahedron* with dihedral angles $\lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{03}, \lambda_{13}, \lambda_{23}$. Unfortunately, the volume function of spherical tetrahedra seems to be a non-elementary function and numerical integration should be used to evaluate it ([Sch58],[Cox35],[Hsi88],[Vin93]).

It is not difficult to verify, using inclusion-exclusion, that

$$prob_H(v_0, v_1, v_2, v_3, v_4) = \frac{1}{2} \sum_{i<j<k<l} prob_H(v_i, v_j, v_k, v_l) - \frac{1}{4} \sum_{i<j} prob_H(v_i, v_j) .$$

Thus, the probability that certain five unit vectors are separated by a random hyperplane can be expressed as a combination of probabilities of events that involve at most four vectors and no further numerical integration is needed. More generally, $prob_H(v_0, \ldots, v_k)$, for any *even* k, can be expressed as combinations of probabilities involving at most k vectors. The same does not hold, unfortunately, when k is odd.

We let $relax(v_0, v_1, \ldots, v_i)$, where $1 \leq i \leq 4$, denote the 'relaxed' value of the clause $x_1 \vee \ldots \vee x_i$, i.e., the maximal value to which the scalar $z_{12\ldots i}$ can be set given the unit vectors $v_0, v_1, \ldots, v_i$ while still satisfying all the relevant constraints . We let

$$ratio_H(v_0, v_1, \ldots, v_i) \;=\; \frac{prob_H(v_0, v_1, \ldots, v_i)}{relax(v_0, v_1, \ldots, v_i)} .$$

For every $1 \leq i \leq 4$, we then let

$$\alpha_i \;=\; \min\, ratio_H(v_0, v_1, \ldots, v_i) ,$$

where the minimum is over all configurations of unit vectors that satisfy the constraints described in the previous section, and for which $relax(v_0, v_1, \ldots, v_i) > 0$. (More formally, we should write here 'inf' instead of 'min'. The minimums are attained, however, in all the cases that we consider.) Finally, we let

$$\alpha = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} .$$

As follows from straightforward arguments (see [GW95] or [KZ97]), α is a lower bound on the performance guarantee of the approximation algorithm for MAX 4-SAT that uses the semidefinite relaxation of the previous section and then rounds the solution using a random hyperplane.

It is shown in [GW95] that $\alpha_1 = \alpha_2 \simeq 0.87856$. It is shown in [KZ97] that $\alpha_3 = 7/8$. Unfortunately, it turns out that $\alpha_4 \simeq 0.845173$. The minimum is attained when the angle between each pair of vectors among $v_0, v_1, \ldots, v_4$ is exactly $\arccos(1/5) \simeq 1.369438$. It is not difficult to check that when $v_i \cdot v_j = 1/5$, for every $0 \le i < j \le 4$, all the inequalities on z_{1234} simplify to $z_{1234} \le 1$ and therefore $relax(v_0, v_1, v_2, v_3, v_4) = 1$, while $prob_H(v_0, v_1, v_2, v_3, v_4) \simeq 0.845173$.

3.2 Pre-rounding rotations

Feige and Goemans [FG95] introduced the following variation of random hyperplane rounding. Let $f : [0, \pi] \to [0, \pi]$ be a continuous function satisfying $f(0) = 0$ and $f(\pi - \theta) = \pi - f(\theta)$, for $0 \le \theta \le \pi$. Before rounding the vectors using a random hyperplane, the vector v_i is rotated into a new vector v_i', in the plane spanned by v_0 and v_i, so that the angle θ'_{0i} between v_0 and v_i' would be $\theta'_{0i} = f(\theta_{0i})$. The rotations of the vectors $v_1, \ldots, v_n$ affects, of course, the angles between these vectors. Let θ'_{ij} be the angle between v_i' and v_j'. It is not difficult to see (see [FG95]), that for $i, j > 0$, $i \ne j$, we have

$$\cos \theta'_{ij} \;=\; \cos \theta'_{0i} \cdot \cos \theta'_{0j} + \frac{\cos \theta_{ij} - \cos \theta_{0i} \cdot \cos \theta_{0j}}{\sin \theta_{0i} \cdot \sin \theta_{0j}} \cdot \sin \theta'_{0i} \cdot \sin \theta'_{0j} \; .$$

The vectors $v_0, v_1', \ldots, v_n'$ are then rounded using a random hyperplane. The condition $f(0) = 0$ is required to ensure the continuity of the transformation $v_i \to v_i'$. The condition $f(\pi - \theta) = \pi - f(\theta)$ ensures that unnegated and negated literals are treated in the same manner.

Feige and Goemans [FG95] use rotations to obtain a 0.931-approximation algorithm for MAX 2-SAT. Rotations are also used in [Zwi98a] and [Zwi98b]. Can we use rotations to get a better approximation algorithm for MAX 4-SAT? The answer is that rotations on their own help, but very little. Consider the configuration v_0, v_1, v_2, v_3, v_4 in which $\theta_{0i} = \pi/2$, for $1 \le i \le 4$, and $\theta_{ij} = \arccos(1/3)$, for $1 \le i < j \le 4$. For this configuration we get $relax(v_0, v_1, v_2, v_3, v_4) = 1$ and $ratio_H(v_0, v_1, v_2, v_3, v_4) \simeq 0.8503$. As every rotation function f must satisfy $f(\pi/2) = \pi/2$, rotations have no effect on this configuration.

3.3 Rounding the vectors independently

The semidefinite programming relaxation of MAX 4-SAT that we are using here is stronger than the linear programming relaxation suggested by Goemans and Williamson [GW94]. It is nonetheless interesting to consider an adaptation of the rounding procedures used in [GW94] to the present context. The rounding procedures of [GW94] are based on the *randomized rounding* technique of Raghavan and Thompson [RT87],[Rag88].

