Exercise 1.1  Show that matrix multiplication and matrix squaring have the same asymptotic complexity.

Exercise 1.2  Give an $O(n^2)$-time randomized algorithm for checking whether $C = AB$, where $A, B, C$ are three $n \times n$ matrices with, say, integer entries. The algorithm should always say 'yes' if $C = AB$, and should say 'no' with a probability of at least $1/2$ if $C \neq AB$.

Exercise 1.3 (a) Give an $O(n^w)$-time algorithm for finding a triangle (a 3-cycle), if there exists one, in a directed graph on $n$ vertices. (b) Give an $O(n^w)$-time algorithm for finding a simple 4-cycle. (c) (Bonus) Give an $O(n^w)$-time algorithm for finding a simple 5-cycle. (A cycle is simple if all vertices on it are distinct.)

Exercise 1.4  Let $G = (V,E)$ be a graph and let $w : E \to \mathbb{N}$ be an integer weight function. Suppose that $G$ has a unique perfect matching $M$ of minimum weight. Let $A$ be the Tutte matrix of $G$ in which $x_{i,j}$ is replaced by $2^{w_{i,j}}$, where $w_{i,j}$ is the weight of the edge $\{i,j\} \in E$. (a) Show that $2^{2W} \left| \det(A) \right|$, but that $2^{2W+1} \nmid \det(A)$. (b) Show that $\{i,j\} \in M$ iff $2^{w_{i,j}} \det(A^{ij})/2^{2W}$ is odd.

Exercise 1.5  Let $G = ([n],E)$ be a directed graph. The symbolic adjacency matrix of $G$ is a matrix $A = (a_{i,j})$, where $a_{i,j} = x_{i,j}$, if $\{i,j\} \in E$, and $a_{i,j} = 0$, otherwise, where the $x_{i,j}$'s are indeterminants. Also, $a_{i,i} = 1$, for every $i \in [n]$. Show that there is a directed path from $i$ to $j$ in $G$ iff $(\text{adj}(A))_{i,j} \neq 0$.

Exercise 1.6  (Bonus) Give a randomized polynomial time algorithm for the exact matching problem.