Coloring k-colorable graphs using smaller palettes

Eran Halperin * *Ram Nathaniel* * *Uri Zwick* *

Abstract

We obtain the following new coloring results:

- A 3-colorable graph on n vertices with maximum degree Δ can be colored, in polynomial time, using $O((\Delta \log \Delta)^{1/3} \cdot \log n)$ colors. This slightly improves an $O((\Delta^{1/3} \log^{1/2} \Delta) \cdot \log n)$ bound given by Karger, Motwani and Sudan. More generally, k-colorable graphs with maximum degree Δ can be colored, in polynomial time, using $O((\Delta^{1-2/k} \log^{1/k} \Delta) \cdot \log n)$ colors.
- A 4-colorable graph on n vertices can be colored, in polynomial time, using $\tilde{O}(n^{7/19})$ colors. This improves an $\tilde{O}(n^{2/5})$ bound given again by Karger, Motwani and Sudan. More generally, k-colorable graphs on n-vertices can be colored, in polynomial time, using $\tilde{O}(n^{\alpha_k})$ colors, where $\alpha_5 = 97/207$, $\alpha_6 = 43/79$, $\alpha_7 = 1391/2315$, $\alpha_8 = 175/271, \ldots$

The first result is obtained by a slightly more refined probabilistic analysis of the semidefinite programming based coloring algorithm of Karger, Motwani and Sudan. The second result is obtained by combining the coloring algorithm of Karger, Motwani and Sudan, the combinatorial coloring algorithms of Blum and an extension of a technique of Alon and Kahale (which is based on the Karger, Motwani and Sudan algorithm) for finding relatively large independent sets in graphs that are guaranteed to have very large independent sets. The extension of the Alon and Kahale result may be of independent interest.

1 Introduction

Finding a 3-coloring of a given 3-colorable graph is a well known NP-hard problem. Finding a 4-coloring of such a graph is also known to be NP-hard (Khanna, Linial and Safra [KLS00] and Guruswami and Khanna [GK00]). Karger, Motwani and Sudan [KMS98] show, on the other hand, using semidefinite programming, that a 3-colorable graph on n vertices with maximum degree Δ can be colored, in polynomial time, using $O((\Delta^{1/3} \log^{1/2} \Delta) \cdot \log n)$ colors. Combining this result with an old coloring algorithm of Wigderson [Wig83] they also obtain an algorithm for coloring arbitrary 3-colorable graphs on n vertices using $O(n^{1/4} \log^{1/2} n)$ colors. By combining the result of [KMS98] with a coloring algorithm of Blum [Blu94], Blum and Karger [BK97] obtain a polynomial time algorithm that can color a 3-colorable graph using $\tilde{O}(n^{3/14})$ colors.

The semidefinite programming based coloring algorithm of Karger, Motwani and Sudan [KMS98] can also be used to color k-colorable graphs of maximum degree Δ using $\tilde{O}(\Delta^{1-2/k})$ colors. Combined again with the technique of Wigderson [Wig83] this gives a polynomial time algorithm for coloring k-colorable graph using $\tilde{O}(n^{1-3/(k+1)})$ colors. Blum [Blu94] gives a combinatorial algorithm for coloring k-coloring graphs using $\tilde{O}(n^{\beta_k})$ color, where the β_k's satisfy a complicated recurrence relation. The first values in the sequence are $\beta_3 = \frac{3}{8}$, $\beta_4 = \frac{3}{5}$, $\beta_5 = \frac{91}{131}$, $\ldots$ The algorithm of [KMS98] uses less colors than the algorithm of Blum [Blu94] for any $k \geq 3$. No combination of the semidefinite programming based coloring algorithm of [KMS98] with the combinatorial algorithm of Blum [Blu94] was given, prior to this work, for $k \geq 4$.

In this paper we present several improved coloring algorithms. Our improvements fall into two different categories. We first consider the semidefinite programming based coloring algorithm of Karger, Motwani and Sudan [KMS98]. We show that the number of colors used by this algorithm can be reduced, alas, by only a polylogarithmic factor. Though the improvement obtained here is not very significant, we believe that it is interesting as it is obtained not using tedious calculations but rather using a simple refinement of the probabilistic analysis given by [KMS98]. Furthermore, we can show that this refined analysis is tight.

Having considered the algorithm of [KMS98] on its own, we turn our attention to possible combinations of that algorithm with the combinatorial algorithm of Blum [Blu94]. The $\tilde{O}(n^{3/14})$ result of Blum and Karger [BK97] for $k = 3$ is an example of such a combination. Although no such combinations were previously reported for $k > 3$, it is not difficult to construct simple combinations of these algorithms that would yield

*School of Computer Science, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: {heran,ramn,zwick}@cs.tau.ac.il.