Let $g : [0, \pi] \to [0, \pi]$ be a continuous function such that $g(\pi - \theta) = \pi - g(\theta)$, for $0 \le \theta \le \pi$. Note again that we must have $g(\pi/2) = \pi/2$. We do not require $g(0) = 0$ this time. The rounding procedure described here rounds each vector *independently*. The variable x_i is assigned the value 1 with probability $g(\theta_{0i})/\pi$, and the value 0 with the complementary probability. Note that this is equivalent to first rotating the vector v_i into v_i', as in the previous subsection, using the function g, and then assigning x_i the value 1 with probability θ'_{0i}/π.

The probability $prob_I(v_0, v_1, \ldots, v_i)$ that a clause $x_1 \vee \ldots \vee x_i$ is satisfied is now

$$prob_I(v_0, v_1, \ldots, v_i) = 1 - \prod_{j=1}^{i} \left(1 - \frac{g(\theta_{0j})}{\pi}\right) \; .$$

Note that as each vector is rounded independently, the angles θ_{ij}, where $i, j > 0$, between the vectors, have no effect this time.

It may be worthwhile to note that the choice $g(\theta) = \pi/2$, for every $0 \le \theta \le \pi$, corresponds to choosing the assignment to the variables $x_1, x_2, \ldots, x_n$ uniformly at random, a 'rounding procedure' that yields a ratio of 7/8 for clauses of size 3 and 15/16 for clauses of size 4. Goemans and Williamson [GW94] describe several functions g using which a 3/4-approximation algorithm for MAX SAT may be obtained.

Independent rounding performs well on long clauses. It cannot yield a ratio larger than 3/4, however, for clauses of size 2. To see this, consider the configuration v_0, v_1, v_2 in which $\theta_{01} = \theta_{02} = \pi/2$ and $\theta_{12} = \pi$. We have $relax(v_0, v_1, v_2) = 1$ and $prob_I(v_0, v_1, v_2) = 3/4$, for any function g satisfying $g(\pi/2) = \pi/2$.

3.4 Simple combinations

We have seen that hyperplane rounding works well for short clauses and that independent rounding works well for long clauses. It is therefore natural to consider a combination of the two.

Perhaps the most natural combination of hyperplane rounding and independent rounding is the following. Let $0 \le \epsilon \le 1$. With probability $1 - \epsilon$ round the vectors using a random hyperplane. With probability ϵ choose a random assignment. It turns out that the best choice of ϵ here is $\epsilon \simeq 0.086553$. With this value of ϵ, we get $\alpha_1 = \alpha_2 = \alpha_4 \simeq 0.853150$ while $\alpha_3 = 7/8$. Thus, we again get a small improvement but we are still far from 7/8.

Instead of rounding all vectors using a random hyperplane, or choosing random values to all variables, we can round some of the vectors using a random hyperplane, and assign some of the variables random values. More precisely, we choose one random hyperplane. Each vector is now rounded using this random hyperplane with probability $1 - \epsilon$, or is assigned a random value with probability ϵ. The decisions for the different variables made independently. Letting $\epsilon \simeq 0.073609$, we get $\alpha_1 = \alpha_2 = \alpha_4 \simeq 0.856994$, while $\alpha_3 \simeq 0.874496$. This is again slightly better but still far from 7/8.

3.5 More complicated combinations

We have seen above that a combination of hyperplane rounding and independent rounding performs better than each one of these methods on its own. The improvement obtained so far, from 0.845 to 0.856, is quite modest. It was obtained, however, using very simple combinations. Can we get a more substantial improvement using more sophisticated combinations? To answer this question we introduce the following family of rounding procedures. The new family seems to include all the natural combinations of the rounding procedures mentioned above.

Each rounding procedure in the new family is characterized by three continuous functions $f, g : [0, \pi] \to [0, \pi]$ and $\epsilon : [0, \pi] \to [0, 1]$. The function f is used for rotating the vectors before rounding them using a random hyperplane, as described in Section 3.2. The function g is used to round the vectors independently, as described in Section 3.3. The function ϵ is used to decide which of the two roundings should be used. The decision is made *independently* for each vector, depending on the angle between it and v_0. The function $\epsilon : [0, \pi] \to [0, 1]$ is a continuous function satisfying $\epsilon(\pi - \theta) = \epsilon(\theta)$, a condition that ensures that negated and unnegated literals are treated in the same manner. The vector v_i is rounded using a random hyperplane, shared by all the vectors rounded using a random hyperplane, with probability $1 - \epsilon(\theta_{0i})$, and is rounded independently, with probability $\epsilon(\theta_{0i})$. Vectors rounded using the shared hyperplane are rotated before the rounding. Let $v_1', v_2', \ldots, v_n'$ be the vectors obtained by rotating the vectors $v_1, v_2, \ldots, v_n$, as specified by the rotation function f. The probability that a clause $x_1 \vee x_2 \vee \ldots \vee x_i$ is satisfied by the assignment produced by this combined rounding procedure is given by the following expression:

$$prob_C(v_0, v_1, \ldots, v_i) = 1 - \sum_S p(S) \cdot (1 - prob_H(v'(S))) \cdot (1 - prob_I(v(S^c))) \ ,$$

where

$$p(S) = [\prod_{i \in S} (1 - \epsilon(\theta_{0i}))] \cdot [\prod_{i \notin S} \epsilon(\theta_{0i})],$$

$$v'(S) = \{v_0\} \cup \{v_i' \mid i \in S\} \quad , \quad v(S^c) = \{v_0\} \cup \{v_i \mid i \notin S\} \ ,$$

and where S ranges over all subsets of $\{1, 2, \ldots, i\}$. Recall that $prob_H(u_1, u_2, \ldots, u_k)$ is the probability that the set of vectors $u_1, u_2, \ldots, u_k$ is separated by a random hyperplane, and that $prob_I(v_0, u_1, \ldots, u_k)$ is the probability that at least one of the vectors $u_1, u_2, \ldots, u_k$ is assigned the value 1 when all these vectors are rounded independently using the function g.

