Problem Set no. 1

Given: March 6, 2007
Due: March 20, 2007

Exercise 1.1 Let $G = (V, E, w)$ be a connected weighted graph. (a) The graph G may have many minimum spanning trees. Show, however, that all these minimum spanning trees have the same (multi-)set of edge weights. (b) Show that if all the edge weights in G are distinct, then the minimum spanning tree is unique. (c) Show that a minimum spanning tree of G is also a spanning tree of G whose maximal edge weight is minimal.

Exercise 1.2 Describe a *deterministic* linear time algorithm for finding a spanning tree whose maximal weight is minimal. (Hint: Start by computing the median of the edge weights.)

Exercise 1.3 Show that Prim's algorithm, which runs in $O(m + n \log n)$ time, can be combined with Borůvka's algorithm to yield an $O(m \log\log n)$ algorithm for finding a minimum spanning tree.

Exercise 1.4 Describe a *deterministic* linear time algorithm algorithm for finding a minimum spanning tree in a *planar* graph. (Hint: a contraction of a planar graph is planar graph. A planar graph on n vertices with no parallel edges has at most $3n - 6$ edges.)

Exercise 1.5 In this exercise we obtain an alternative proof, due to Chan, of the following variant of the sampling lemma used to obtain the randomized linear MST algorithm:

> Let $G = (V, E)$ be a weighted graph with distinct edge weights. Let $G' = (V, R)$ be a random subgraph of G containing *exactly* r edges. Let F be a minimum spanning forest of G'. Then, the expected number of edges of G that are F-light is at most nm/r.

(a) Show that $e \in E$ is F-light if and only if $e \in MSF(R \cup \{e\})$, where $MSF(R \cup \{e\})$ is the minimum spanning forest of the subgraph $(V, R \cup \{e\})$.

(b) Show that if e is a random edge of G and R is a random subset of exactly r edges of G then $\Pr[e \in MSF(R \cup \{e\})] < n/r$. (Hint: note that e is a random element of $R \cup \{e\}$, a random set of size r or $r + 1$. How many edges from this set are in $MSF(R \cup \{e\})$?)

(c) Finish of the proof of the lemma.