Lecture notes on:
Maximum matching in non-bipartite graphs

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Abstract

We prove the correctness of Edmonds' *blossom shrinking* algorithm for finding a maximum cardinality matching in a general graph.

1 The maximum matching problem

Let $G = (V, E)$ be an undirected graph. A set $M \subseteq E$ is a *matching* if no two edges in M touch each other or, in other words, if the degree of every vertex in the subgraph (V, M) is at most 1. A vertex v is *matched* by M if there is an edge of M that touches v. Otherwise, v in *unmatched*. In the *maximum matching problem* we are asked to find a matching M of maximum size in a given input graph $G = (V, E)$.

2 Alternating and augmenting paths

We begin with a definition of alternating paths and cycles.

Definition 2.1 (Alternating paths and cycles) *Let $G = (V, E)$ be a graph and let M be a matching in M. A path P is said to be an alternating path with respect to M if and only if among every two consecutive edges along the path, exactly one belongs to M. An alternating cycle C is defined similarly.*

Some alternating paths and an alternating cycle are shown in Figure 1. We use the convention that edges that belong to a matching M are shown as *thick* edges, while edges not belonging to M are shown as *thin* edges.

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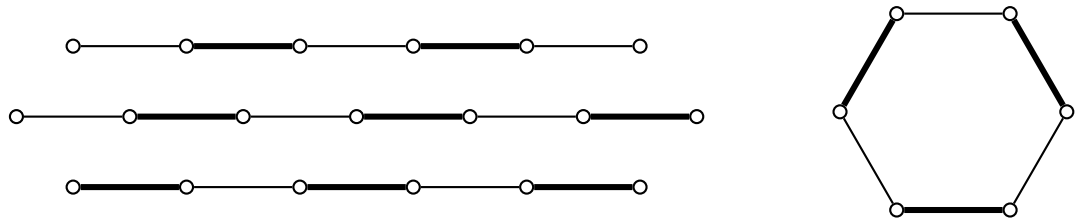

Figure 1: Alternating paths and cycles

Definition 2.2 (Symmetric difference) *If A and B are sets, we let $A \oplus B = (A - B) \cup (B - A)$ be their* symmetric difference.

The following lemma is now obvious.

Lemma 2.3 *If M is a matching and P is an alternating path with respect to M, where each endpoint of P is either unmatched by M or matched by the edge of P touching it, then $M \oplus P$ is also a matching.*

Note that if P starts and ends in vertices unmatched by M (e.g., the top path in Figure 1), then $|M \oplus P| = |M| + 1$, i.e., $M \oplus P$ is a larger matching. If P starts with an edge that does not belong to M and ends with an edge of M (e.g., the middle path in Figure 1), then $|M \oplus P| = |M|$. Finally, if P starts and ends with edges of M (see the bottom path in Figure 1, then $|M \oplus P| = |M| - 1$.

Lemma 2.4 *Let $G = (V, E)$ be an undirected graph and let M_1 and M_2 be matchings in G. Then, the subgraph $(V, M_1 \oplus M_2)$ is composed of isolated vertices, alternating paths and alternating cycles with respect to both M_1 and M_2.*

Definition 2.5 (Augmenting paths) *An* augmenting path *P with respect to a matching M is an alternating path that starts and ends in* unmatched *vertices.*

Note that an augmenting path in necessarily of odd length and that the number of edges on P that do not belong to M is by one larger than the number of edges that do belong to M.

Theorem 2.6 *Let $G = (V, E)$ be an undirected graph and let M be a matching in G. Then, M is a maximum matching in G if and only if there are no augmenting paths with respect to M in G.*

Proof: If P is an augmenting path with respect to M, then $M \oplus P$ is also a matching and $|M \oplus P| > |M|$, so M is not a maximum cardinality matching of G.

Conversely, suppose that M is not a maximum matching in G. Let M' be a matching with $|M'| > |M|$. By Lemma 2.4 $M \oplus M'$ is composed of alternating paths and cycles. All the alternating cycles

are of even length and contain the same number of edges from M and M'. At least one of the alternating paths must contain more edges from M' and this path is then an augmenting path with respect to M. $\square$

Theorem 2.6 suggests the following simple algorithm for finding a maximum matching. Start with some initial matching M, possibly the empty one. As long as there is an augmenting path P with respect to M, augment M using P and repeat. All that remains, therefore, is to devise a procedure for finding augmenting paths, if they exist.