We have made some experiments with an even wider family of rounding procedures but we were not able to improve on the results obtained using rounding procedures selected from the family described here.

Can we select a rounding procedure from the proposed family of rounding procedures using which we can get an optimal, or an almost optimal, approximation algorithm for MAX 4-SAT?

4 The search for good rounding procedures

The new family of rounding procedures defined in the previous section is huge. How can we expect to select the best, or almost the best, rounding procedure from this family? As it turns out, although each rounding procedure is defined by three continuous functions f, g and ϵ, most of the values of these functions do not matter much. What really matter are the values of these functions at several 'important' angles. We therefore restrict ourselves to rounding procedures defined by *piecewise linear* functions f, g and ϵ with a relatively small number of bends. By placing these bends at the 'important' angles, we can find, as we shall see, a rounding procedure which is close to being the best rounding procedure from this family.

More specifically, we consider functions obtained by connecting k given points $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ by straight line segments, where $x_1 = 0$ and $x_k = \pi/2$. For f we also require $y_1 = 0$ and $y_k = \pi/2$. For g we also require $y_k = \pi/2$. The values of the functions f, g and ϵ for $\pi/2 < \theta \le \pi$ are determined by the conditions $f(\pi - \theta) = \pi - f(\theta)$, $g(\pi - \theta) = \pi - g(\theta)$ and $\epsilon(\pi - \theta) = \epsilon(\theta)$. We usually worked with $k \le 5$.

For a given value of k we are now faced with a very difficult optimization problem in $6k-9$ real variables, the variables being the x and y coordinates of the points through which the functions f, g and ϵ are required to pass. By allowing the x-coordinates, as well as the y-coordinates, of these points to vary, we are not forced to choose the 'important angles' in advance. The objective is to maximize $\alpha(C(f, g, \epsilon))$, the performance guarantee obtained by using the rounding procedure defined by the functions f, g and ϵ that pass through the points. Recall that evaluating $\alpha(C(f, g, \epsilon))$ for a given set of functions f, g and ϵ is already a difficult task that requires finding the global minimum of a rather complicated function of 10 real variables.

We have written a Matlab program, called opt_fun, that tries to find a close to optimal rounding procedure that uses functions specified using at most k points. This is quite a non-trivial task and, as mentioned in the introduction, it required about 3000 lines of code, in addition to the sophisticated numerical optimization routines of Matlab's optimization toolbox. A description of the program is given in Section 6.

Although numerical methods were used to evaluate the performance guarantees of the different rounding procedures, we believe that the 0.8721 and 7/8 performances ratio claimed for the two rounding procedures that will be described shortly are the correct performance ratios. There is, in fact, a completely mechanical way of generating a (long and tedious) rigorous proof of these claims. As mentioned in the introduction, we believe that it would be more fruitful to look for an algorithm that seems to achieve a performance ratio of 7/8 before taking on the task of producing rigorous proofs. We believe that the use of numerical methods would be inevitable in the search for optimal algorithms for MAX 4-SAT and MAX SAT, at least using the current techniques.

5 Almost optimal or optimal approximation algorithms

In this section we present some optimal or close to optimal approximation algorithms obtained using rounding procedures from the new family of rounding procedures.

5.1 MAX 4-SAT

Using the semidefinite programming relaxation of Section 2 and the rounding procedure defined by the three piecewise linear functions passing through the points given in Figure 1 we seem to obtain a 0.8721-approximation algorithm for MAX 4-SAT. As we shall see in Section 7, this is essentially the best approximation ratio that we can obtain using a rounding procedure from the family considered.

It is interesting to note that $g(\theta) = 0$ for $0 \le \theta \le 1.072646$ and that $0.13 \le \epsilon(\theta) \le 0.36$ for $0 \le \theta \le \pi$. This means that if the angle θ_{0i} between v_i and v_0 is less than about $\pi/3$, then with a probability of about 1/4, the variable x_i is assigned the value 0, without any further consideration of the angle θ_{0i}. It is also interesting to note that the function $f(\theta)$ is not monotone.

f	g	ϵ
$(0, 0)$	$(0, 0)$	$(0, 0.250000)$
$(0.777843, 1.210627)$	$(0.750000, 0)$	$(0.744611, 0.357201)$
$(1.038994, 1.445975)$	$(1.072646, 0)$	$(1.039987, 0.255183)$
$(1.248362, 1.394099)$	$(1.248697, 0.872552)$	$(1.072689, 0.222928)$
$(\pi/2, \pi/2)$	$(\pi/2, \pi/2)$	$(\pi/2, 0.131681)$

Figure 1: The rounding procedure that seems to yield a 0.8721-approximation algorithm for MAX 4-SAT

f	g	ϵ
(0 , 0)	(0 , 0.550000)	(0 , 0.650000)
(1.394245 , 1.544705)	(1.155432 , 1.154866)	(0.413021 , 0.163085)
($\pi/2$, $\pi/2$)	(1.394111 , 0.931661)	($\pi/2$, 0.160924)
	($\pi/2$, $\pi/2$)	

Figure 2: The rounding procedure that seems to yield an optimal 7/8-approximation algorithm for MAX $\{2,3,4\}$-SAT.

f
(0 , 0)
(0.834994 , 0.676086)
(1.456735 , 1.456733)
($\pi/2$, $\pi/2$)

Figure 3: The pre-rounding rotation f that seems to yield an approximation algorithm for MAX 3-SAT with $\alpha_1 = \alpha_2 \simeq 0.9197$ and $\alpha_3 = 7/8$.

