Exercise 4.1  a) Let $P$ be an irreducible and aperiodic finite Markov chain and let $A \subseteq S$ be a subset of its states. Prove that if we pick a state according to the stationary distribution and make a step according to $P$ then the probability that we leave a state of $A$ is the same as the probability that we enter a state of $A$.
b) Consider a bounded queue $Q$ containing at most $n$ elements (it has $n+1$ states according to the number of elements it has). If $Q$ has less than $n$ elements then with probability $\lambda$ a new element is inserted into $Q$. If $Q$ is not empty then with probability $\mu$ an element leaves $Q$. Otherwise, $Q$ does not change. Prove that $Q$ has a unique stationary distribution and find it.

Exercise 4.2  Let $P$ be an irreducible Markov chain with $n$ states.
a) For two states $x$ and $y$, let $\tau_{xy}$ be number of steps that we do starting from $x$ until the first time we get to $y$. Prove that $E(\tau_{xy})$ is finite for any two states $x$ and $y$.
b) Prove that the stationary distribution of $P$ is unique. (You do not need to prove that $P$ has a stationary distribution.)
(Hint: One way to do this is by proving that the rank of $P-I$ is $n-1$.)

Exercise 4.3  Let $P$ be a Markov chain obtained from an undirected, non-bipartite, $d$-regular (all vertices are of the same degree $d$) and connected graph. (i.e. $P$ picks a neighbor uniformly at random from the $d$ neighbors of $v$)
a) Prove that $P$ is irreducible and aperiodic.
b) Prove that for any probability distribution $x^0$, $||x^0P^t - \pi||_2 \leq |\lambda_2|^t$, where $\pi$ is the stationary distribution of $P$ and $\lambda_2$ is the second largest eigenvalue of $P$ in absolute value. ($|| \cdot ||_2$ is the Euclidean $L_2$ norm).
c) Prove that the mixing time of $P$ is at most $[\log(4\sqrt{n})/ \log(1/|\lambda_2|)]$.

Exercise 4.4  In the Traveling Salesman Problem (TSP) we are given a set $\{1, \ldots, n\}$ of $n$ cities and the distances $d(i, j)$ between any pair $i, j$ of cities. Our goal is to find a permutation $\pi_1, \ldots, \pi_n$ of the cities that minimized $\sum_{i=1}^n d(\pi_i, \pi_{i+1})$ (where we define $\pi_{n+1} = \pi_1$). A popular local search algorithm for TSP, called 2OPT, defines two permutations $\pi^1$ and $\pi^2$ as neighbors if $\pi^2$ can be obtained from $\pi^1$ by reversing an interval. I.e. if there exist two indices $k$ and $\ell$, $1 \leq k < \ell \leq n$, such that $\pi^2_j = \pi^1_{k+\ell-j}$ for $k \leq j \leq \ell$ and $\pi^2_j = \pi^1_j$ for $j < k$ and $j > \ell$. Describe a simulated annealing algorithm for TSP which is based on a random walk on the graph which is defined by this local search scheme. Write down the transition matrix of the underlying chain at a fixed temperature $T$. Prove that this chain is irreducible.

Exercise 4.5  Let $S = \{s_1, s_2, s_3, s_4\}$ and let $f : S \rightarrow R$ be given by $f(s_1) = 1$, $f(s_2) = 2$, $f(s_3) = 0$, $f(s_4) = 2$. Suppose we want to find the minimum of $f(s_i)$ using simulated annealing.
a) Construct the Metropolis chain for the Boltzman distribution with respect to $f$ with parameter $T$ using an underlying chain which is a random walk on the cycle $(s_1, s_2), (s_2, s_3), (s_3, s_4), (s_4, s_1)$ (when at $s_i$ you choose each of your two neighbors with the same probability). Write the transition probabilities for this Metropolis chain.
b) Suppose we set the temperature at step $k$ to be $T_k$, what is the probability, $P_n$, that if we start at state $s_1$, we never leave $s_1$ during the first $n$ steps.

c) Suppose that $T_k = 1/(2 \ln(k + 1))$ for $k = 1, 2, \ldots$, what is $\lim_{n \to \infty} P_n$? is it good or bad?