Problem Set no. 1 — Minimum spanning trees

Given: Movember 11, 2009

Exercise 1.1 Prove the correctness of the blue (cut) rule and the red (cycle) rule.

Exercise 1.2 Let $G = (V, E, w)$ be a connected weighted graph. (a) The graph G may have many minimum spanning trees. Show, however, that all these minimum spanning trees have the same (multi-)set of edge weights. (b) Show that if all the edge weights in G are distinct, then the minimum spanning tree is unique. (c) Show that a minimum spanning tree of G is also a spanning tree of G whose maximal edge weight is minimal.

Exercise 1.3 Let ALG be an algorithm for computing minimum spanning trees that is guaranteed to work only if all edges weights in the input graph are distinct. Suppose that the only operation performed by ALG on edge weights are *comparisons*. Show how to convert ALG into an algorithm ALG$'$, with the same asymptotic running time, that works even if some of the edge weights are equal.

Exercise 1.4 Describe a *deterministic* linear time algorithm for finding a spanning tree whose maximal weight is minimal. (Hint: Start by computing the median of the edge weights.)

Exercise 1.5 Describe a simple implementation of Borůvka's algorithm that runs in $O(n^2)$ time, and another simple implementation that runs in $O(m \log n)$ time. Show next that these two implementations can be combined to yield an implementation whose complexity is $O(m \log \frac{n^2}{m})$ time.

Exercise 1.6 Show that Prim's algorithm, which runs in $O(m + n \log n)$ time, can be combined with Borůvka's algorithm to yield an $O(m \log \log n)$ algorithm for finding a minimum spanning tree.

Exercise 1.7 Describe a *deterministic* linear time algorithm algorithm for finding a minimum spanning tree in a *planar* graph. (Hint: a contraction of a planar graph is planar graph. A planar graph on n vertices with no parallel edges has at most $3n - 6$ edges.)

Exercise 1.8 Let $G = (V, E)$ be an undirected graph on n vertices and let $0 < p < 1$. Let $H = (V, E')$ be a random subgraph of G to which each edge of G is added, independently, with probability p. Show that the expected number of edges of G that connect different connected components of H is at most n/p.

Exercise 1.9 The randomized linear time algorithm of Karger, Klein and Tarjan for finding a minimum spanning tree uses sampling steps in which each edge of the graph is chosen, independently, with probability $1/2$. What would be the expected running time of the algorithm if this sampling probability were changed to p, where $0 \le p \le 1$?

Exercise 1.10 In this exercise we obtain an alternative proof, due to Chan, of the following variant of the sampling lemma used to obtain the randomized linear MST algorithm:

> Let $G = (V, E)$ be a weighted graph with distinct edge weights. Let $G' = (V, R)$ be a random subgraph of G containing *exactly* r edges. Let F be a minimum spanning forest of G'. Then, the expected number of edges of G that are F-light is at most nm/r.

(a) Show that $e \in E$ is F-light if and only if $e \in \mathit{MSF}(R \cup \{e\})$, where $\mathit{MSF}(R \cup \{e\})$ is the minimum spanning forest of the subgraph $(V, R \cup \{e\})$.

(b) Show that if e is a random edge of G and R is a random subset of exactly r edges of G then $\Pr[e \in \mathit{MSF}(R \cup \{e\})] < n/r$. (Hint: note that e is a random element of $R \cup \{e\}$, a random set of size r or $r + 1$. How many edges from this set are in $\mathit{MSF}(R \cup \{e\})$?)

(c) Finish of the proof of the lemma.