5.2 MAX {2,3,4}-SAT

Using the semidefinite programming relaxation of Section 2 and the rounding procedure defined by the three piecewise linear functions passing through the points given in Figure 2 we believe we obtain a 7/8-approximation algorithm for MAX $\{2,3,4\}$-SAT. We get in fact, an approximation algorithm for MAX 4-SAT with $\alpha_2 \simeq 0.8751$, $\alpha_3 = 7/8$, $\alpha_4 \simeq 0.8755$ but with $\alpha_1 \simeq 0.8352$. It is interesting to note the non-monotonicity of the function $g(\theta)$ and the fact that only one intermediate point is needed for $f(\theta)$ and $\epsilon(\theta)$ and only two intermediate points are needed for $g(\theta)$.

A 7/8-approximation algorithm for MAX $\{2,3,4\}$-SAT is of course optimal as a ratio better than 7/8 cannot be obtained even for MAX $\{3\}$-SAT, which is a subproblem of MAX $\{2,3,4\}$-SAT.

5.3 MAX 3-SAT

The optimal 7/8-approximation algorithm for MAX 3-SAT presented in [KZ97] has $\alpha_1 = \alpha_2 \simeq 0.87856$ and $\alpha_3 = 7/8$. Using pre-rounding rotations we can obtain an approximation algorithm for MAX 3-SAT with $\alpha_1, \alpha_2 \simeq 0.9197$ and $\alpha_3 = 7/8$. This algorithm would perform better than the algorithm of [KZ97] on instances in which some of the contribution to the optimal value of their semidefinite programming relaxation comes from clauses of size one or two. The pre-rounding rotation function f used to obtain this result is given in Figure 3. We cannot improve this result using a more complicated rounding procedure. Independent rounding seems to help only when clauses of length four are also allowed. When we try to

f
(0 , 0)
(0.379897 , 0.124860)
(0.754251 , 0.453529)
(1.049671 , 0.824970)
(1.325123 , 1.214945)
($\pi/2$, $\pi/2$)

Figure 4: The pre-rounding rotation f that seems to yield an approximation algorithm for MAX 2-SAT with $\alpha_1 \simeq 0.9828$ and $\alpha_2 \simeq 0.9309$.

optimize over the three function f,g and ϵ, we get that the optimal choice for ϵ is the all zero function.

5.4 MAX 2-SAT

Feige and Goemans [FG95] obtained an approximation algorithm for MAX 2-SAT with $\alpha_1 \simeq 0.976$ and $\alpha_2 \simeq 0.931$. Although we cannot improve α_2, the performance ratio on clauses of size two, we can obtain, using pre-rounding rotations, an approximation algorithm for MAX 2-SAT with $\alpha_1 \simeq 0.9828$ and $\alpha_2 \simeq 0.9309$. The rotation function f used to obtain this result is given in Figure 4. As was the case with MAX 3-SAT, combined rounding procedures that round vectors independently, with some positive probability, do not help here.

6 A description of `opt_fun`

As explained in Section 4, finding functions f, g and ϵ that define a good rounding procedure for MAX 4-SAT is not an easy task. In this section we describe the Matlab program `opt_fun` that helped us accomplish this task.

6.1 The performance ratio of a given rounding procedure

Before describing the search for a good rounding procedure, let us consider the more basic problem of finding, or more precisely estimating, the performance ratio of a given rounding procedure.

Each rounding procedure of the family we considered is specified using three functions $f, g : [0,\pi] \to [0,\pi]$ and $\epsilon : [0,\pi] \to [0,1]$. The functions f, g and ϵ are usually piecewise linear, and thus, a finite set of points determines the functions. We let $C = C(f, g, \epsilon)$ denote the rounding procedure obtained using the functions f, g and ϵ. As explained in Section 3, a lower bound $\alpha_i(C)$ on the performance ratio of the rounding procedure C on clauses of size i can be obtained as follows:

$$\alpha_i(C) \;=\; \min_{v_0,\ldots,v_i} \; ratio_C(v_0, \ldots, v_i) \;,$$

where

$$ratio_C(v_0,\ldots,v_i) \;=\; \frac{prob_C(v_0,\ldots,v_i)}{relax_C(v_0,\ldots,v_i)}$$

and where the minimum, or infimum, is taken over all configurations of the vectors $(v_0,\ldots,v_i)$ that satisfy the constraints given in Section 2 and for which $relax_C(v_0,\ldots,v_i) > 0$. The quantity

$$\alpha(C) \;=\; \min\{\,\alpha_1(C)\,,\,\alpha_2(C)\,,\,\alpha_3(C)\,,\,\alpha_4(C)\,\}\,,$$

is then a lower bound on the performance ratio of the MAX 4-SAT approximation algorithm that uses C.

Each configuration (v_0,v_1,v_2,v_3,v_4) is characterized, up to rotations, by the 10 angles $\Theta = (\theta_{ij})_{i<j}$, where $\theta_{ij} = \arccos(v_i \cdot v_j)$ is the angle between v_i and v_j. A setting Θ can be realized if and only if the matrix $(\cos\theta_{ij})$ is positive semidefinite. The functions $prob_C(v_0,v_1,v_2,v_3,v_4)$ and $relax_C(v_0,v_1,v_2,v_3,v_4)$ are therefore functions of 10 real variables. In general, a configuration of k vectors is described, up to rotations, by $\binom{k}{2}$ angles.

Evaluating $relax_C(v_0,\ldots,v_i)$ for a given configuration, where $1 \le i \le 4$, is a simple task. Evaluating $prob_C(v_0,v_1)$ and $prob_C(v_0,v_1,v_2)$ is also easy. Evaluating $prob_C(v_0,v_1,v_2,v_3)$ and $prob_C(v_0,v_1,v_2,v_3,v_4)$ is more difficult as it involves the evaluation of spherical volumes using numerical integration. We evaluated spherical volumes using Hsiang's method of infinitesimal symmetrization (see [Hsi88]).