Coloring algorithm	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
Wigderson	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
[Wig83]	0.5	0.6666666	0.75	0.8	0.8333333	0.8571428
Blum	$\frac{3}{8}$	$\frac{3}{5}$	$\frac{91}{131}$	$\frac{105}{137}$	$\frac{5301}{6581}$	$\frac{10647}{12695}$
[Blu94]	0.375	0.6	0.6946564	0.7664233	0.8055006	0.8386766
Karger, Motwani, Sudan	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{2}{3}$
[KMS98]	0.25	0.4	0.5	0.5714285	0.625	0.6666666
Our Results	$\left[\frac{3}{14}\right]$	$\frac{7}{19}$	$\frac{97}{207}$	$\frac{43}{79}$	$\frac{1391}{2315}$	$\frac{175}{271}$
$\star$	(0.2142857)	0.3684210	0.4685990	0.5443037	0.6008639	0.6457564

Table 1: The exponents of the new coloring algorithms, and of the previously available algorithms, for $3 \leq k \leq 8$. The $3/14$ exponent for $k=3$ is from Blum and Karger [BK97].

improved results. We go one step further and present, what we believe, are non-trivial combinations of these algorithms that yield even further improved results. In particular, our combinations use a third ingredient, an extension of algorithm of Alon and Kahale [AK98] that can be used to find large independent sets in graphs that contain very large independent sets. More specifically, Alon and Kahale [AK98] show that if a graph on n vertices contains an independent set of size $n/k+m$, for some fixed integer $k \geq 3$ and some $m > 0$, then an independent set of size $\tilde{\Omega}(m^{3/(k+1)})$ can be found in (random) polynomial time. We extend this result and show that if a graph on n vertices contains an independent set of size n/α, where $\alpha \geq 1$ is not necessarily integral, then an independent set of size $\tilde{\Omega}(n^{f(\alpha)})$ can be found in (random) polynomial time, where $f(\alpha)$ is a continuous function, to be described explicitly, that satisfies $f(k) = 3/(k+1)$ for every integer $k \geq 2$. This result may be of independent interest. Interestingly, the Alon and Kahale [AK98] result, and its extension, are based on the algorithm of Karger, Motwani and Sudan [KMS98] that may also be viewed as an algorithm for finding large independent sets.

Equiped with this new ingredient, we describe a combined coloring algorithm that uses ideas from [Blu94], [KMS98] and [AK98] to color a k-colorable graph using $\tilde{O}(n^{\alpha_k})$ colors, where $\alpha_4 = 7/19$, $\alpha_5 = 97/207$, $\alpha_6 = 43/79$, $\alpha_7 = 1391/2315$, $\alpha_8 = 175/271$, ... (See Table 1 for a comparison of these bounds with the previously available bounds.) An explicit, but complicated, recurrence relation defining α_k for every k is given later in the paper. The new algorithm performs better than all the previously available algorithms for $k \geq 4$. We obtain no improvement over the $\tilde{O}(n^{3/14})$ bound of Blum and Karger [BK97] for $k=3$ (other than the polylogarithmic improvement mentioned earlier).

The rest of this paper is organized as follows. In Section 2 we present our refinement to the algorithm of Karger *et al.* [KMS98]. In Section 3 we present our extension of the technique of Alon and Kahale [AK98]. In Section 4 we describe some coloring tools of Blum [Blu94]. Finally, in Section 5 we describe our new coloring algorithm. We end in Section 6 with some concluding remarks and open problems.

2 The Karger-Motwani-Sudan algorithm and its refinement

Karger, Motwani and Sudan introduce *vector colorings* of a graph, a notion which is closely related to Lovász's *orthogonal representations* and to its ϑ-function (Lov'asz [Lov79], Grötschel *et al.* [GLS93]):

Definition 2.1 ([KMS98]) *A* vector α-coloring *of a graph* $G=(V,E)$*, where* $V=\{1,2,\ldots,n\}$*, is sequence* *of unit vectors* $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ *such that if* $(i,j) \in E$*, then* $v_i \cdot v_j \leq -\frac{1}{\alpha-1}$.

It is easy to see that if G is k-colorable then G also has a vector k-coloring. There are, however, graphs that are vector k-colorable but are not k-colorable. A vector k-coloring of a graph $G = (V,E)$, if one exists, can be found, in polynomial time, by solving a semidefinite program. See [KMS98] for details.[1] Karger, Motwani and Sudan [KMS98] also present the following lemma which we use in Section 5.

Lemma 2.2 ([KMS98]) *Let* $G = (V,E)$ *be a vector* α*-colorable graph, where* $\alpha > 2$*. Then, for every vertex* $v \in V$*, the subgraph of* G *induced by the neighbors of* v *is vector* $(\alpha-1)$*-colorable, and a vector* $(\alpha-1)$*-coloring of it can be found in polynomial time.*

Karger *et al.* [KMS98] show next that if $G = (V,E)$ is a vector k-colorable graph on n vertices with maximum degree Δ, then an independent set of G of size $\Omega(\frac{n}{\Delta^{1-2/k}\log^{1/2}\Delta})$ can be found in polynomial time. This easily implies that a vector k-colorable graph on n vertices with maximum degree Δ may be colored, in polynomial time, using $O((\Delta^{1-2/k}\log^{1/2}\Delta)\cdot\log n)$ colors. We present the following refinement of this result:

Theorem 2.3 *Let* $\alpha \geq 2$ *and let* $G = (V,E)$ *be vector* α*-colorable graph on* n *vertices with average degree* D*. Then, an independent set of* G *of size* $\Omega(\frac{n}{D^{1-2/\alpha}\log^{1/\alpha} D})$ *can be found in polynomial time.*

There are two minor differences, and one more substantial difference, between this formulation and the result of [KMS98]. The first is that α is not assumed to be integral. The second is that the *maximum* degree Δ is replaced by the *average* degree D. (The Δ in the $\Omega((\Delta^{1-2/k}\log^{1/2}\Delta)\cdot\log n)$ bound *cannot* be replaced by D, as the average degree, unlike the maximum degree, may increase when vertices are removed from the graph.) More interestingly, the exponent of $\log\Delta$ is reduced from $1/2$ to $1/\alpha$, thus obtaining a logarithmic improvement in the number of colors needed to color low degree graphs. This improvement, as we mentioned, is obtained using a simple modification of the probabilistic argument of [KMS98].

