Finding and counting given length cycles *

Noga Alon † *Raphael Yuster* † *Uri Zwick* †

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Abstract

We present an assortment of methods for finding and counting simple cycles of a given length in directed and undirected graphs. Most of the bounds obtained depend solely on the number of edges in the graph in question, and not on the number of vertices. The bounds obtained improve upon various previously known results.

1 Introduction

The problem of deciding whether a given graph $G = (V, E)$ contains a simple cycle of length k is among the most natural and easily stated algorithmic graph problems. If the cycle length k is part of the input then the problem is clearly NP-complete as it includes in particular the Hamiltonian cycle problem. For every fixed k however, the problem can be solved in either $O(VE)$ time (Monien [10]) or $O(V^\omega \log V)$ [1], where $\omega < 2.376$ is the exponent of matrix multiplication.

The main contribution of this paper is a collection of new bounds on the complexity of finding simple cycles of length exactly k, where $k \geq 3$ is a fixed integer, in a directed or an undirected graph $G = (V, E)$. These bounds are of the form $O(E^{\alpha_k})$ or of the form $O(E^{\beta_k} \cdot d(G)^{\gamma_k})$, where $d(G)$ is the *degeneracy* of the graph (see below). The bounds improve upon previously known bounds when the graph in question is relatively sparse or relatively degenerate.

We let C_k stand for a simple cycle of length k. When considering directed graphs, a C_k is assumed to be directed. We show that a C_k in a directed or undirected graph $G = (V, E)$, if one exists, can be found in $O(E^{2-\frac{2}{k}})$ time, if k is even, and in $O(E^{2-\frac{2}{k+1}})$ time, if k is odd. For finding triangles (C_3's), we get the slightly better bound of $O(E^{\frac{2\omega}{\omega+1}}) = O(E^{1.41})$, where $\omega < 2.376$ is the exponent of matrix multiplication.

Even cycles in undirected graphs can be found even faster. A C_{4k-2} in an undirected graph $G = (V, E)$, if one exists, can be found in $O(E^{2-\frac{1}{2k}(1+\frac{1}{k})})$ time. A C_{4k}, if one exists, can be found in $O(E^{2-(\frac{1}{k}-\frac{1}{2k+1})})$ time. In particular, we can find an undirected C_4 in $O(E^{4/3})$ time and an undirected C_6 in $O(E^{13/8})$ time.

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†School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, ISRAEL. E-mail addresses: {noga,raphy,zwick}@math.tau.ac.il .

cycle	complexity		cycle	complexity	
C_3	$E^{1.41}$	, $E \cdot d(G)$	C_7	$E^{1.75}$	, $E^{3/2} \cdot d(G)$
C_4	$E^{1.5}$	, $E \cdot d(G)$	C_8	$E^{1.75}$	, $E^{3/2} \cdot d(G)$
C_5	$E^{1.67}$	, $E \cdot d(G)^2$	C_9	$E^{1.8}$	, $E^{3/2} \cdot d(G)^{3/2}$
C_6	$E^{1.67}$	, $E^{3/2} \cdot d(G)^{1/2}$	C_{10}	$E^{1.8}$	, $E^{5/3} \cdot d(G)^{2/3}$

Table 1: Finding small cycles in directed graphs – some of the new results

cycle	complexity	cycle	complexity
C_4	$E^{1.34}$	C_8	$E^{1.7}$
C_6	$E^{1.63}$	C_{10}	$E^{1.78}$

Table 2: Finding small cycles in undirected graphs – some of the new results

The *degeneracy* $d(G)$ of an undirected graph $G = (V, E)$ is the smallest number d for which there exists an acyclic orientation of G in which all the out-degrees are at most d. The degeneracy $d(G)$ of a graph G is linearly related to the *arboricity* $a(G)$ of the graph, i.e., $a(G) = \Theta(d(G))$, where $a(G)$ is the minimal number of forests needed to cover all the edges of G. The degeneracy of a directed graph $G = (V, E)$ is defined to be the degeneracy of the undirected version of G. The degeneracy of a graph is an important parameter of the graph that appears in many combinatorial results. It is easy to see that for any graph $G = (V, E)$ we have $d(G) \leq 2E^{1/2}$. For graphs with relatively low degeneracy we can improve upon the previously stated results. A C_{4k} in a directed or undirected graph $G = (V, E)$ that contains one can be found in $O(E^{2-\frac{1}{k}} \cdot d(G))$ time. A C_{4k+1}, if one exists, can be found in $O(E^{2-\frac{1}{k}} \cdot d(G)^{1+\frac{1}{k}})$ time. Similar results are obtained for finding C_{4k-2}'s and C_{4k-1}'s. In particular, C_3's and C_4's can be found in $O(E \cdot d(G))$ time and C_5's in $O(E \cdot d(G)^2)$ time. Some of the results mentioned are summarized in Tables 1 and 2.

As any planar graph has a vertex whose degree is at most 5, the degeneracy of any planar graph is at most 5. As a consequence of the above bounds we get, in particular, that C_3's, C_4's and C_5's in planar graphs can be found in $O(V)$ time. This in fact holds not only for planar graphs but for any non-trivial *minor-closed* family of graphs.

Another contribution of this paper is an $O(V^\omega)$ algorithm for *counting* the number of C_k's, for $k \leq 7$, in a graph $G = (V, E)$.

A preliminary version of this work appeared in [2].