To compute $\alpha_i(C(f,g,\epsilon))$, for $1 \le i \le 4$, we used the function `minimax` from Matlab's optimization toolbox. (The function $relax_C(v_0,v_1,v_2,v_3,v_4)$ is defined as the minimum of 23 different expressions. The function $ratio_C(v_0,v_1,v_2,v_3,v_4)$ is therefore the maximum of 23 expressions, and to find $\alpha_4(C(f,g,\epsilon))$ we want to find the configurations that minimize the maximum of these expressions.)

The function `minimax` tries to find a *local* minimum of a function $f = \max\{f_1,f_2,\ldots,f_k\}$, where each $f_i = f_i(x_1,\ldots,x_n)$ is a function of n real variables, subject to the constraints $g_i(x_1,\ldots,x_n) \ge 0$, for $1 \le i \le m$. (For a more exact description of `minimax` see [Mat96].) It receives an initial point $x^0 = (x_1^0,\ldots,x_n^0)$ and tries to converge to a local minimum of $f = \max\{f_1,f_2,\ldots,f_k\}$ that satisfies all the constraints.

To find $\alpha_i(C(f,g,\epsilon))$, we use `minimax` to minimize the function $ratio_C(v_0,\ldots,v_i)$, starting from hundreds of different starting points. We used both *random* starting points as well as many other starting points, such as points on which similar rounding procedures behaved badly. The constraints we used are the constraints of the semidefinite programming, that is, $relax_C(v_0,\ldots,v_i) > 0$, as well as the condition that the matrix $(\cos\theta_{ij})$ be positive semidefinite

It is important to note that for optimal, or almost optimal, rounding procedures, such as the rounding procedures described in Section 5, the minimum $\alpha_i(C(f,g,\epsilon))$ is usually attained at many different configurations. Changing the value of one of the function f,g and ϵ at an 'important' point, would usually improve the ratio attained on some of the configurations but worsen the ratio obtained on others.

6.2 Optimizing the functions

`opt_fun` tries to find functions f,g and ϵ that define good rounding procedures. It receives initial suggestions f_0,g_0 and ϵ_0 for the functions f,g and ϵ, each specified by giving a set of k points $(x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)$ through which the function is required to pass. The goal is to change the values of these $6k$ parameters so that the performance ratio of the resulting rounding procedure is maximized.

`opt_fun` works in an iterative manner. Each iteration is composed of the following two stages:

1. Find bad configurations for the current functions f, g and ϵ.

2. Optimize the functions f, g and ϵ with respect to the bad configurations found.

The bad configurations for the current functions f, g and ϵ are found as described in Section 6.1 above, using both random starting points and the bad configurations of the previous iteration. Both the number of random starting points and the number of bad configurations from the previous iteration are parameters of the program. The output of the first stage is a list $\Theta_1, \Theta_2, \ldots, \Theta_\ell$ of bad configurations. The maximal number of configurations used is a parameter supplied to the program. Each one of these configurations is a configuration of either 2,3,4 or 5 vectors, corresponding to clauses of sizes 1,2,3 and 4. By excluding configurations of certain sizes, we can deal with subproblems of MAX 4-SAT, such as MAX $\{2, 3, 4\}$-SAT and MAX $\{1, 4\}$-SAT.

In the second stage, we try to modify the functions f, g and ϵ so that the minimum ratio achieved on the configurations $\Theta_1, \Theta_2, \ldots, \Theta_\ell$ is maximized, or at least increased. This can be cast as a maximin problem. The function to be maximized is $\min\{ratio_C(\Theta_1), \ldots, ratio_C(\Theta_\ell)\}$, where this function is now considered to be a function of the $6k$ parameters that determine the functions f, g and ϵ. To solve this problem we again use Matlab's `minimax` (it is easy to convert a maximin problem into a minimax problem by negating the functions). The user of the program can ask that some of the $6k$ parameters determining the functions will not be changed by the optimization process. This is used to maintain conditions like $f(0) = 0$, $f(\pi/2) = \pi/2$, etc.

As the optimization of the functions f, g and ϵ in each iteration is done only with respect to the configurations $\Theta_1, \Theta_2, \ldots, \Theta_\ell$ obtained in the first stage of the iteration, and not with respect to all possible configurations, we are only allowing small changes to the functions f, g and ϵ to take place in each iteration. This ensures, according to our experience, the convergence of the iterative process, after not too many iterations. The process usually does not converge when large changes are allowed. The functions f, g and ϵ produced by this process seem to be the optimal choice of these functions, given the allowable number of bends.

Many technical details are omitted from the above discussion. It is, for example, necessary to make sure that the collection of bad configurations $\Theta_1, \Theta_2, \ldots, \Theta_\ell$ does not contain almost identical configurations. As the number of configurations held is bounded, such identical configurations may cause more interesting configurations to be removed from the list.

The rounding procedures for MAX 3-SAT and MAX 2-SAT, mentioned in Sections 5.3 and 5.4, were obtained by asking `opt_fun` to optimize the ratios obtained on clauses of certain sizes, while supplying lower bounds on ratios that should be obtained on clauses of other sizes. These lower bounds were achieved by adding constraints to `minimax`.

The exact operation of `opt_fun` is guided by dozens of parameters, many of them have to be tuned by trial and error. `opt_fun` is very time consuming. The rounding procedures of Section 5 were obtained by running evolving versions of `opt_fun` around the clock for months. However, verifying (numerically) that specific functions achieve a certain bound is less time consuming, as discussed in Section 6.1. After obtaining rounding procedures, such as those that are given in Figures 1,2,3 and 4, we made extensive efforts to verify the performance ratios claimed for them. But, as mentioned several times, we have no rigorous proofs that the performance ratios given are the correct ones.