We present here a proof of Theorem 2.3 for the case $\alpha = 3$. This allows us to explain the refined argument in the simplest possible setting. The proof for general α is almost identical.

Proof: (of Theorem 2.3 for $\alpha = 3$**)** Let $v_1, v_2, \ldots, v_n$ be a vector 3-coloring of G. Let D be the average degree of G. Let $c = \sqrt{\frac{2}{3}\ln D - \frac{1}{3}\ln\ln D}$. Choose a random vector r according to the standard n-dimensional normal distribution. Let $I = \{i \in V \mid v_i \cdot r \geq c\}$. Let $n' = |I|$ be size of I and let $m' = |\{(i,j) \in E \mid i,j \in I\}|$ be the number of edges contained in I. An independent set I' of size $n' - m'$ is then easily obtained by removing one vertex from each edge contained in I. We show that the expected size of I' is $\Omega(\frac{n}{(D\log D)^{1/3}})$.

Let $N(x) = \int_x^\infty \phi(y)dy$, where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, denote the tail of the standard normal distribution. It is well known that $(\frac{1}{x} - \frac{1}{x^3})\phi(x) \leq N(x) \leq \frac{1}{x}\phi(x)$, for every $x > 0$. It is also known that if v is an arbitrary unit vector in $\mathbb{R}^n$, and r is a random vector chosen according to the standard n-dimensional normal distribution, then the inner product $v \cdot r$ is distributed according to the standard one dimensional normal distribution. Furthermore, if v_1 and v_2 are *orthogonal* unit vectors then the two random variables $v_1 \cdot r$ and $v_2 \cdot r$ are independent. It is easy to see, then, that:

$$\begin{aligned} E[n'] &= n\Pr[v_1 \cdot r \geq c] = nN(c)\,, \\ E[m'] &= m\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c]\,, \end{aligned}$$

where v_1 and v_2 are two unit vectors such that $v_1 \cdot v_2 \leq -\frac{1}{2}$, and n and m, respectively, are the number of vertices and edges in the graph. It is not difficult to see that the probability $\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c]$ is a monotone increasing function of the angle between v_1 and v_2. As we would like to obtain an upper bound on the probability, we may assume, therefore, that $v_1 \cdot v_2 = -\frac{1}{2}$. Karger *et al.* [KMS98] argue that

$$\begin{aligned} &\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c] \\ \leq\ &\Pr[(v_1 + v_2) \cdot r \geq 2c) = N(2c)\,, \end{aligned}$$

where the rightmost equality follows from the fact that $v_1 + v_2$ is also a unit vector. We obtain a slightly sharper upper bound on this probability:

Claim 2.4 *If* v_1 *and* v_2 *are unit vectors such that* $v_1 \cdot v_2 = -\frac{1}{2}$ *then*

$$\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c] \leq N(\sqrt{2}c)^2\,.$$

Proof: Let v_1 and v_2 be two unit vectors such that $v_1 \cdot v_2 = -\frac{1}{2}$. Note that v_1 and v_2 form an angle of $120°$. Let D be the tip of cv_1. Draw a line perpendicular

[1] This statement is not completely accurate. What can be found in polynomial time is a vector (k+ϵ)-coloring of the graph for, say, $\epsilon = 2^{-n}$. The technical difficulties caused by this can be easily overcome. See [KMS98] for details.

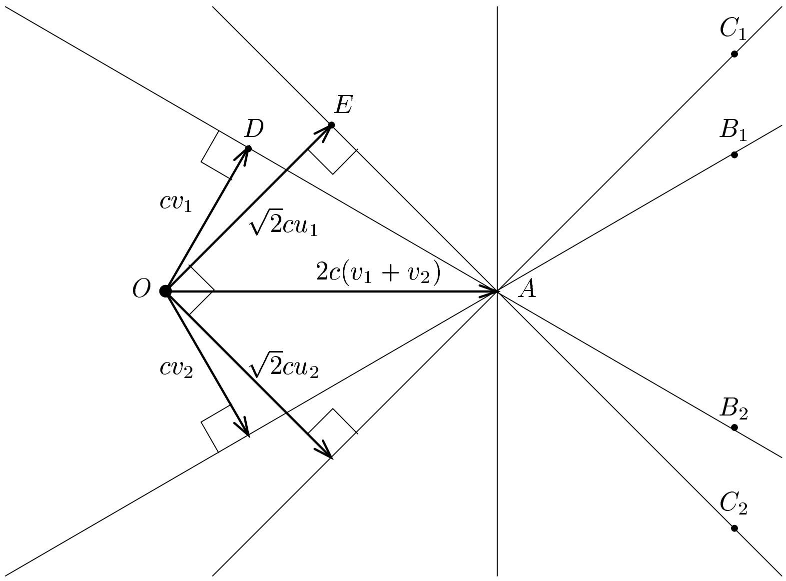

Figure 1: Upper bounding $\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c]$ when $||v_1|| = ||v_2|| = 1$ and $v_1 \cdot v_2 = -\frac{1}{2}$.