2 Comparison with previous works

Monien [10] obtained, for any fixed $k \geq 3$, an $O(VE)$ algorithm for finding C_k's in a directed or undirected graph $G = (V, E)$. In a previous work [1] we showed, using the *color-coding* method, that a C_k, for any fixed $k \geq 3$, if one exists, can also be found in $O(V^\omega)$ expected time or in $O(V^\omega \log V)$ worst-case time, where $\omega < 2.376$ is the exponent of matrix multiplication.

Our new $O(E^{2-\frac{2}{k}})$ algorithm is better than both the $O(VE)$ and the $O(V^\omega)$ algorithms when the input graph $G=(V,E)$ is sufficiently sparse. It is interesting to note that for $k \leq 6$, Monien's $O(VE)$ bound is superseded by either the $O(V^\omega)$ algorithm, when the graph is dense, or by the $O(E^{2-1/\lceil \frac{k}{2} \rceil})$ algorithm, when the graph is sparse. For every $k \geq 7$, each one of the four bounds (including the bound that involves the degeneracy) beats the others on an appropriate family of graphs.

In a previous work [15] we have also shown that cycles of an *even* length in *undirected* graphs can be found even faster. Namely, for any even $k \geq 4$, if an undirected graph $G=(V,E)$ contains a C_k then such a C_k can be found in $O(V^2)$ time. Our $O(E^{2-\frac{1}{2k}(1+\frac{1}{k})})$ bound for C_{4k-2} and $O(E^{2-(\frac{1}{k}-\frac{1}{2k+1})})$ bound for C_{4k} are again better when the graph is sparse enough.

Itai and Rodeh [8] showed that a *triangle* (a C_3) in a graph $G=(V,E)$ that contains one can be found in $O(V^\omega)$ or $O(E^{3/2})$ time. We improve their second result and show that the same can be done, in directed or undirected graphs, in $O(E^{\frac{2\omega}{\omega+1}}) = O(E^{1.41})$ time.

Chiba and Nishizeki [6] showed that triangles (C_3's) and quadrilaterals (C_4's) in graphs that contain them can be found in $O(E \cdot d(G))$ time. As $d(G) = O(E^{1/2})$ for any graph G, this extends the result of Itai and Rodeh. We extend the result of Chiba and Nishizeki and show that C_{4k-1}'s and C_{4k}'s can be found in $O(E^{2-\frac{1}{k}} \cdot d(G))$ time. We also show that C_{4k+1}'s can be found in $O(E^{2-\frac{1}{k}} \cdot d(G)^{1+\frac{1}{k}})$ time. This gives, in particular, an $O(E \cdot d(G)^2)$ algorithm for finding pentagons (C_5's). Our results apply to both directed and undirected graphs.

Itai and Rodeh [8] and also Papadimitriou and Yannakakis [12] showed that C_3's in planar graphs can be found in $O(V)$ time. Chiba and Nishizeki [6] showed that C_3's as well as C_4's in planar graphs can be found in $O(V)$ time. Richards [13] showed that C_5's and C_6's in planar graphs can be found in $O(V \log V)$ time. We improve upon the result of Richards and show that C_5's in planar graphs can be found in $O(V)$ time. In a previous work [1] we showed, using color-coding, that for any $k \geq 3$, a C_k in a planar graph, if one exists, can be found in either $O(V)$ *expected* time or $O(V \log V)$ worst case time.

The fact that the number of triangles in a graph can be counted in $O(V^\omega)$ time is trivial. In [1] we showed, using color-coding, that for any $k \geq 3$, a C_k, if one exists, can be found in either $O(V^\omega)$ *expected* time or in $O(V^\omega \log V)$ worst case time. Here we show that for any $k \leq 7$ the number of C_k's in a graph can be counted in $O(V^\omega)$ time. The counting method used here yields, in particular, a way of finding C_k's for $k \leq 7$, in $O(V^\omega)$ worst case time.

Sundaram and Skiena [14] have recently presented some more fixed-subgraph isomorphism algorithms. The results presented here, and in [1] and [15], improve some of their results.

Eppstein [7] has recently shown that the fixed-subgraph isomorphism problem for *planar* graphs, i.e., given a fixed graph H and a planar graph $G=(V,E)$, find a subgraph of G isomorphic to H, can be solved, for every fixed H, in $O(V)$ time.

3 Finding cycles in sparse graphs

Monien [10] obtained his $O(VE)$ algorithm by the use of *representative collections*. Such collections are also used by our algorithms. In the sequel, a p-set is a set of size p.

Definition 3.1 ([10]) *Let $\mathcal{F}$ be a collection of p-sets. A sub-collection $\hat{\mathcal{F}} \subseteq \mathcal{F}$ is q-representative for $\mathcal{F}$, if for every q-set B, there exists a set $A \in \mathcal{F}$ such that $A \cap B = \emptyset$ if and only if there exists a set $A \in \hat{\mathcal{F}}$ with this property.*

It follows from a combinatorial lemma of Bollobás [3] that any collection $\mathcal{F}$ of p-sets, no matter how large, has a q-representative sub-collection of size at most $\binom{p+q}{p}$. Monien [10] describes an $O(pq \cdot \sum_{i=0}^{q} p^i \cdot |\mathcal{F}|)$ time algorithm for finding a q-representative sub-collection of $\mathcal{F}$ whose size is at most $\sum_{i=0}^{q} p^i$. Relying on Monien's result we obtain the following lemma:

Lemma 3.2 *Let $\mathcal{F}$ be a collection of p-sets and let $\mathcal{G}$ be a collection of q-sets. Consider p and q to be fixed. In $O(|\mathcal{F}| + |\mathcal{G}|)$ time, we can either find two sets $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $A \cap B = \emptyset$ or decide that no two such sets exist.*

Proof : We use Monien's algorithm to find a q-representative sub-collection $\hat{\mathcal{F}}$ of $\mathcal{F}$ whose size is at most $\sum_{i=0}^{q} p^i$ and a p-representative sub-collection $\hat{\mathcal{G}}$ of $\mathcal{G}$ whose size is at most $\sum_{i=0}^{p} q^i$. This takes only $O(|\mathcal{F}| + |\mathcal{G}|)$ time (as p and q are constants).

