Problem Set no. 6

Given: January 23, 2020

**Exercise 6.1** (a) Prove that if $A$ is an $n$-USO and $B(x) = A(x) \oplus e_i$ for every $x \in \{0,1\}^n$, where $e_i$ is the $i$-th unit vector and $i \in [n]$, then $B$ is also a USO. (b) Prove that if $A$ is an $n$-USO and $B(x) = A(x) \oplus c$ for every $x \in \{0,1\}^n$, where $c \in \{0,1\}^n$, then $B$ is also a USO. (See slide 14 of lecture 7.)

**Exercise 6.2** Let $A : \{0,1\}^n \to \{0,1\}^n$ be an orientation of the $n$-cube, i.e., $A(x)_i \neq A(x \oplus e_i)_i$, for every $x \in \{0,1\}^n$ and $i \in [n]$. Prove that $A$ is an $n$-USO if and only if for every $x \neq y \in \{0,1\}^n$ we have $(x \oplus y) \land (A(x) \oplus A(y)) \neq 0^n$. (Here $\oplus$ and $\land$ are applied bit-wise on the two vectors.)

**Exercise 6.3** Let $A : \{0,1\}^n \to \{0,1\}^n$ be an acyclic orientation of the $n$-cube, where $n \geq 2$. Prove that $A$ is an $n$-AUSO if and only if every 2-dimensional subcube of $A$ has a unique sink.

**Exercise 6.4** Let $A$ be an $n$-USO and let $y$ be the sink of $A$. Prove that from any vertex $x$ there is a directed path of length $|x \oplus y|$ from $x$ to $y$.

**Exercise 6.5** Let $x \neq y \in \{0,1\}^n$. Prove that there is an $n$-AUSO whose source is at $x$ and its sink is at $y$.

**Exercise 6.6** Let $\Gamma = (S = S_0 \cup S_1, A = \cup_{i \in S} A_i, p, c)$ be a binary TBSG, i.e., for every $i \in S_0$ we have $|A_i| = 2$. Furthermore, assume that for every two positional strategies $\pi$ and $\pi'$ of player 0 that differ in exactly one action we have $y^{\pi,\tau(\pi)} \neq y^{\pi',\tau(\pi')}$, where $\tau(\pi)$ is an optimal counter-strategy of player 1 for $\pi$. Describe a way of constructing an $n$-AUSO, where $n = |S_0|$ that 'encodes' $\Gamma$.