3 Blossoms

Definition 3.1 (Flowers and blossoms) *A* flower *with respect to a matching M is composed of a* stem*, which is an alternating path of even length from an unmatched vertex r, called the* root*, to vertex b, and an 'alternating' cycle of odd length that passes through b, called a* blossom*. The last edge on the stem belongs to M. The two edges of the blossom touching b are not in M. Other than that, every second edge on the blossom belongs to M. The vertex b is said to be the* base *of the blossom.*

If $G = (V, E)$ is a graph, $M \subseteq E$ is a matching, and B is a blossom, we let G_B be the graph obtained from G by contracting G, and by $M_B = M - B$ the matching corresponding to M in G_B. We use B to denote the vertex of G_B obtained by contracting B.)

Lemma 3.2 *Let $G = (V, E)$ be an undirected graph and let M be a matching in G. Let B be a blossom in G. If there is an augmenting path with respect to M_B in G_B, then there is also an augmenting path with respect to M in G.*

Proof: Let P be an augmenting path in G_B with respect to M_B. If P does not pass through b, then P is also an augmenting path in G with respect to M, and we are done.

Suppose, therefore, that P does pass through B. We consider two cases:

Case 1: P starts (or ends) at B.

Let (B, c) be the first edge on P and let P_c be the part of P that starts at c. Clearly $(B, c) \notin M_B$. Also, there is a vertex $v \in B$ such that $(v, c) \in E$ and $(v, c) \notin M$. In the blossom B, we can find an even length alternating path Q from b to v. This path ends with an edge of M. The path $Q, (v, c), P_c$ is then an augmenting path in G with respect to M, as required.

Case 2: B is not the first or last vertex on P.

Let $(a, B), (B, c) \in P$ be the edges of P that touch B. Assume that $(a, B) \in M_B$ and $(B, c) \notin M_B$. Let P_a be the part of P up to a, and let P_c be the part of P from c. As $(a, B) \in M_b$, the edge (a, b) must be present in the original graph and $(a, b) \in M$, as the only edge of the matching that enters a blossom enters it at its base. As before, there is a vertex $v \in B$ such that $(v, c) \in E$ and

$(v, c) \notin M$. We can again find an even alternating path Q in the blossom from b to v. The path $P_a, (a, b), Q, (v, c), P_c$ is then an augmenting path in G with respect to M, as required. $\square$

Lemma 3.3 *Let $G = (V, E)$ be an undirected graph and let M be a matching in G. Let B be a blossom in G which is part of a flower with stem Q. If there is an augmenting path with respect to M in G, then there is also an augmenting path with respect to M_B in G_B.*

Proof: Let b be the base of the blossom B. We consider two cases:

Case 1: b is unmatched by M.

Let P be an augmenting path in G with respect to M. If P does not pass through any vertex of B, then P is also an augmenting path with respect to M in G. Suppose therefore that P does pass through B. As P starts and ends at unmatched vertices, and only one vertex on B is unmatched, we may assume that P does not start on B. Let P' be the prefix of P until the first encounter with a vertex on B. Let $P' = P'', (a, v)$, where $v \in B$. Then, $P'', (a, B)$ is an augmenting path with respect to M_B in G_B, as B is unmatched by M_B.

Case 2: b is matched by M.

Let $M' = M \oplus Q$, where Q is a stem of a flower with blossom B. As Q is an even alternating path starting at an unmatched vertex, we get that M' is a matching and that $|M'| = |M|$. Note that B is also a blossom with respect to M' and that b is unmatched with respect to M'. We also have $|M'_B| = |M_B|$.

As M' and M are matching of the same size and as there is an augmenting path with respect to M, there must also be an augmenting path with respect to M'. As b in unmatched by M', we get by Case 1 that there must be augmenting path with respect to M'_B in G_B, and as M'_B and M_B are again of the same size, we get that there must also be an augmenting path with respect to M_B in G_B, as required. $\square$

For more information see Edmonds' original paper [Edm65]. For a very clear description of the whole algorithm, with C++ code, see Section 7.7 of LEDA book [MN99]. Various versions of the algorithm are also described in the books [Law76], [Tar83] and [AMO93].

References

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