It is easy to see that if there exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $A \cap B = \emptyset$, then there also exist $A' \in \hat{\mathcal{F}}$ and $B' \in \hat{\mathcal{G}}$ such that $A' \cap B' = \emptyset$. To see this note that if $A \cap B = \emptyset$ then by the definition of q-representatives, there must exist a set $A' \in \hat{\mathcal{F}}$ such that $A' \cap B = \emptyset$ and then, there must exist a set $B' \in \hat{\mathcal{G}}$ such that $A' \cap B' = \emptyset$ as required.

After finding the representative collections $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ it is therefore enough to check whether they contain two disjoint sets. This can be easily done in constant time (as p and q are constants). □

We also need the following lemma that follows immediately from the work of Monien [10].

Lemma 3.3 ([10]) *Let $G = (V, E)$ be a directed or undirected graph, let $v \in V$ and let $k \geq 3$. A C_k that passes through v, if one exists, can be found in $O(E)$ time.*

We are finally able to present our improved algorithm.

Theorem 3.4 *Deciding whether a directed or undirected graph $G = (V, E)$ contains simple cycles of length exactly $2k - 1$ and of length exactly $2k$, and finding such cycles if it does, can be done in $O(E^{2-\frac{1}{k}})$ time.*

Proof : We describe an $O(E^{2-\frac{1}{k}})$ time algorithm for finding a C_{2k} in a directed graph $G = (V, E)$. The details of all the other cases are similar. Let $\Delta = E^{\frac{1}{k}}$. A vertex in G whose degree is at least Δ is said to be of *high degree*. The graph $G = (V, E)$ contains at most $2E/\Delta = O(E^{1-\frac{1}{k}})$ high degree vertices. We check, using Monien's algorithm (Lemma 3.3), whether any of these high degree vertices lies on a simple cycle of length $2k$. For each vertex this costs $O(E)$ operations and the total cost is $O(E^2/\Delta) = O(E^{2-\frac{1}{k}})$. If one of these vertices does lie on a cycle of length $2k$ we are done. Otherwise, we remove all the high degree vertices and all the edges adjacent to them from G and obtain a subgraph G' that contains a C_{2k} if and only if G does. The maximum degree of G' is at most $\Delta = E^{\frac{1}{k}}$ and there are therefore at most $E \cdot \Delta^{k-1} = E^{2-\frac{1}{k}}$ simple directed paths of length k in G'. We can find all these simple paths in $O(E^{2-\frac{1}{k}})$ time. We divide these paths into groups according to their endpoints. This can be done using radix sort in $O(E^{2-\frac{1}{k}})$ time and space. We get

a list of all the pairs of vertices connected by simple directed paths of length exactly k. For each such pair u, v, we get a collection $\mathcal{F}_{u,v}$ of $k-1$-sets. Each $k-1$-set in $\mathcal{F}_{u,v}$ corresponds to the $k-1$ intermediate vertices that appear on simple directed paths of length k from u to v. For each pair u, v that appears on the list, we check whether there exist two directed paths of length k, one from u to v and the other from v to u, that meet only at their endpoints. Such two paths exist if there exist $A \in \mathcal{F}_{u,v}$ and $B \in F_{v,u}$ such that $A \cap B = \emptyset$. This can be checked, as shown in Lemma 3.2, in $O(|\mathcal{F}_{u,v}| + |\mathcal{F}_{v,u}|)$ time. As the sum of the sizes of all these collections is $O(E^{2-\frac{1}{k}})$, the total complexity is again $O(E^{2-\frac{1}{k}})$. This completes the proof. □

In the case of triangles, we can get a better result by using fast matrix multiplication.

Theorem 3.5 *Deciding whether a directed or an undirected graph $G = (V, E)$ contains a triangle, and finding one if it does, can be done is $O(E^{\frac{2\omega}{\omega+1}}) = O(E^{1.41})$ time.*

Proof : Let $\Delta = E^{\frac{\omega-1}{\omega+1}}$. A vertex is said to be of *high degree* if its degree is more than Δ and of *low degree* otherwise. Consider all directed paths of length two in G whose intermediate vertex is of low degree. There are at most $E \cdot \Delta$ such paths and they can be found in $O(E \cdot \Delta)$ time. For each such path, check whether its endpoints are connected by an edge in the appropriate direction. If no triangle is found in this way, then any triangle in G must be composed of three high degree vertices. As there are at most $2E/\Delta$ high degree vertices, we can check whether there exists such a triangle using matrix multiplication in $O((E/\Delta)^\omega)$ time. The total complexity of the algorithm is therefore

$$O(E \cdot \Delta + (E/\Delta)^\omega) = O(E^{\frac{2\omega}{\omega+1}}).$$

This completes the proof. □

We have not been able to utilize matrix multiplication to improve upon the result of Theorem 3.4 for $k \geq 4$. This constitutes an interesting open problem.

