Problem Set no. 6

Given: January 23, 2020

Exercise 6.1 (a) Prove that if A is an n-USO and $B(x) = A(x) \oplus e_i$ for every $x \in \{0,1\}^n$, where e_i is the i-th unit vector and $i \in [n]$, then B is also a USO. (b) Prove that if A is an n-USO and $B(x) = A(x) \oplus c$ for every $x \in \{0,1\}^n$, where $c \in \{0,1\}^n$, then B is also a USO. (See slide 14 of lecture 7.)

Exercise 6.2 Let $A : \{0,1\}^n \to \{0,1\}^n$ be an orientation of the n-cube, i.e., $A(x)_i \neq A(x \oplus e_i)_i$, for every $x \in \{0,1\}^n$ and $i \in [n]$. Prove that A is an n-USO if and only if for every $x \neq y \in \{0,1\}^n$ we have $(x \oplus y) \wedge (A(x) \oplus A(y)) \neq 0^n$. (Here $\oplus$ and $\wedge$ are applied bit-wise on the two vectors.)

Exercise 6.3 Let $A : \{0,1\}^n \to \{0,1\}^n$ be an *acyclic* orientation of the n-cube, where $n \geq 2$. Prove that A is an n-AUSO if and only if every 2-dimensional subcube of A has a unique sink.

Exercise 6.4 Let A be an n-USO and let y be the sink of A. Prove that from any vertex x there is a directed path of length $|x \oplus y|$ from x to y.

Exercise 6.5 Let $x \neq y \in \{0,1\}^n$. Prove that there is an n-AUSO whose source is at x and its sink is at y.

Exercise 6.6 Let $\Gamma = (S = S_0 \uplus S_1, A = \uplus_{i \in S} A_i, p, c)$ be a binary TBSG, i.e., for every $i \in S_0$ we have $|A_i| = 2$. Furthermore, assume that for every two positional strategies π and π' of player 0 that differ in exactly one action we have $\mathbf{y}^{\pi,\tau(\pi)} \neq \mathbf{y}^{\pi',\tau(\pi')}$, where $\tau(\pi)$ is an optimal counter-strategy of player 1 for π. Describe a way of constructing an n-AUSO, where $n = |S_0|$ that 'encodes' Γ.