Problem Set no. 3

Given: June 12, 2007
Due: June 26, 2007

Exercise 3.1 Show that a flow $f : E \to \mathbb{R}^+$ is a minimum cost flow in a flow network $G = (V, E)$ with cost function $a : E \to \mathbb{R}$ and capacity function $c : E \to \mathbb{R}^+$ if and only if there exists a potential function $\pi : V \to \mathbb{R}$ such that

1. if $a_\pi(e) > 0$, then $f(e) = 0$,
2. if $0 < f(e) < c(e)$, then $a_\pi(e) = 0$,
3. if $a_\pi(e) < 0$, then $f(e) = c(e)$,

where $a_\pi(e) = a(e) + \pi(u) - \pi(v)$, for every $e = (u, v) \in E$. (Hint: we have shown in class that f is a minimum cost flow if and only if there are no negative cycles in the *residual* network.)

Exercise 3.2 A *circulation* is a flow of value 0, i.e., a flow in which the conservation constraints are satisfied at all vertices. Describe a simple reduction from the problem of finding maximum flow to the problem of finding minimum cost circulation.

Exercise 3.3 In the minimum cost flow problem that we defined in class, there was a unique source s and unique sink t and we were asked to find a flow of minimum cost of value b_0. A slight generalization of the minimum cost is obtained when additional *demand* function $b : V \to \mathbb{R}$ is given. To goal is to find a flow in which the *excess* at every vertex $v \in V$, is $b(v)$, where the access at v is the total amount of flow going into v, minus the total amount of flow leaving v. (Note that for such a problem to be feasible, we must have $\sum_{v \in V} b(v) = 0$.) In the original version of the problem considered in class we have $b(s) = -b_0$, $b(t) = b_0$, and $b(v) = 0$, for every $v \in V - \{s, t\}$. Describe a simple reduction from the this more general version of the minimum cost flow problem to the original version.

Exercise 3.4 An instance of the *uncapacitated* minimum cost flow problem with demands is specified by giving a graph $G = (V, E)$, a cost function $a : E \to \mathbb{R}$ and demand function $b : V \to \mathbb{R}$. (See the previous exercise.) The capacity of each edge is *infinite*. Describe a reduction from the standard capacitated minimum cost flow problem, with a source and a sink, to the uncapacitated version of the minimum cost flow problem with demands. (Hint: For every edge $e = (u, v) \in E$, introduce a new *vertex* $\bar{e}$. Replace the edge e by the two edges $(u, \bar{e})$ and $(v, \bar{e})$. Define the costs of these edges appropriately, and adjust the demands at u, v and $\bar{e}$.)

Exercise 3.5 Complete the proof of Lemma 7.5 in the lecture notes for the case $a_{\pi_i}(e) \le -2n\varepsilon(f_i)$.