4 Finding cycles in graphs with low degeneracy

An undirected graph $G = (V, E)$ is *d-degenerate* (see Bollobás [4], p. 222) if there exists an acyclic orientation of it in which $d_{out}(v) \leq d$ for every $v \in V$. The smallest d for which G is d-degenerate is called the *degeneracy* or the *max-min degree* of G and is denoted by $d(G)$. It can be easily seen (see again [4]) that $d(G)$ is the maximum of the minimum degrees taken over all the subgraphs of G. The degeneracy $d(G)$ of a graph G is linearly related to the *arboricity* $a(G)$ of the graph, i.e., $a(G) = \Theta(d(G))$, where $a(G)$ is the minimal number of forests needed to cover all the edges of G. The degeneracy of a directed graph $G = (V, E)$ is defined to be the degeneracy of the undirected version of G. It is easy to see that the degeneracy of any planar graph is at most 5. Clearly, if G is d-degenerate then $|E| \leq d \cdot |V|$. The following simple lemma, whose proof is omitted, is part of the folklore (see, e.g., [9]).

Lemma 4.1 *Let $G = (V, E)$ be a connected undirected graph $G = (V, E)$. An acyclic orientation of G such that for every $v \in V$ we have $d_{out}(v) \leq d(G)$ can be found in $O(E)$ time.*

The main result of this section is the following Theorem:

Theorem 4.2 *Let $G = (V, E)$ be a directed or an undirected graph.*

(i) Deciding whether G contains a simple cycle of length exactly $4k-2$, and finding such a cycle if it does, can be done in $O(E^{2-\frac{1}{k}} \cdot d(G)^{1-\frac{1}{k}})$ time.

(ii) Deciding whether G contains simple cycles of length exactly $4k-1$ and of length exactly $4k$, and finding such cycles if it does, can be done in $O(E^{2-\frac{1}{k}} \cdot d(G))$ time.

(iii) Deciding whether G contains a simple cycle of length exactly $4k+1$, and finding such a cycle if it does, can be done in $O(E^{2-\frac{1}{k}} \cdot d(G)^{1+\frac{1}{k}})$ time.

Proof : We show how to find a C_{4k+1} in a directed graph $G = (V, E)$, if one exists, in $O(E^{2-\frac{1}{k}} \cdot d(G)^{1+\frac{1}{k}})$ time. The proofs of the other claims are easier. If $d(G) \geq E^{\frac{1}{2k+1}}$, we can use the algorithm of Theorem 3.4 whose complexity is $O(E^{2-\frac{1}{2k+1}}) \leq O(E^{2-\frac{1}{k}} \cdot d(G)^{1+\frac{1}{k}})$. Assume therefore that $d(G) \leq E^{\frac{1}{2k+1}}$.

Let $\Delta = E^{\frac{1}{k}}/d(G)^{1+\frac{1}{k}}$. As $d(G) \leq E^{\frac{1}{2k+1}}$, we have that $d(G) \leq \Delta$. A vertex is said to be of *high degree* if its degree is more than Δ and of *low degree* otherwise. As in the proof of Theorem 3.4, we can check in $O(E^2/\Delta)$ time whether any of the high degree vertices lies on a C_{4k+1}. If none of them lies on a C_{4k+1}, we can remove all the high degree vertices along with the edges adjacent to them from G and obtain a graph whose maximal degree is at most Δ. The degeneracy of a graph can only decrease when vertices and edges are deleted and $d(G)$ is therefore an upper bound on the degeneracy of the graph obtained.

Suppose therefore that G is a graph with maximal degree Δ and degeneracy $d(G)$. To find a C_{4k+1} in G, it is enough to find all directed simple paths of length $2k$ and $2k+1$ in G and then check, using the algorithm described in the proof of Lemma 3.2, whether there exist a path of length $2k$ and a path of length $2k+1$ that meet only at their endpoints.

In $O(E)$ time we can get an acyclically oriented version G' of G in which the out-degree of each vertex is at most $d(G)$. The orientations of the edges in G and G' may be completely different.

The number of paths, not necessarily directed, of length $2k+1$ in G, is at most

$$2 \cdot 2E \cdot \sum_{i=0}^{k} \binom{2k}{i} \Delta^i d(G)^{2k-i} = O(E\Delta^k d(G)^k).$$

To see this, consider the orientations, in G', of the edges on a $2k+1$-path in G. In at least one direction, at most k of the edges are counter directed. The number of paths of length $2k+1$ in which exactly i among the last $2k$ edges are counter directed is at most $2E \cdot \binom{2k}{i} \Delta^i d(G)^{2k-i}$. The binomial coefficient $\binom{2k}{i}$ stands for the possible choices for the position of the counter directed edges in the path. Similarly, the number of paths of length $2k$ in G is $O(E\Delta^k d(G)^{k-1})$.

We can lower the number of paths of length $2k+1$ and $2k$ we have to consider by utilizing the fact that a C_{4k+1} can be broken into two paths of length $2k+1$ and $2k$ in many different ways. In particular, let C be a directed C_{4k+1} in G and consider the orientations of its edges in G'. As $4k+1$ is odd and as G' is acyclic, C must contain three consecutive edges e_{2k}, e_{2k+1} and e_{2k+2}, the first two of which have the same orientation while the third one has an opposite orientation. It is therefore

enough to consider all $2k+1$-paths that start with at least two backward oriented edges and all $2k$-paths that start with at least one backward oriented edge. The orientations referred to here are in G'.

The number of paths of length $2k+1$ in G whose first two edges are backward oriented in G' is $O(E\Delta^{k-1}d(G)^{k+1})$. To see this, note that any such path is composed of a directed path $\{e_{2k}, e_{2k+1}\}$ of length two, attached to an arbitrarily oriented path $\{e_1, \ldots, e_{2k-1}\}$ of length $2k-1$. The number of paths of length $2k-1$ is, as shown earlier, at most $O(E\Delta^{k-1}d(G)^{k-1})$ and the number of directed path of length two with a specified starting point is at most $d(G)^2$. Similarly, there are at most $O(E\Delta^{k-1}d(G)^k)$ $2k$-paths that start with a backward oriented edge.