7 Limitations of current rounding procedures

We presented above a rounding procedure using which we seem to get a 0.8721-approximation algorithm for MAX 4-SAT. This is extremely close to 7/8. Could it be that by searching a little bit harder, or perhaps allowing more bends, we could find a rounding procedure from the family defined in Section 3.5 using which we could obtain an optimal 7/8-approximation algorithm for MAX 4-SAT? Unfortunately, the answer is no, at least not using the type of local analysis used here. We show in this section that the rounding procedure described in Section 5.1 is close to being the best rounding procedure of the family considered.

Let θ_{ij}, for $0 \le i < j \le 4$, be the angles between the five unit vectors v_0, v_1, v_2, v_3, v_4. Let $c_{ij} = \cos\theta_{ij}$. It is not difficult to check that if

$$c_{12} = \tfrac{1+c_{01}+c_{02}-2c_{03}-2c_{04}}{3} \quad , \quad c_{13} = \tfrac{1+c_{01}-2c_{02}+c_{03}-2c_{04}}{3} \quad , \quad c_{23} = \tfrac{1-2c_{01}+c_{02}+c_{03}-2c_{04}}{3}$$

$$c_{14} = \tfrac{1+c_{01}-2c_{02}-2c_{03}+c_{04}}{3} \quad , \quad c_{24} = \tfrac{1-2c_{01}+c_{02}-2c_{03}+c_{04}}{3} \quad , \quad c_{34} = \tfrac{1-2c_{01}-2c_{02}+c_{03}+c_{04}}{3}$$

then $relax(v_0, v_1, v_2, v_3, v_4) = 1$. Let $0 < \theta_1 < \theta_2 \le \pi/2$ be two angles. Consider the configuration (v_0, v_1) in which $\theta_{01} = \pi - \theta_1$, and the two configurations $(v_0, v_1^1, v_2^1, v_3^1, v_4^1)$ and $(v_0, v_1^2, v_2^2, v_3^2, v_4^2)$ in which

$$(\theta_{01}^1, \theta_{02}^1, \theta_{03}^1, \theta_{04}^1) \;=\; (\theta_1, \theta_1, \theta_1, \pi-\theta_2) \quad , \quad (\theta_{01}^2, \theta_{02}^2, \theta_{03}^2, \theta_{04}^2) \;=\; (\theta_2, \theta_2, \theta_2, \theta_2)$$

and in which the angles θ_{jk}^i, for $1 \le j < k \le 4$ are determined according the relations above so that $relax(v_0, v_1^i, v_2^i, v_3^i, v_4^i) = 1$, for $i = 1, 2$. It is not difficult to check that

$$\theta_{12}^1 = \theta_{13}^1 = \theta_{23}^1 = \arccos\left(\tfrac{1+2\cos\theta_2}{3}\right) ,$$
$$\theta_{14}^1 = \theta_{24}^1 = \theta_{34}^1 = \arccos\left(\tfrac{1-3\cos\theta_1-\cos\theta_2}{3}\right) ,$$
$$\theta_{ij}^2 = \arccos\left(\tfrac{1-2\cos\theta_2}{3}\right) ,$$

for $1 \le i \le j \le 4$. Assume that the configurations $(v_0, v_1^i, v_2^i, v_3^i, v_4^i)$, for $i = 1, 2$, are feasible. For every rounding procedure C we have

$$\alpha(C) \;\le\; \min\{\, ratio_C(v_0, v_1) \,,\, ratio_C(v_0, v_1^1, v_2^1, v_3^1, v_4^1) \,,\, ratio_C(v_0, v_1^2, v_2^2, v_3^2, v_4^2) \,\} \,.$$

As the only angles between v_0 and and other vectors in these three configurations are $\theta_1, \theta_2, \pi-\theta_1$ and $\pi-\theta_2$, and as $f(\pi-\theta) = \pi - f(\theta)$, $g(\pi-\theta) = \pi - g(\theta)$ and $\epsilon(\pi-\theta) = \epsilon(\theta)$, we get that for every rounding procedure from our family, the three ratios $ratio_C(v_0, v_1)$, $ratio_C(v_0, v_1^1, v_2^1, v_3^1, v_4^1)$ and $ratio_C(v_0, v_1^2, v_2^2, v_3^2, v_4^2)$ depend on only six parameters, namely, on $f(\theta_1), f(\theta_2), g(\theta_1), g(\theta_2), \epsilon(\theta_1)$ and $\epsilon(\theta_2)$.

Take $\theta_1 = 0.95$ and $\theta_2 = \arccos(1/5) \simeq 1.369438$. It is possible to check that the resulting two configurations $(v_0, v_1^1, v_2^1, v_3^1, v_4^1)$ and $(v_0, v_1^2, v_2^2, v_3^2, v_4^2)$ are feasible. The choice of the six parameters that maximizes the minimum ratio of the three configurations is:

$$f(\theta_1) \simeq 1.410756 \quad , \quad g(\theta_1) \simeq 0 \quad , \quad \epsilon(\theta_1) \simeq 0.309376$$

$$f(\theta_2) \simeq 1.448494 \quad , \quad g(\theta_2) \simeq 1.233821 \quad , \quad \epsilon(\theta_2) \simeq 0.122906$$

With this choice of parameters, the three ratios evaluate to about 0.8724. No rounding procedure from the family can therefore attain a ratio of more than 0.8724 simultaneously on these three specific configurations. No rounding procedure from the family can therefore yield a performance ratio greater than 0.8724 for MAX 4-SAT, even if the functions f, g and ϵ are not restricted to be piecewise linear.

