Problem Set no. 5
Given: January 23, 2020

Exercise 5.1 Consider the separating automaton of [CDFJLP (2018)]. (a) Let \( w \in \text{AllCycEven}_{n,d} \). Derive an upper bound on the number of steps after which the separating automaton accepts \( w \). (b) Let \( w \in \text{LimsupEven}_d \). Does the automaton necessarily accepts \( w \)? After how many steps?

Exercise 5.2 Let \( G = (V, E, p) \), where \( p : E \to [d] \), be an even graph, i.e., a graph in which the maximum priority on every cycle is even. Assume that \( d \) is even. Let \( L_i \) be the set of vertices from which there is a path containing \( i \) edges of priority \( d - 1 \) and no edges of priority \( d \), but there is no such path containing \( i + 1 \) edges of priority \( d - 1 \). Prove that: (1) \( V = \bigcup_{i=0}^{n-1} L_i \). (2) If \( (u, v) \in E, p(u, v) = d - 1, u \in L_j, v \in L_i \), then \( j > i \). (3) If \( (u, v) \in E, p(u, v) \leq d - 1, u \in L_j, v \in L_i \), then \( j \geq i \).

Exercise 5.3 Describe a polynomial time algorithm that given an even graph \( G = (V, E, p) \), where \( p : E \to [d] \) and \( d \) even, finds a progress measure \( f : V \to [0, n-1]^d/2 \cup \{\infty\} \). What is the complexity of the algorithm?

Exercise 5.4 Let \( G = (V = V_0 \sqcup V_1, E, p) \), where \( p : E \to [d] \) and \( d \) even, be a parity game. Let \( f^*(u) = \min \{ f(u) \mid f \) is a progress measure of \( G \} \), for every \( u \in V \). (a) Prove that \( f^* \) is a progress measure. (b) Prove that for every \( u \in V \), EVEN can win from \( u \) if and only if \( f^*(u) < \infty \).

Exercise 5.5 Prove that a maximal even \((n,d)\)-multigraph is tree-like. (See slide 57 of Lecture 6.)

Exercise 5.6 Let \( G = (V, E) \) be a tree-like \((n,d)\)-multigraph, where \( E \subseteq V \times [0,d] \times V \). Let \( E_p = \{ (u,v) \in V \times V \mid (u,p,v) \in E \} \). Show that each equivalence class of \( E_{p} \), where \( p > 0 \) is even, is composed of a disjoint union of equivalence classes of \( E_{p-2} \). (See slide 59 of Lecture 6.)

Exercise 5.7 Prove that a tree \( t \) is an ordered subtree of tree \( T \) if and only if there is a homomorphism from \( G(t) \) to \( G(T) \).