It should be clear from the above discussion that all the required $2k+1$- and $2k$-paths, whose total number is $O(E\Delta^{k-1}d(G)^{k+1})$, can be found in $O(E\Delta^{k-1}d(G)^{k+1})$ time. Paths which are not properly directed, in G, are thrown away. Properly directed paths are sorted, using radix sort, according to their endpoints. Using Lemma 3.2 we then check whether there exist a directed $2k+1$-path and a directed $2k$-path that close a directed simple cycle. All these operations can again be performed in $O(E\Delta^{k-1}d(G)^{k+1})$ time.

Recalling that $\Delta = E^{\frac{1}{k}}/d(G)^{1+\frac{1}{k}}$, we get that the over all complexity of the algorithm is

$$O(\frac{E^2}{\Delta} + E\cdot\Delta^{k-1}d(G)^{k+1}) = O(E^{2-\frac{1}{k}}d(G)^{1+\frac{1}{k}}) \quad .$$

This completes the proof of the Theorem. □

As an immediate corollary we get:

Corollary 4.3 *If a directed or undirected planar graph $G=(V,E)$ contains a pentagon (a C_5), then such a pentagon can be found in $O(V)$ worst-case time.*

By combining the ideas of this section, the $O(E^{\frac{2\omega}{\omega+1}})$ algorithm of Theorem 3.5 and the color-coding method [1] we can also obtain the following result.

Theorem 4.4 *Let $G=(V,E)$ be a directed or undirected graph. A C_6 in G, if one exists, can be found in either $O((E\cdot d(G))^{\frac{2\omega}{\omega+1}}) = O((E\cdot d(G))^{1.41})$ expected time or $O((E\cdot d(G))^{\frac{2\omega}{\omega+1}}\cdot\log V) = O((E\cdot d(G))^{1.41}\cdot\log V)$ worst case time.*

Proof : We show how to find a C_6 in an undirected graph $G=(V,E)$, if one exists, in $O((E\cdot d(G))^{\frac{2\omega}{\omega+1}})$ expected time, or $O((E\cdot d(G))^{\frac{2\omega}{\omega+1}}\log V)$ worst case time. The proof of the directed case is similar.

In $O(E)$ time, we can get an acyclically oriented version $G'=(V,E')$ of G in which the out-degree of each vertex is at most $d(G)$. Suppose that G contains a C_6. The six possible non-isomorphic orientations of this C_6 in G' are shown in Figure 1. We refer to these orientations as $A_1, \ldots, A_6$. Our algorithm checks, for each $1 \le i \le 6$, whether G' contains an A_i and if so finds one.

We show how to find an A_1 in G', if one exists. The other cases are similar, and in fact easier. As in [1], we color the vertices of G' randomly using six colors (i.e., every vertex receives a number between 1 and 6, all numbers equally likely). Let $c(v)$ denote the color of vertex v. Let A be a specific copy of an A_1 in G'. We say that A is *well-colored* if its vertices are consecutively colored

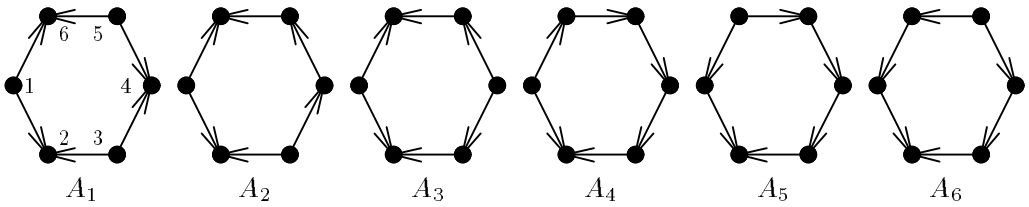

Figure 1: The possible orientations of C_6 in G'.

by 1 through 6, and the color 1 is assigned to one of the three vertices having only outgoing edges in A. (By "consecutively colored" we mean that each $v \in A$ with $c(v) < 6$ has a neighbor $u \in A$ with $c(u) = c(v) + 1$). The probability that A is well-colored is $6/6^6$. We now show how to detect a well-colored copy of an A_1 deterministically, if one exists.

Create a new undirected graph $G^* = (V^*, E^*)$ defined as follows:

$$V^* = \{v \in V \ : \ c(v) \in \{2, 4, 6\}\}$$

$$\begin{aligned} E^* &= \{(u,v) \ : \ c(u) = 6,\ c(v) = 2,\ \ (\exists\, w \in V)\ (c(w) = 1,\ (w,u),(w,v) \in E')\} \\ &\cup \{(u,v) \ : \ c(u) = 2,\ c(v) = 4,\ \ (\exists\, w \in V)\ (c(w) = 3,\ (w,u),(w,v) \in E')\} \\ &\cup \{(u,v) \ : \ c(u) = 4,\ c(v) = 6,\ \ (\exists\, w \in V)\ (c(w) = 5,\ (w,u),(w,v) \in E')\}\ . \end{aligned}$$

Clearly, $V^* \leq V$. To create G^*, we examine each edge $(w,u) \in E'$ with $c(w)$ odd. Suppose $c(w) = 1$ and $c(u) = 6$. We create edges in G^* between u and all vertices v such that $(w,v) \in E'$ and $c(v) = 2$. There are at most $d(G) - 1$ such vertices. We therefore have $E^* < E{\cdot}d(G)$ and G^* can be constructed in $O(E \cdot d(G))$ time from G'. Clearly, there exists an undirected triangle in G^* iff there exists a well-colored A_1 in G'. We can detect such a triangle in G^* in $O((E^*)^{\frac{2\omega}{\omega+1}}) = O((Ed(G))^{\frac{2\omega}{\omega+1}})$ time using the algorithm of Theorem 3.5. If such a triangle is not found, we repeat the whole process using a new random coloring. If G' contains an A_1, then such an A_1 will be found after an expected number of $6^5 = 7776$ attempts.