to cv_1 that passes through D. Similarly, draw a line perpendicular to cv_2 that passes through the tip of cv_2, as shown in Figure 1. It is easy to see that these two lines intersect at the point A which is $2c(v_1 + v_2)$. (This follows from the fact that $\angle DOA = 60°$ so that $\angle DAO = 30°$ and the fact that $\sin 30° = \frac{1}{2}$. Note that $v_1 + v_2$ is also a unit vector.) The projection of a standard n-dimensional normal vector r on the plane spanned by v_1 and v_2 is a standard 2-dimensional normal vector which we denote by r'. Note that $v_1 \cdot r = v_1 \cdot r'$ and $v_2 \cdot r = v_2 \cdot r'$. The probability that we have to bound is therefore the probability that the random vector r' falls into the wedge defined by the angle $\angle B_1AB_2$. Karger, Motwani and Sudan [KMS98] bound this probability by the probability that r' falls to the right of the vertical line that passes through A, which is $N(2c)$.

Let u_1 and u_2 be unit vectors in the plane spanned by v_1 and v_2 such that the angle formed by them and $v_1 + v_2$ is 45° (see Figure 1). Draw a line through A which is perpendicular to u_1. Similarly, draw a line through A which is perpendicular to u_2. Let E be the point on the first line in the direction of u_1. A simple calculation shows that $OE = \sqrt{2}c$. We bound the probability that r' falls into the wedge formed by $\angle B_1AB_2$ by the probability that it falls into the wedge formed by $\angle C_1AC_2$. This probability is just $\Pr[u_1 \cdot r' \geq \sqrt{2}c \text{ and } u_2 \cdot r' \geq \sqrt{2}c]$. As $u_1 \cdot u_2 = 0$, the events $u_1 \cdot r' \geq \sqrt{2}c$ and $u_2 \cdot r' \geq \sqrt{2}c$ are *independent*. Thus, this probability is just $N(\sqrt{2}c)^2$. ∎

Using a more complicated analysis, which is omitted from this extended abstract, we can show that $\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c] = \Omega(\frac{1}{c^2}\mathrm{e}^{-2c^2})$. Thus, the bound given in Claim 2.4 is asymptotically tight.

We are now back in the proof of Theorem 2.3. As $m \leq nD/2$, we get that

$$\begin{aligned} E[n' - m'] &\geq nN(c) - \tfrac{nD}{2}N(\sqrt{2}c)^2 \\ &= n\left(N(c) - \tfrac{D}{2}N(\sqrt{2}c)^2\right) . \end{aligned}$$

With $c = \sqrt{\frac{2}{3}\ln D - \frac{1}{3}\ln\ln D}$ we have $\mathrm{e}^{3c^2/2} = \frac{D}{\ln^{1/2} D}$ and therefore

$$\frac{N(c)}{N(\sqrt{2}c)^2} > \frac{(\frac{1}{c} - \frac{1}{c^3})\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-c^2/2}}{\frac{1}{2c^2}\frac{1}{2\pi}\mathrm{e}^{-2c^2}} > \sqrt{2\pi}\cdot c\,\mathrm{e}^{3c^2/2} > D .$$

Thus,

$$E[n' - m'] \geq \frac{n}{2}(\frac{1}{c} - \frac{1}{c^3})\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-c^2/2} = \Omega(\frac{n}{(D\ln D)^{1/3}}) ,$$

and the proof of the theorem (for $\alpha = 3$) is completed. ∎

The proof of the theorem for general α is very similar. We choose $c = \sqrt{(1 - \frac{2}{\alpha})(2\ln D - \ln\ln D)}$. It is then not difficult to see that the probability $\Pr[v_1 \cdot r \geq c \text{ and } v_2 \cdot r \geq c]$, when $v_1 \cdot v_2 = -\frac{1}{\alpha - 1}$, is upper bounded by $N(\sqrt{\frac{k-1}{k-2}}c)^2$, and the expected size of the independent set I' is indeed $\Omega(\frac{n}{D^{1-2/\alpha}\log^{1/\alpha} D})$.

3 The Alon-Kahale algorithm and its extension

Alon and Kahale [AK98] obtained the following result:

Theorem 3.1 *Let $G = (V, E)$ be a graph on n vertices that contains an independent set of size at least $\frac{n}{k} + m$, where $k \geq 3$ is an integer. Then, an independent set of G of size $\tilde{\Omega}(m^{3/(k+1)})$ can be found in polynomial time.*

Here we prove the following extension of their result:

Theorem 3.2 *Let $G = (V, E)$ be a graph on n vertices that contains an independent set of size at least $\frac{n}{\alpha}$, where $\alpha \geq 1$. Let $k = \lfloor \alpha \rfloor$. Then, an independent set of G of size $\tilde{\Omega}(n^{f(\alpha)})$ can be found in polynomial time, where*

$$f(\alpha) \;=\; \frac{\alpha(\alpha-1)}{k\left(\alpha(\alpha-k) + \frac{(k-1)(k+1)}{3}\right)} \;.$$

In particular, $f(\alpha) = 1$, if $1 \leq \alpha \leq 2$, $f(\alpha) = \frac{\alpha}{2(\alpha-1)}$, if $2 \leq \alpha \leq 3$, and $f(k) = \frac{3}{k+1}$ for every integer $k \geq 1$. Also, the function $f(\alpha)$ satisfies the functional equation $f(\alpha) = 1 \Big/ \left(1 + \frac{1-2/\alpha}{f(\alpha-1)}\right)$, for every $\alpha \geq 2$.