8 The quality of the relaxation

Let I be an instance of MAX 4-SAT. Let $opt(I)$ be the value of the optimal assignment for this instance. Let $opt^*(I)$ be the value of the optimal solution of the canonical semidefinite programming relaxation of the instance given in Section 2. Clearly $opt(I) \leq opt^*(I)$ for every instance I. The *integrality ratio* of the relaxation is defined to be $\gamma = \inf_I opt(I)/opt^*(I)$, where the infimum is taken over all the instances.

In Section 3, when we analyzed the performance of different rounding procedures, we compared the value, or rather the expected value, of the assignment produced by a rounding procedure to $opt^*(I)$, the optimal value of the semidefinite programming relaxation. It is not difficult to see that any lower bound α on the performance ratio of a rounding procedure obtained in this way would satisfy $\alpha \leq \gamma$. Thus, the rounding procedure of Section 5.1 seem to imply that $\gamma \geq 0.8721$. In this section we describe *upper bounds* on the integrality ratio γ, thereby obtaining upper bounds on the performance ratios that can be obtained by *any* approximation algorithm that uses the relaxation of Section 2, at least using the type of analysis used in Section 3.

It is shown in [KZ97] that the integrality ratio of the canonical semidefinite programming relaxation of MAX 3-SAT is exactly $\gamma_3 = 7/8$. As the canonical relaxations of MAX 3-SAT and MAX 4-SAT coincide on instances of MAX 3-SAT, we get that $\gamma = \gamma_4 \leq 7/8$.

We next show, that the integrality ratio of the canonical relaxation of MAX 4-SAT, given in Section 2, is smaller than 0.8754, even when restricted to instances of MAX $\{1,4\}$-SAT, i.e., to instances of MAX 4-SAT in which all clauses are of size 1 or 4. In other words, we show that there are instances of MAX $\{1,4\}$-SAT for which the ratio between the value of the optimal solution of the instance and the value of the optimal solution of the semidefinite programming relaxation of the instance is smaller than 0.8754. More precisely, we describe a sequence of instances $\{I_n\}_{n\geq 1}$ such that the above ratio approaches, as n tends to infinity, a limit which is below 0.8754.

The variables appearing in the instance I_n are $x_1, \ldots, x_n, y_1, \ldots, y_n$. The clauses appearing in I_n are as follows, where c_1, c_2, c_3 are some constants to be chosen later:

- $\bar{x}_i$ with weight $\frac{c_1}{n}$, for $1 \leq i \leq n$.
- $x_{i_1} \vee x_{i_2} \vee x_{i_3} \vee \bar{y}_{i_4}$ with weight $\frac{c_2}{n\binom{n}{3}}$, for $1 \leq i_1 < i_2 < i_3 \leq n$ and $1 \leq i_4 \leq n$.
- $y_{i_1} \vee y_{i_2} \vee y_{i_3} \vee y_{i_4}$ with weight $\frac{c_3}{\binom{n}{4}}$, for $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$.

We first find the optimal assignment for I_n in terms of c_1, c_2 and c_3. Because of the symmetry of I_n, we only have to decide how many x_i's should get the value 1 and how many y_i's should get the value 1. Assume that k_1 and k_2 are the number of x_i's and y_i's, respectively, that get the value 1 in an assignment. The value of the assignment is then:

$$(n-k_1) \cdot \frac{c_1}{n} + \left(n\binom{n}{3} - k_2 \cdot \binom{n-k_1}{3}\right) \cdot \frac{c_2}{n\binom{n}{3}} + \left(\binom{n}{4} - \binom{n-k_2}{4}\right) \cdot \frac{c_3}{\binom{n}{4}} .$$

Let $p = \frac{k_1}{n}$ and $q = \frac{k_2}{n}$. As n tends to infinity the value of the assignment tends to

$$c_1 \cdot (1-p) + c_2 \cdot (1 - q \cdot (1-p)^3) + c_3 \cdot (1 - (1-q)^4) . \quad (1)$$

For each c_1, c_2, c_3 we choose p and q that maximize this expression.

We now construct a good feasible solution of the semidefinite programming relaxation of I_n. The value of this solution provides a lower bound on the value of the optimal solution of the semidefinite program. For every $1 \le i \le n$, let u_i and w_i be the vectors corresponding to x_i and y_i , respectively. In our solution, there is a fixed angle θ_{01} between every u_i and v_0, a fixed angle θ_{02} between every w_i and v_0, a fixed angle θ_{11} between every u_i and u_j for $i \ne j$, a fixed angle θ_{22} between every w_i and w_j for $i \ne j$, and a fixed angle θ_{12} between every u_i and w_j. Let

$$a = \cos\theta_{01} \quad , \quad b = \cos\theta_{02} \quad , \quad c = \cos\theta_{11} \quad , \quad d = \cos\theta_{12} \quad , \quad e = \cos\theta_{22} \; .$$

We want every 4-clause to have a relaxed value of 1. We therefore take:

$$c = \tfrac{1+2b}{3} \quad , \quad d = \tfrac{b+3a-1}{3} \quad , \quad e = \tfrac{1-2b}{3} \; . \tag{2}$$

It is not difficult to check, as was done in the previous section, that this ensures that the relaxed value of all the 4-clauses is indeed 1. To maximize the value of the solution we therefore maximize the value of the 1-clauses. The value of each 1-clause is $\frac{1+a}{2}$ and thus, we want to maximize a. In other words, we want to solve the following semidefinite program:

$$\begin{array}{ll}
\text{Maximize} & a \\
\text{subject to :} & y_{ii} = 1 \quad , \quad 0 \le i \le 2n \\
& y_{0i} = a \quad , \quad 1 \le i \le n \\
& y_{0i} = b \quad , \quad n+1 \le i \le 2n \\
& y_{ij} = c \quad , \quad 1 \le i,j \le n \; , \; i \ne j \\
& y_{ij} = d \quad , \quad 1 \le i \le n \; , \; n+1 \le j \le 2n \\
& y_{ij} = e \quad , \quad n+1 \le i,j \le 2n \; , \; i \ne j \\
& c = \frac{1+2b}{3} \quad , \quad d = \frac{b+3a-1}{3} \quad , \quad e = \frac{1-2b}{3} \\
& Y = (y_{ij}) \text{ is symmetric positive semidefinite}
\end{array}$$