We have thus shown how to detect an A_1 in G', if one exists, in $O((E{\cdot}d(G))^{\frac{2\omega}{\omega+1}}) = O((E \cdot d(G))^{\frac{2\omega}{\omega+1}})$ expected time. As shown in [1], such a coloring scheme can be derandomized at the price of an $O(\log V)$ factor. □

5 Finding cycles in sparse undirected graphs

To obtain the results of this section we rely on the following combinatorial lemma of Bondy and Simonovits [5].

Lemma 5.1 ([5]) *Let $G = (V, E)$ be an undirected graph. If $|E| \geq 100k{\cdot}|V|^{1+\frac{1}{k}}$ then G contains a $C_{2\ell}$ for every integer $\ell \in [k, n^{1/k}]$.*

By combining the algorithm described in the proof of Theorem 4.2 with an algorithm given in [15] we obtain the following theorem.

Theorem 5.2 *Let $G = (V, E)$ be an undirected graph.*

(i) A C_{4k-2} in G, if one exists, can be found in $O(E^{2-\frac{1}{2k}(1+\frac{1}{k})})$ time.

(ii) A C_{4k} in G, if one exists, can be found in $O(E^{2-(\frac{1}{k}-\frac{1}{2k+1})})$ time.

Proof : We prove the second claim. The proof of the first claim is similar. Let $d = 200k \cdot E^{\frac{1}{2k+1}}$. If $d(G) \geq d$ then, by the definition of degeneracy, there is a subgraph $G' = (V', E')$ of G in which the minimal degree is at least d. Such a subgraph can be easily found in $O(E)$ time (see, e.g., [9]). Clearly $E' \geq dV'/2 \geq 100k \cdot V' \cdot E'^{\frac{1}{2k+1}}$ and therefore $E' \geq (100k \cdot V')^{1+\frac{1}{2k}} \geq 100k \cdot (V')^{1+\frac{1}{2k}}$. By Lemma 5.1 we get that G' contains a C_{4k} and such a C_{4k} can be found in $O(V'^2) = O(E^{2-\frac{2}{2k+1}})$ time using the algorithm given in [15]. If, on the other hand, $d(G) \leq d$, then a C_{4k} in G, if one exists, can be found in $O(E^{2-\frac{1}{k}} \cdot d) = O(E^{2-(\frac{1}{k}-\frac{1}{2k+1})})$ time using the algorithm of Theorem 4.2. It is easy to check that $E^{2-\frac{2}{2k+1}} \leq E^{2-(\frac{1}{k}-\frac{1}{2k+1})}$ with equality holding only if $k = 1$. In both cases, the complexity is therefore $O(E^{2-(\frac{1}{k}-\frac{1}{2k+1})})$ as required. □

Corollary 5.3 *Let $G = (V, E)$ be an undirected graph.*

(i) A quadrilateral (C_4) in G, if one exists, can be found in $O(E^{4/3})$ time.

(ii) A hexagon (C_6) in G, if one exists, can be found in $O(E^{13/8})$ time.

6 Counting small cycles

Let $G = (V, E)$ be a simple undirected graph and let $A = A_G$ be the adjacency matrix of G. Assume for simplicity that $V = \{1, \ldots, n\}$. Denote by $a_{ij}^{(k)} = (A^k)_{ij}$ the elements of the k-th power of A. The trace $tr(A^k)$ of A^k, which is the sum of the entries along the diagonal of A^k, gives us the number of closed walks of length k in G. If we could also compute the number of *non-simple* closed walks of length k in G we would easily obtain the number of *simple* closed paths of length k in G. This number is just $2k$ times the number of C_k's in G.

Before describing a way of counting the number of non-simple closed walks of length k, where $k \leq 7$, in a graph G in $O(V^\omega)$ time, we need the following definitions:

Definition 6.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. A mapping $f : V_1 \cup E_1 \rightarrow V_2 \cup E_2$ is a* homomorphism *if for every $v \in V_1$ we have $f(v) \in V_2$ and for every $e = (u, v) \in E_1$ we have $f(e) = (f(u), f(v)) \in E_2$. If f is onto $V_2 \cup E_2$, we say that G_2 is a* homomorphic image *of G_1.*

Definition 6.2 *A graph $H = (V_H, E_H)$ is said to be* k-cyclic, *for $k \geq 3$, if it is a homomorphic image of the cycle C_k. The number of different homomorphisms from C_k to H is denoted by $c_k(H)$. Clearly, H is k-cyclic if and only if $c_k(H) > 0$.*

It is easy to check, for example, that C_3 is k-cyclic for every $k \geq 3$ except $k = 4$. It is also not difficult to check that $c_3(C_3) = 6$ (and more generally $c_k(C_k) = 2k$ for every $k \geq 3$) and that $c_5(C_3) = 30$. The only 3-cyclic graph is C_3 itself. The k-cyclic graphs, for $4 \leq k \leq 7$, are given in Figures 2–5.

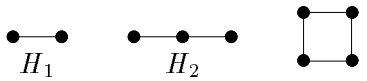

Figure 2: The 4-cyclic graphs.