We only use this result for $\alpha = k + O(\frac{1}{\log n})$, where $k \geq 2$ is an integer. As $f(k+O(\frac{1}{\log n})) = \frac{3}{k+1} + O(\frac{1}{\log n})$, we still get in this case an independent set of size $\tilde{\Omega}(n^{3/(k+1)})$. For completeness, we give a proof of the more general result. The proof of Theorem 3.2 follows from the following two lemmas:

Lemma 3.3 *Let $G = (V, E)$ be a graph on n vertices with an independent set of size at least $\frac{n}{\alpha}$, where $\alpha \geq 2$. Then, a subset $S \subseteq V$ of size $|S| \geq \frac{n}{\log n}$, and a vector $(\alpha + O(\frac{1}{\log n}))$-coloring of $G[S]$, the subgraph of G induced by S, can be found in polynomial time.*

Proof: Assume that $V = \{1, 2, \ldots, n\}$ and consider the natural semidefinite programming relaxation of the independent set problem:

$$\begin{array}{lc}
\text{Maximize} & \displaystyle\sum_{i=1}^{n} \frac{1 + v_0 \cdot v_i}{2} \\
\text{s.t.} & (v_0 + v_i) \cdot (v_0 + v_j) = 0 \;,\; (i,j) \in E \\
 & \|v_i\| = 1 \;,\; 1 \leq i \leq n
\end{array}$$

An almost optimal solution $v_0, v_1, \ldots, v_n$ of this semidefinite program can be found in polynomial time. As G is assumed to contain an independent set of size at least n/α, we may assume that

$$\sum_{i=1}^{n} v_0 \cdot v_i \;\geq\; \left(\frac{2}{\alpha} - 1 - \frac{1}{\log n}\right) n \;.$$

(The $-1/\log n$ term comes from the fact that $v_0, v_1, \ldots, v_n$ is only an almost optimal solution of the program. We can make this term much smaller if we wish, but $1/\log n$ is small enough for our purposes.) We now use the following simple facts:

Claim 3.4 *If $\sum_{i=1}^{n} x_i \geq \gamma n$ and $x_i \leq 1$, for every $1 \leq i \leq n$, then for any $\epsilon \geq 0$, at least ϵn of the x_i's satisfy $x_i > (\gamma - \epsilon)/(1 - \epsilon)$.*

Indeed, if the claim is not satisfied then $\sum_{i=1}^{n} x_i < (1-\epsilon)n \cdot (\gamma - \epsilon)/(1-\epsilon) + \epsilon n = \gamma n$, a contradiction. It is easy to check that $(\gamma - \epsilon)/(1-\epsilon) > \gamma - 2\epsilon$, if $\epsilon < (1+\gamma)/2$. Using this fact with $x_i = v_0 \cdot v_i$, $\gamma = \frac{2}{\alpha} - 1 - \frac{1}{\log n}$, and $\epsilon = \frac{1}{\log n}$, we get that for at least $\frac{n}{\log n}$ of the vectors satisfy $v_0 \cdot v_i > \frac{2}{\alpha} - 1 - \frac{3}{\log n}$. Thus, if $S = \{i \mid v_0 \cdot v_i > \frac{2}{\alpha} - 1 - \frac{3}{\log n}\}$, then $|S| \geq \frac{n}{\log n}$.

Claim 3.5 *Let v_0, v_i and v_j be unit vectors such that $v_i \neq v_0$, $v_j \neq v_0$ and $(v_0 + v_i) \cdot (v_0 + v_j) = 0$. Let v_i' and v_j', respectively, be the normalized projections of v_i and v_j on the space orthogonal to v_0. Then*

$$v_i' \cdot v_j' \;=\; -\sqrt{\frac{1 + (v_0 \cdot v_i)}{1 - (v_0 \cdot v_i)}} \cdot \sqrt{\frac{1 + (v_0 \cdot v_j)}{1 - (v_0 \cdot v_j)}} \;.$$

Proof: Let $a_i = v_0 \cdot v_i$ and $a_j = v_0 \cdot v_j$. Then

$$\begin{aligned} v_i' \;&=\; \frac{v_i - a_i v_0}{||v_i - a_i v_0||} \\ =\; \frac{v_i - a_i v_0}{\sqrt{(v_i - a_i v_0) \cdot (v_i - a_i v_0)}} \;&=\; \frac{v_i - a_i v_0}{\sqrt{1 - a_i^2}} \;. \end{aligned}$$

(Recall that v_i is a unit vector so $v_i \cdot v_i = 1$.) . Thus,

$$v_i' \cdot v_j' = \frac{(v_i - a_i v_0) \cdot (v_j - a_j v_0)}{\sqrt{(1 - a_i^2)(1 - a_j^2)}} \;.$$

