Problem Set no. 3 — Dynamic All-Pairs Shortest Paths

Given: January 13, 2010

Exercise 3.1 Describe a simple dynamic all-pairs shortest path algorithm that can handle *decreasing* updates in $O(n^2)$ *worst-case* time. (A decreasing update of a vertex v, as defined in class, is an update in which new edges entering v or emanating from v are added and/or weights of existing edges entering v or emanating from v are *decreased*.)

Exercise 3.2 Show that any fully dynamic algorithm that explicitly maintains a *distance matrix* must have an update time of $\Omega(n^2)$.

Exercise 3.3 (a) Construct a weighted n-vertex graph with unique shortest paths in which there are $\Omega(n^3)$ locally shortest paths. (b) Show that a decreasing update can generate at most $O(n^2)$ new locally shortest paths. (c) Show that an increasing update may generate $\Omega(n^3)$ new locally shortest paths.

Exercise 3.4 Let π_1 and π_2 be two distinct locally historical paths from x to y passing through v at some time t. Let v_i be the last updated vertex, other than x, on π_i, for $i = 1, 2$. Suppose that v_i appears before v on π_i, or $v_i = v$, for $i = 1, 2$. Let π'_i be the subpath of π_i from x to v, for $i = 1, 2$. Show that $\pi'_1 \neq \pi'_2$.

Guidance: Suppose for the sake of contradiction that $\pi'_1 = \pi'_2$. In particular, $v_1 = v_2$. Let π''_i be the subpath of π_i from v_i to y, for $i = 1, 2$. Then, π''_1 and π''_2 are distinct historical paths from $v_1 = v_2$ to y with the same last updated vertex, a contradiction.

Exercise 3.5 Suppose that at a certain moment there at most z historical paths between any pair of vertices of the graph. Show that there can be at most $O(zn^2)$ locally historical paths passing through a given vertex v.

Guidance: Showing that there are at most $O(zn^2)$ such paths that start or end at v is relatively straightforward. The harder part of the proof is showing that there only $O(zn^2)$ locally historical paths having v as an intermediate vertex. To show that, prove that for every $x, y \neq v$, there are only $O(z)$ locally historical paths from x to y that pass through v. Use the previous exercise to show that there can be at most z locally historical shortest paths from x to y passing through v whose last updated vertices, other than x and y, lie *before* v on these paths, and at most z such paths whose last updated vertices, other than x and y, lie *after* v on these paths.

Exercise 3.6 Show that if the Demetrescu-Italiano algorithm described in class is run on graph in which shortest paths are not necessarily unique, with ties broken arbitrarily, then the algorithm may fail to find paths, let alone shortest paths, between all pairs of vertices in the graph.