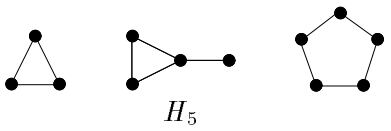

Figure 3: The 5-cyclic graphs.

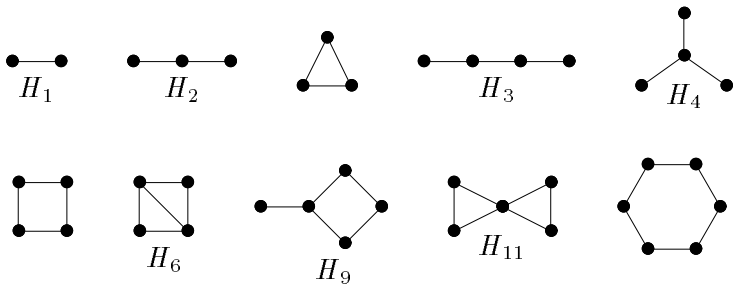

Figure 4: The 6-cyclic graphs.

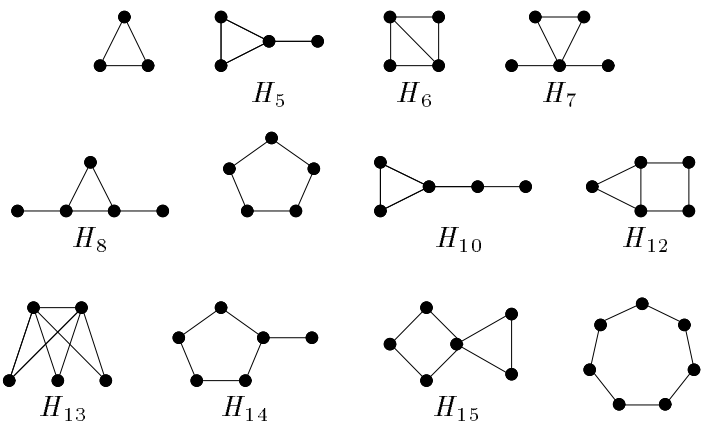

Figure 5: The 7-cyclic graphs.

Let $n_G(H)$ denote the number of subgraphs of G isomorphic to H. Clearly, the total number of closed walks of length k in G is

$$tr(A^k) = \sum_H c_k(H) n_G(H).$$

If $c_k(H) > 0$ then H is connected and has at most k edges. Also, H cannot be a tree on $k+1$ vertices as each edge leading to a leaf must be the image of at least two edges in C_k. Hence, $|V_H| \leq k$ and in fact, $|V_H| < k$ unless $H = C_k$. We therefore obtain, for an undirected graph $G = (V, E)$:

$$n_G(C_k) = \frac{1}{2k} \cdot [tr(A^k) - \sum_{|V_H|<k} c_k(H) n_G(H)]. \tag{1}$$

A very similar formula can be obtained for directed graphs. We show how to compute $n_G(H)$, for all k-cyclic graphs H with $3 \leq k \leq 7$, in $O(V^\omega)$ time. Hence, we obtain the following theorem.

Theorem 6.3 *The number of C_k's, for $3 \leq k \leq 7$ in an undirected (or directed) graph $G = (V, E)$ can be found in $O(V^\omega)$ time.*

Proof : We consider the undirected case. The directed case is similar and, in fact, slightly simpler (as there are less k-cyclic graphs). Clearly, the traces $tr(A^k)$, for $3 \leq k \leq 7$, can be computed in $O(V^\omega)$ time using fast matrix multiplication. It remains to show how to find $n_G(H)$ for all k-cyclic graphs H, where $3 \leq k \leq 7$, exluding the cycles $C_3, \ldots, C_7$ themselves, in $O(V^\omega)$ time.

The k-cyclic graphs shown in Figures 2–5, which are not simple cycles, are denoted by $H_1, \ldots, H_{15}$ (they are ordered according to the number of edges they contain).

The following list shows how to obtain $n_G(H_i)$, for $1 \leq 1 \leq 15$, and $n_G(C_k)$, for $3 \leq k \leq 7$.

In all cases, the formulae reference at most $O(V^2)$ values of $a_{ij}^{(p)}$ for some $1 \leq p \leq k$ and can hence be computed in $O(V^\omega)$ time. We let $d_i = a_{ii}^{(2)}$ denote the degree of vertex i.

1. $$n_G(C_3) = \frac{1}{6} \cdot tr(A^3).$$

2. $$n_G(H_1) = |E| = \sum_{1 \leq i < j \leq n} a_{ij}^{(1)}.$$

3. $$n_G(H_2) = \sum_{i=1}^{n} \binom{d_i}{2}.$$

4. $$n_G(C_4) = \frac{1}{8} \cdot [tr(A^4) - 4n_G(H_2) - 2n_G(H_1)].$$

5. $$n_G(H_3) = \sum_{(i,j) \in E} (d_i - 1)(d_j - 1) - 3n_G(C_3).$$

6.
$$n_G(H_4) = \sum_{i=1}^{n} \binom{d_i}{3}.$$

7.
$$n_G(H_5) = \frac{1}{2} \cdot [\sum_{i=1}^{n} a_{ii}^{(3)}(d_i - 2)].$$

Note that $a_{ii}^{(3)}$ is twice the number of triangles that pass through vertex i.

8.
$$n_G(C_5) = \frac{1}{10} \cdot [tr(A^5) - 10n_G(H_5) - 30n_G(C_3)].$$

9.
$$n_G(H_6) = \sum_{(i,j) \in E} \binom{a_{ij}^{(2)}}{2}.$$

Note that $a_{ij}^{(2)}$ is the number of common neighbors of i and j, which is also the number of paths of length 2 between i and j.