As $(v_0 + v_j) \cdot (v_0 + v_i) = 0$, we get that $v_i \cdot v_j = -1 - v_0 \cdot v_i - v_0 \cdot v_j = -1 - a_i - a_j$, and the numerator of the expression given above for $v_i' \cdot v_j'$ can be simplified as follows:

$$\begin{aligned} (v_i - a_i v_0) \cdot (v_j - a_j v_0) \;&=\; v_i \cdot v_j - a_i a_j \\ =\; -1 - a_i - a_j - a_i a_j \;&=\; -(1 + a_i)(1 + a_j) \;, \end{aligned}$$

and the claim follows. ■

We continue now with the proof of Lemma 3.3. Recall that $S = \{i \mid v_0 \cdot v_i > \beta\}$, where $\beta = \frac{2}{\alpha} - 1 - \frac{2}{\log n}$, and that $|S| \geq \frac{n}{\log n}$. Suppose now that $i,j \in S$ and $(i,j) \in E$. Thus $v_0 \cdot v_i > \beta$, $v_0 \cdot v_j > \beta$ and $(v_0+v_i)\cdot(v_0+v_j) = 0$. Let v_i' and v_j' be the normalized projections of v_i and v_j on the space orthogonal to v_0. The expression given for $v_i' \cdot v_j'$ in Claim 3.5 is *decreasing* in both $v_0 \cdot v_i$ and $v_0 \cdot v_j$. Thus,

$$v_i' \cdot v_j' \;\leq\; -\frac{1+\beta}{1-\beta} \;=\; -\frac{1}{\alpha - 1 + O(\frac{1}{\log n})}\;.$$

We obtained, therefore, a vector $(\alpha + O(\frac{1}{\log n}))$-coloring of $G[S]$. This completes the proof. ■

Lemma 3.6 *Let $\alpha \geq 1$, and let $G = (V,E)$ a vector α-colorable graph on n vertices. Then, an independent set of G of size $\tilde{\Omega}(n^{f(\alpha)})$ can be found in polynomial time, where $f(\alpha)$ is as in Theorem 3.2.*

Proof: The proof is by induction on $k = \lfloor \alpha \rfloor$. Assume at first that $k = 1$. It is easy to see that a graph is vector α-colorable, for some $\alpha < 2$, if and only if the graph contains no edges. Thus, V is an independent set of size n.

Assume, therefore, that $k \geq 2$. Let Δ be the maximum degree of G. We describe two ways of finding independent sets of G. Using the algorithm of Karger, Motwani and Sudan [KMS98] (Theorem 2.3), we can find, in polynomial time, an independent set of G of size $\tilde{\Omega}(n/\Delta^{1-2/\alpha})$. Alternatively, let v be a vertex of G of degree Δ and let $N(v)$ be the set of its neighbors. It follows from Lemma 2.2 that the subgraph $G[N(v)]$ induced by $N(v)$ is vector $(\alpha - 1)$-colorable. By the induction hypothesis, we can find in $G[N(v)]$, in polynomial time, an independent set of size $\tilde{\Omega}(\Delta^{f(\alpha-1)})$. This independent set is also an independent set of G. Taking the larger of these two independent sets, we obtain an independent set of G of size

$$\tilde{\Omega}(\max\{\,\frac{n}{\Delta^{1-2/\alpha}}\,,\,\Delta^{f(\alpha-1)}\,\}) \geq \tilde{\Omega}(n^{1/\left(1+\frac{1-2/\alpha}{f(\alpha-1)}\right)}) \;=\; \tilde{\Omega}(n^{f(\alpha)})\;,$$

as required. It is easy to verify, by induction, that $f(\alpha) = \frac{\alpha(\alpha-1)}{k\left(\alpha(\alpha-k)+\frac{(k-1)(k+1)}{3}\right)}$, where $k = \lfloor \alpha \rfloor$. We omit the straightforward details. This completes the proof of the lemma. ■

We now present a proof of Theorem 3.2.

Proof: **(of Theorem 3.2)** Suppose that $G = (V,E)$ contains an independent set of size n/α. By Lemma 3.3, we can find, in polynomial time, a subset $S \subseteq V$ of size $|S| \geq \frac{n}{\log n}$ and a vector α'-coloring of $G[S]$, where $\alpha' = \alpha + O(\frac{1}{\log n})$. By Lemma 3.6, we can find, in polynomial time, an independent set of $G[S]$ of size $\tilde{\Omega}(|S|^{f(\alpha')})$. As $|S| = \tilde{\Omega}(n)$, and $f(\alpha') = f(\alpha) - O(\frac{1}{\log n})$, we get that the size of this independent set, which is also an independent set of G, is $\tilde{\Omega}(n^{f(\alpha)})$, as required. ■

4 The coloring tools of Blum

Blum [Blu94] makes the following simple observation:

Lemma 4.1 ([Blu94]) *Let $k \geq 3$ be an integer and let $0 < \alpha < 1$. If in any k-colorable graph $G = (V,E)$ on n vertices we can find, in polynomial time, at least one of the following:*

1. *Two vertices $u,v \in V$ that have the same color under some valid k-coloring of G* (Same color),

2. *An independent set $I \subseteq V$ of size $\tilde{\Omega}(n^{1-\alpha})$* (Large independent set),

then, we can color every k-colorable graph, in polynomial time, using $\tilde{O}(n^{\alpha})$ colors.

