Analysis of Algorithms

Problem Set no. 3 — Dynamic All-Pairs Shortest Paths

Given: June 16, 2009

Exercise 3.1 Describe a simple dynamic all-pairs shortest path algorithm that can handle decreasing updates in $O(n^2)$ worst-case time. (A decreasing update of a vertex $v$, as defined in class, is an update in which new edges entering $v$ or emanating from $v$ are added and/or weights of existing edges entering $v$ or emanating from $v$ are decreased.)

Exercise 3.2 Show that any fully dynamic algorithm that explicitly maintains a distance matrix must have an update time of $\Omega(n^2)$.

Exercise 3.3 (a) Construct a weighted $n$-vertex graph with unique shortest paths in which there are $\Omega(n^3)$ locally shortest paths. (b) Show that a decreasing update can generate at most $O(n^2)$ new locally shortest paths. (c) Show that an increasing update may generate $\Omega(n^3)$ new locally shortest paths.

Exercise 3.4 Let $\pi_1$ and $\pi_2$ be two distinct locally historical paths from $x$ to $y$ passing through $v$ at some time $t$. Let $v_i$ be the last updated vertex, other than $x$, on $\pi_i$, for $i = 1, 2$. Suppose that $v_i$ appears before $v$ on $\pi_i$, or $v_i = v$, for $i = 1, 2$. Let $\pi'_i$ be the subpath of $\pi_i$ from $x$ to $v$, for $i = 1, 2$. Show that $\pi'_1 \neq \pi'_2$.

Guidance: Suppose for the sake of contradiction that $\pi'_1 = \pi'_2$. In particular, $v_1 = v_2$. Let $\pi''_i$ be the subpath of $\pi_i$ from $v_i$ to $y$, for $i = 1, 2$. Then, $\pi''_1$ and $\pi''_2$ are distinct historical paths from $v_1 = v_2$ to $y$ with the same last updated vertex, a contradiction.

Exercise 3.5 Suppose that at a certain moment there at most $z$ historical paths between any pair of vertices of the graph. Show that there can be at most $O(zn^2)$ locally historical paths passing through a given vertex $v$.

Guidance: Showing that there are at most $O(zn^2)$ such paths that start or end at $v$ is relatively straightforward. The harder part of the proof is showing that there only $O(zn^2)$ locally historical paths having $v$ as an intermediate vertex. To show that, prove that for every $x, y \neq v$, there are only $O(z)$ locally historical paths from $x$ to $y$ that pass through $v$. Use the previous exercise to show that there can be at most $z$ locally historical shortest paths from $x$ to $y$ passing through $v$ whose last updated vertices, other than $x$ and $y$, lie before $v$ on these paths, and at most $z$ such paths whose last updated vertices, other than $x$ and $y$, lie after $v$ on these paths.

Exercise 3.6 Show that if the Demetrescu-Italiano algorithm described in class is run on graph in which shortest paths are not necessarily unique, with ties broken arbitrarily, then the algorithm may fail to find paths, let alone shortest paths, between all pairs of vertices in the graph.