10.
$$n_G(H_7) = \frac{1}{2} \cdot [\sum_{i=1}^{n} a_{ii}^{(3)} \binom{d_i - 2}{2}].$$

11.
$$n_G(H_8) = \sum_{(i,j) \in E} a_{ij}^{(2)}(d_i - 2)(d_j - 2) - 2n_G(H_6).$$

Note that we must subtract $2n_G(H_6)$ to avoid the case in which the two degree-one vertices of H_8 are, actually, the same vertex.

12.
$$n_G(H_9) = \sum_{i=1}^{n} (d_i - 2) \sum_{j \neq i} \binom{a_{ij}^{(2)}}{2}.$$

Note that $\sum_{j \neq i} \binom{a_{ij}^{(2)}}{2}$ is exactly the number of quadrilaterals in which i participates.

13.
$$n_G(H_{10}) = \sum_{i=1}^{n} (\frac{1}{2} a_{ii}^{(3)})(\sum_{j \neq i} a_{ij}^{(2)}) - 6n_G(C_3) - 2n_G(H_5) - 4n_G(H_6).$$

Note that $(\frac{1}{2} a_{ii}^{(3)})(\sum_{j \neq i} a_{ij}^{(2)})$ is simply the number of triangles through i times the number of paths of length two that begin with i. However, we must only count such a triangle and such a path if they are disjoint, so we must subtract appropriate occurrences of C_3, H_5 and H_6.

14.
$$n_G(H_{11}) = \sum_{i=1}^{n} \binom{\frac{1}{2} a_{ii}^{(3)}}{2} - 2n_G(H_6).$$

15. Since we have already shown how to compute $n_G(H)$ for all the 6-cyclic graphs, exluding C_6, we can use equation (1) to compute $n_G(C_6)$.

16. $$n_G(H_{12}) = \sum_{(i,j)\in E} a_{ij}^{(2)} \cdot a_{ij}^{(3)} - 9n_G(C_3) - 2n_G(H_5) - 4n_G(H_6).$$

    Here we count the number of triangles through (i,j) and multiply each triangle by the number of walks of length 3 between i and j. Since these walks need not be simple, or may intersect the triangle, we may be actually counting C_3's, H_5's or H_6's. Therefore, we subtract the appropriate values.

17. $$n_G(H_{13}) = \sum_{(i,j)\in E} \binom{a_{ij}^{(2)}}{3}.$$

18. $$n_G(H_{14}) = \sum_{i=1}^{n} (d_i - 2) \cdot B_i - 2n_G(H_{12})$$

    where B_i is the number of C_5 passing through i. The expression for B_i is:

$$B_i = \frac{1}{2}[a_{ii}^{(5)} - 10a_{ii}^{(3)} - 4a_{ii}^{(3)}(d_i - 2) - 2 \cdot \sum_{(i,j)\in E} a_{ij}^{(2)}(d_j - 2) - 2 \cdot \sum_{(i,j)\in E} (\frac{1}{2}a_{jj}^{(3)} - a_{ij}^{(2)})].$$

19. $$n_G(H_{15}) = \sum_{i=1}^{n} (\frac{1}{2}a_{ii}^{(3)})(\sum_{j\neq i} \binom{a_{ij}^{(2)}}{2}) - 6n_G(H_6) - 2n_G(H_{12}) - 6n_G(H_{13}).$$

□

Using slightly more effort, it can be shown that in $O(V^\omega)$ time, we can also count the number of C_k's, for $3 \leq k \leq 7$, that pass through each vertex of G. We have, in fact, done this in the preceding proof for $k = 3, 4, 5$. If the graph G contains a C_k, for some $3 \leq k \leq 7$, we can therefore find, in $O(V^\omega)$ time, a vertex through which such a C_k passes. We can then locate a C_k in the graph in additional $O(E)$ time using Monien's method (Lemma 3.3).

Similar formulae can be obtained, of course, for the number of octagons (C_8's) and even larger cycles. To compute the number of octagons, however, we have to compute first the number of K_4's in the graph, since a K_4 is 8-cyclic. We do not know how to do this is $O(V^\omega)$ time.

It is easy to count the number of K_4's in a graph in $O(V^{\omega+1})$ time: for each vertex, count the number of triangles among its neighbors, sum these numbers and divide by 4. Counting the number of K_4's in a graph, or in fact, deciding whether a graph contains a K_4, in $o(V^{\omega+1})$ time is an interesting open problem.

For counting the number of larger cycles using our method, we would have to count the number of larger cliques in the graph. Nešetřil and Poljak [11] give an $O(V^{\omega\lceil\frac{\ell}{3}\rceil})$ time algorithm for deciding whether a graph $G = (V, E)$ contains a K_l. It is easy to check that their method can also be used to

count the number of such cliques contained in the graph. By combining the method of Nešetřil and Poljak [11] with the ideas used in Section 4, we get the following result.

Theorem 6.4 *The number of K_l's in an undirected graph $G = (V, E)$ can be counted in either $O(V \cdot (d(G))^{\omega\lceil\frac{l-1}{3}\rceil})$ or $O(E \cdot (d(G))^{\omega\lceil\frac{l-2}{3}\rceil})$ time.*

Using an idea similar to the one used in Theorem 3.5, we can also get an $O(E^{\frac{3\omega+3}{\omega+3}}) = O(E^{1.89})$ algorithm for deciding whether a graph $G = (V, E)$ contains a K_4.

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