

## Problem Set no. 3

Given: December 24, 2019

**Exercise 3.1** Let  $G = (V_0, V_1, E, c)$  be an energy game. Recall that a function  $f : V \rightarrow \mathbb{R}^+ \cup \{\infty\}$ , where  $\mathbb{R}^+ = [0, \infty)$ , is *feasible* iff (1) If  $u \in V_0$  then there exists  $(u, v) \in E$  such that  $f(u) \geq f(v) - c(u, v)$ ; (2) If  $u \in V_1$  then for every  $(u, v) \in E$  we have  $f(u) \geq f(v) - c(u, v)$ . Show that:

- (a) The infimum  $f^*$ , vertex-wise, of all feasible functions is feasible.
- (b) For every  $u \in V$ ,  $f^*(u)$  is the value of the game that starts at  $u$ .

**Exercise 3.2** Let  $G = (V_0, V_1, E, c)$  be an energy game. For every  $u \in V$ , let  $f^*(u)$  be the value of the energy game that starts at  $u$ . Show that:

- (a) If there is a vertex  $u \in V$  such that  $f^*(u) < \infty$ , then there is a vertex  $v \in V$  such that  $f^*(v) = 0$ .
- (b) For every  $u \in V$ , if  $f^*(u) < \infty$  then  $f^*(u) < nW$ , where  $W = \max_{(u,v) \in E, c(u,v) < 0} c(u, v)$ .

**Exercise 3.3** (a) Give a polynomial time algorithm for finding the values of all vertices in an energy game in which all vertices are controlled by player 0 (the minimizer).

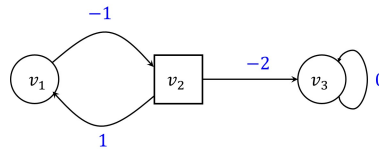
(b) Give a polynomial time algorithm for finding the values of all vertices in an energy game in which all vertices are controlled by player 1 (the maximizers).

Bonus: How efficient can you make these algorithms?

**Exercise 3.4** Use the reduction from MPGs to EGs to obtain the following algorithms for the exact solution of a MPG  $G = (V_0, V_1, E, c)$ , where  $c : E \rightarrow \{-W, \dots, 0, \dots, W\}$ .

- (a) An  $O(mn^2W \log(nW))$  algorithm for finding the exact value of a given vertex  $u_0 \in V$ . (Hint: The value of each vertex is a rational number of the form  $a/b$ , where  $a$  and  $b$  are integers and  $-nW \leq a \leq nW$  and  $1 \leq b \leq n$ . Use binary search on the possible rational values.)
- (b) An  $O(mn^2W \log(nW))$  algorithm for finding the exact values of all vertices. (Hint: Do the binary search simultaneously on all vertices. If for some rational value  $r$  we already know which vertices have value at least  $r$  and which vertices have values less than  $r$ , then when we continue the search for the values that are at least  $r$  we can remove from the game all vertices of value less than  $r$ , and vice versa.)
- (c) An  $O(mn^2W \log(nW))$  algorithm for finding the exact values of all vertices and optimal strategies of the two players. (Hint: Extend the algorithm developed in item (b).)

**Exercise 3.5** Consider a variant of energy games in which the battery has a finite capacity  $B$ . The charge of the battery is never allowed to exceed  $B$ . If the electric car is in vertex  $u$  with charge  $a$ ,  $(u, v)$  is an edge and  $a + c(u, v) \geq 0$ , then  $(u, v)$  can be traversed. If it is traversed, the car reaches  $v$  with a charge of  $\min\{a + c(u, v), B\}$ . The value of a vertex  $u$  is the minimum initial charge, not exceeding  $B$ , for which player 0 (the minimizer) has a strategy that enables her to keep playing forever, no matter what player 1 (the maximizer) is doing. If there is no such finite initial charge then the value is  $\infty$ . Consider the following game, where  $v_1, v_3$  belong to player 0 and  $v_2$  belongs to player 1.



- (a) What is the value of  $v_2$  in the standard energy game, i.e., when  $B = \infty$ .
- (b) What is the value of  $v_2$  when  $B = 2$ ?
- (c) Does player 1 have a positional strategy that ensures this value?