If we find one of the objects listed in Lemma 4.1 then, following Blum [Blu94], we say that *progress* was made towards coloring the graph using $\tilde{O}(n^{\alpha})$ colors. (Blum [Blu94] describes several other ways of making progress towards an $\tilde{O}(n^{\alpha})$-coloring of the graph which we do not use here.) We do use the following intricate result which is a small variant of Corollary 17 of Blum [Blu94]:

Theorem 4.2 ([Blu94]) *Let $G = (V,E)$ be a k-colorable graph on n vertices with minimum degree $d_{\min}$ in which no two vertices have more than s common neighbors. Then, it is possible to construct, in polynomial time, a collection $\mathcal{T}$ of $\tilde{O}(n)$ subsets of V, such that at least one $T \in \mathcal{T}$ satisfies the following two conditions: (i) $|T| \geq \tilde{\Omega}(d_{\min}^2/s)$. (ii) T has an independent subset of size at least $(\frac{1}{k-1} - O(\frac{1}{\log n}))|T|$.*

The proof of Theorem 4.2 is quite complicated, though the construction of the collection $\mathcal{T}$ is quite simple. The difficulty is in proving that at least one of the sets in this collection satisfies the required properties.

Algorithm COMBINED-COLOR:

Input: A graph $G = (V, E)$ on n vertices and an integer $k \geq 2$.
Output: An $\tilde{O}(n^{\alpha_k})$ coloring of G, if G is k-colorable.

1. If $k = 2$, color the graph, in linear time, using 2 colors.
2. If $k = 3$, use the algorithm of Blum and Karger [BK97] to color the graph using $\tilde{O}(n^{3/14})$ colors.
3. Assume, therefore, that $k \geq 4$. Repeatedly remove from the graph G vertices of degree less than $n^{\alpha_k/(1-2/k)}$. Let U be the set of vertices so removed, and let $G[U]$ be the subgraph of G induced by U. Let D be the *average* degree of $G[U]$. It is easy to see that $D \leq 2n^{\alpha_k/(1-2/k)}$.
4. If $|U| \geq \frac{n}{2}$ then we can use the Karger, Motwani and Sudan algorithm [KMS98] (Theorem 2.3) to find an independent set of $G[U]$ of size $\tilde{\Omega}(n/D^{1-2/k}) \geq \tilde{\Omega}(n^{1-\alpha_k})$, as $D \leq 2n^{\alpha_k/(1-2/k)}$, and we have made progress of type 2.
5. Otherwise, if $|U| < \frac{n}{2}$, let $W = V - U$. Note that $|W| \geq \frac{n}{2}$ and that the minimum degree $d_{\min}$ in $G[W]$ satisfies $d_{\min} \geq n^{\alpha_k/(1-2/k)}$.
6. For every $u, v \in W$ consider the set $S = N(u) \cap N(v)$. If $|S| \geq n^{(1-\alpha_k)/(1-\alpha_{k-2})}$, then apply the coloring algorithm recursively on $G[S]$ and $k-2$. If $G[S]$ is $(k-2)$-colorable, then the algorithm produces a coloring of $G[S]$ using $\tilde{O}(|S|^{\alpha_{k-2}})$ colors, from which an independent set of size $\tilde{\Omega}(|S|^{1-\alpha_{k-2}}) \geq \tilde{\Omega}(n^{1-\alpha_k})$ is easily extracted, and we have made progress of type 2. If the coloring returned by the recursive call uses more than $\tilde{O}(|S|^{\alpha_{k-2}})$ colors, we can infer that $G[S]$ is *not* $(k-2)$-colorable and thus, u and v must be assigned the same color under any valid k-coloring of G, as we have made progress of type 1.
7. Otherwise, we get that $|N(u) \cap N(v)| < n^{(1-\alpha_k)/(1-\alpha_{k-2})}$, for every $u, v \in W$. Also, we know that the minimum degree in $G[W]$ is at least $d_{\min} \geq n^{\alpha_k/(1-2/k)}$.
8. We can now apply Blum's algorithm [Blu94] (Theorem 4.2), with $d_{\min} \geq n^{\alpha_k/(1-2/k)}$ and $s \leq n^{(1-\alpha_k)/(1-\alpha_{k-2})}$, and obtain a collection $\mathcal{T}$ of $\tilde{O}(n)$ subsets of W such that at least one $T \in \mathcal{T}$ satisfies $|T| \geq \tilde{\Omega}(\frac{d_{\min}^2}{s}) \geq \tilde{\Omega}\left(n^{\frac{2\alpha_k}{(1-2/k)} - \frac{1-\alpha_k}{1-\alpha_{k-2}}}\right)$, and T contains an independent set of size at least $(\frac{1}{k-1} - O(\frac{1}{\log n}))|T|$.
9. We now apply the extension of the Alon and Kahale [AK98] technique (Theorem 3.2) on $G[T]$, for each $T \in \mathcal{T}$. In at least one of these runs we obtain an independent set of size $\tilde{\Omega}\left(n^{\left(\frac{2\alpha_k}{1-2/k} - \frac{1-\alpha_k}{1-\alpha_{k-2}}\right)\cdot\frac{3}{k}}\right)$. It is easy to check that $\left(\frac{2\alpha_k}{1-2/k} - \frac{1-\alpha_k}{1-\alpha_{k-2}}\right) \cdot \frac{3}{k} = 1 - \alpha_k$ (the sequence α_k is defined to satisfy this relation), so we have made progress of type 2.

Figure 2: The new coloring algorithm.

