

Problem Set no. 1 — Minimum spanning trees

Given: March 24, 2009

Exercise 1.1 Let $G = (V, E, w)$ be a connected weighted graph. (a) The graph G may have many minimum spanning trees. Show, however, that all these minimum spanning trees have the same (multi-)set of edge weights. (b) Show that if all the edge weights in G are distinct, then the minimum spanning tree is unique. (c) Show that a minimum spanning tree of G is also a spanning tree of G whose maximal edge weight is minimal.

Exercise 1.2 Describe a *deterministic* linear time algorithm for finding a spanning tree whose maximal weight is minimal. (Hint: Start by computing the median of the edge weights.)

Exercise 1.3 Show that Prim's algorithm, which runs in $O(m + n \log n)$ time, can be combined with Borůvka's algorithm to yield an $O(m \log \log n)$ algorithm for finding a minimum spanning tree.

Exercise 1.4 Describe a *deterministic* linear time algorithm for finding a minimum spanning tree in a *planar* graph. (Hint: a contraction of a planar graph is planar graph. A planar graph on n vertices with no parallel edges has at most $3n - 6$ edges.)

Exercise 1.5 Let $G = (V, E)$ be an undirected graph on n vertices and let $0 < p < 1$. Let $H = (V, E')$ be a random subgraph of G to which each edge of G is added, independently, with probability p . Show that the expected number of edges of G that connect different connected components of H is at most n/p .

Exercise 1.6 The randomized linear time algorithm of Karger, Klein and Tarjan for finding a minimum spanning tree uses sampling steps in which each edge of the graph is chosen, independently, with probability $1/2$. What would be the expected running time of the algorithm if this sampling probability were changed to p , where $0 \leq p \leq 1$?

Exercise 1.7 In this exercise we obtain an alternative proof, due to Chan, of the following variant of the sampling lemma used to obtain the randomized linear MST algorithm:

Let $G = (V, E)$ be a weighted graph with distinct edge weights. Let $G' = (V, R)$ be a random subgraph of G containing *exactly* r edges. Let F be a minimum spanning forest of G' . Then, the expected number of edges of G that are F -light is at most nm/r .

- (a) Show that $e \in E$ is F -light if and only if $e \in MSF(R \cup \{e\})$, where $MSF(R \cup \{e\})$ is the minimum spanning forest of the subgraph $(V, R \cup \{e\})$.
- (b) Show that if e is a random edge of G and R is a random subset of exactly r edges of G then $\Pr[e \in MSF(R \cup \{e\})] < n/r$. (Hint: note that e is a random element of $R \cup \{e\}$, a random set of size r or $r + 1$. How many edges from this set are in $MSF(R \cup \{e\})$?)
- (c) Finish of the proof of the lemma.