The matrix Y has the following form:

$$
Y \;=\; \begin{array}{|c|cccc|cccc|}
\hline
1 & a & \cdots & \cdots & a & b & \cdots & \cdots & b \\
\hline
a & 1 & c & \cdots & c & d & \cdots & d & d \\
a & c & 1 & \cdots & c & d & \cdots & d & d \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a & c & c & \cdots & 1 & d & \cdots & d & d \\
\hline
b & d & d & \cdots & d & 1 & \cdots & e & e \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b & d & d & \cdots & d & e & \cdots & 1 & e \\
b & d & d & \cdots & d & e & \cdots & e & 1 \\
\hline
\end{array}
$$

We want to find settings of a and b, which then determine the values of c, d and e, such that Y is positive semidefinite and a is as large as possible. In other words, we want to maximize a subject to the constraint that all the eigenvalues of Y, are non-negative. It is easy to see that for this extremal setting of a and b, 0 would be an eigenvalue of Y and therefore $\det(Y) = 0$. It is not difficult to check that

$$
\det(Y) = (1-c)^{n-1} \cdot (1-e)^{n-1} \cdot \det \begin{pmatrix} 1 & an & bn \\ a & 1+c(n-1) & dn \\ b & dn & 1+e(n-1) \end{pmatrix} .
$$

By expanding the 3×3 determinant, and expressing c, d and e in terms of a and b, we get the equation:

$$
\begin{aligned}
&(-6\,a + 12\,a^2 - 2\,b + 12\,a\,b - 24\,a^2\,b + 8\,b^2 - 6\,a\,b^2 + 6\,b^3)\,n^2 \\
&+ (-4 + 6\,a^2 + 6\,a^2\,b - 2\,b^2 - 6\,b^3)\;n + (-4 + 4\,b^2) \;=\; 0 \,.
\end{aligned}
$$

As n tends to infinity, the a and b that we are looking for would therefore approach a root of the following equation:

$$
-6\,a + 12\,a^2 - 2\,b + 12\,a\,b - 24\,a^2\,b + 8\,b^2 - 6\,a\,b^2 + 6\,b^3 \;=\; 0 \,.
$$

This is a quadratic equation with respect to a, and it can therefore be used to express a in terms of b. The maximal value of a is then attained when $b \simeq 0.195208$ and then $a \simeq 0.541588$. (The optimal b is a root of the equation $153b^6 - 69b^5 - 149b^4 + 58b^3 + 35b^2 - 13b + 1 = 0$.) One can verify that for these values of a and b, the matrix Y is positive semidefinite for every n.

Now let $c_1 = 0.288608$, $c_2 = 0.520776$ and $c_3 = 0.190616$. The p and q that we get by maximizing (1) are $p \simeq 0.325341$ and $q \simeq 0.405851$. With all these parameters we have that the value of the optimal assignment is about 0.817446 and the value of the semidefinite program is at least 0.933849, and their ratio is, therefore, at most 0.875351.

Though this upper bound does not preclude the possibility of obtaining an optimal 7/8-approximation algorithm for MAX 4-SAT using the canonical semidefinite programming relaxation of the problem, the closeness of this upper bound to 7/8 does indicate that it will not be easy, even if clauses of length 3 are not present. It may be necessary to consider stronger relaxations of MAX 4-SAT, e.g., relaxations obtained by considering several clauses of the instance at once.

9 Concluding remarks

We have come frustratingly close to obtaining an optimal 7/8-approximation algorithm for MAX 4-SAT. We have seen that devising a 7/8-approximation algorithm for MAX $\{1,4\}$-SAT is already a challenging problem. Note that Håstad's 7/8 upper bound for MAX 3-SAT and MAX 4-SAT does not apply to MAX $\{1,4\}$-SAT, as clauses of length three are not allowed in this problem. A gadget (see [TSSW96]) supplied by Greg Sorkin shows that no polynomial time approximation algorithm for MAX $\{1,4\}$-SAT can have a performance ratio greater that 9/10, unless P=NP.

We believe that optimal 7/8-approximation algorithms for MAX 4-SAT and MAX SAT do exist. The fact that we have come so close to obtaining such algorithms may in fact be seen as cause for optimism. There is still a possibility that simple extensions of ideas laid out here could be used to achieve this goal. If this fails, it may be necessary to attack the problems from a more global point of view. Note that the analysis carried out here was very local in nature. We only considered one clause of the instance at a time. As a result we only obtained lower bounds on the performance ratios of the algorithms considered. It may even be the case that the algorithms from the family of algorithms considered here do give a performance ratio of 7/8 for MAX 4-SAT although a more global analysis is required to show it.

We also hope that MAX 4-SAT would turn out to be the last barrier on the road to an optimal approximation algorithm for MAX SAT. The almost optimal algorithms for MAX 4-SAT presented here may be used to obtain an almost optimal algorithm for MAX SAT. We have not worked out yet the exact bounds that we can get for MAX SAT as we still hope to get an optimal algorithm for MAX 4-SAT before proceeding with MAX SAT.

Finally, a word on our methodology. Our work is a bit unusual as we use experimental and numerical means to obtain *theoretical* results. We think that the nature of the problems that we are trying to solve calls for this approach. No one can rule out, of course, the possibility that some clever new ideas would dispense with most of the technical difficulties that we are facing here. Until that happens, however, we see no alternative to the current techniques. The use of experimental and numerical means does not mean that we have to give up the rigorousity of the results. Once we obtain the 'right' result, we can devote efforts to proving it rigorously, possibly using automated means.

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