5 The combined coloring algorithm

We are now able to present the new algorithm for coloring k-colorable graphs using $\tilde{O}(n^{\alpha_k})$ colors, where

$$\alpha_2 = 0 \quad , \quad \alpha_3 = \frac{3}{14} \quad ,$$

$$\alpha_k = 1 - \frac{6}{k + 4 + 3(1 - \frac{2}{k})\frac{1}{1-\alpha_{k-2}}} \quad , \quad \text{for } k \geq 4 \, .$$

The algorithm, which we call COMBINED-COLOR is described in Figure 2. The description of COMBINED-COLOR is annotated with a proof that on any k-colorable graph on n vertices it makes progress towards an $\tilde{O}(n^{\alpha_k})$-coloring of the graph. This, combined with Lemma 4.1 gives us the following:

Theorem 5.1 *Algorithm* COMBINED-COLOR *runs in polynomial time and it colors a k-colorable graph on n vertices using $\tilde{O}(n^{\alpha_k})$ colors, where $\alpha_2 = 0$, $\alpha_3 = \frac{3}{14}$ and $\alpha_k = 1 - \frac{6}{k+4+3(1-\frac{2}{k})\frac{1}{1-\alpha_{k-2}}}$, for $k \geq 4$.*

One comment should be made, however. In step 6 of COMBINED-COLOR we are tacitly assuming that the coloring algorithm is deterministic so that it is guaranteed to produce a coloring using $\tilde{O}(|S|^{\alpha_{k-2}})$ colors, if $G[S]$ is $(k-2)$-colorable. Our algorithm, however, is randomized. There are two ways of overcoming this difficulty. The first is to derandomize it using the technique of Mahajan and Ramesh [MR99]. Alternatively, we can simply repeat the whole algorithm a sufficient number of times so that the error probability is small enough.

6 Concluding remarks

We obtained several improved coloring algorithms. It would be interesting to obtain further improvements. In particular, it would be interesting to obtain more than logarithmic improvements to the $\tilde{O}(\Delta^{1-2/k})$ bound of Karger, Motwani and Sudan [KMS98], and to see whether better combinations between the algorithms of Blum [Blu94], Karger, *et al.* [KMS98] and Alon and Kahale [AK98] are possible.

Halldórsson [Hal93] describes an algorithm for coloring general graphs using a number of colors which is at most $O(n(\log\log n)^2/\log^3 n)$ times the minimal number of colors required. His algorithm is close to being best possible, as it is known that the chromatic number of general graphs cannot be approximated, in polynomial time, to within a ratio of $n^{1-\epsilon}$, for every $\epsilon > 0$, unless $NP = RP$ (Feige and Killian [FK96]).

It is only known, however, that coloring 3-colorable graphs using 4 colors in NP-hard (Khanna, Linial and Safra [KLS00] and Guruswami and Khanna [GK00]). Obtaining improved hardness results for coloring 3-colorable graphs is a challenging open problem.

Another interesting problem is the following: how large can the chromatic number of vector 3-colorable (or vector k-colorable) graphs be? See [KMS98] for a discussion of this problem.

Related to the problem of graph coloring is the problem of hypergraph coloring. See Krivelevich and Sudakov [KS98] and Krivelevich, Nathaniel and Sudakov [KNS01] for the best available results for this problem.

References

[AK98] N. Alon and N. Kahale. Approximating the independence number via the θ-function. *Mathematical Programming*, 80:253–264, 1998.

[BK97] A. Blum and D. Karger. An $\tilde{O}(n^{3/14})$-coloring algorithm for 3-colorable graphs. *Information Processing Letters*, 61:49–53, 1997.

[Blu94] A. Blum. New approximation algorithms for graph coloring. *Journal of the ACM*, 41:470–516, 1994.

[FK96] U. Feige and J. Kilian. Zero knowledge and the chromatic number (preliminary version). In *Proceedings of the 11th Annual Symposium on Structure in Complexity Theory*, 1996.

[GK00] V. Guruswami and S. Khanna. On the hardness of 4-coloring a 3-colorable graph. In *Proceedings of the 15th Annual IEEE Conference on Computational Complexity, Florence, Italy*, 2000.

[GLS93] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization.* Springer Verlag, 1993. Second corrected edition.

[Hal93] M.M. Halldórsson. A still better performance guarantee for approximate graph coloring. *Information Processing Letters*, 45:19–23, 1993.

[KLS00] S. Khanna, N. Linial, and S. Safra. On the hardness of approximating the chromatic number. *Combinatorica*, 20:393–415, 2000.

[KMS98] D. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. *Journal of the ACM*, 45:246–265, 1998.

[KNS01] M. Krivelevich, R. Nathaniel, and B. Sudakov. Approximating coloring and independent sets in 3-uniform hypergraphs In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms, Washington, D.C.*, 2001.

[KS98] M. Krivelevich and B. Sudakov. Approximate coloring of uniform hypergraphs. Technical Report 98-31, DIMACS, 1998.

[Lov79] L. Lovász. On the shannon capacity of a graph. *IEEE Transactions on Information Theory*, IT-25:1–7, 1979.

[MR99] S. Mahajan and H. Ramesh. Derandomizing approximation algorithms based on semidefinite programming. *SIAM Journal on Computing*, 28:1641–1663, 1999.

[Wig83] A. Wigderson. Improving the performance guarantee for approximate graph coloring. *Journal of the ACM*, 30:729–735, 1983.