Construction of wavelet analysis in the space of discrete splines using Zak transform

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Abstract

We consider equidistant discrete splines $S(j)$, $j \in \mathbb{Z}$, which may grow as $O(|j|^s)$ as $|j| \to \infty$. Such splines present a relevant tool for digital signal processing. The Zak transforms of B-splines yield the integral representation of discrete splines. We define the wavelet space as a weak orthogonal complement of the coarse-grid space in the fine-grid space. We establish the integral representation of the elements of the wavelet space. We define and characterize the wavelets whose shifts form bases of the wavelet space. By this means we design a wide library of bases for the space of discrete-time signals of power growth construct multiscale representation of this space. We provide formulas for processing such the signals by discrete spline wavelets. Constructed bases are at the same time the Riesz bases for the space l_2.

Keywords: Discrete spline, Zak transform, wavelets, signal processing.

1 Introduction

In this paper we continue our study of discrete cardinal splines of power growth that was started in [18]. We construct the wavelet analysis in the space of such splines and suggest some algorithms for signal transforms by discrete spline wavelets. We stress that discrete splines and spline wavelets are defined on the set $\mathbb{Z}$ of integers and, as such, yield a natural tool for digital signal processing. In this area these splines and wavelets offer obvious advantages over the splines and wavelets of the continuous argument. The term *cardinal spline* means a spline with equidistant nodes kn where $k \in \mathbb{Z}$, n is a fixed natural number. Cardinal splines of continuous argument of power growth were investigated in [21], [20] and [27]. In [27] the author presented an integral representation of such splines which allowed construction of wavelet analysis in the space of continuous splines of power growth [28]. In the present paper we develop a similar theory in the space of discrete splines of power growth using as a tool the Zak transforms ([25, 5]) of the discrete B–splines.

Discrete splines first appeared in early seventies ([22]), but recently became the subject of extensive investigation ([7] , [8, Chapter 6], [15], [16], [17], [18], [4]). We also mention a related work [19] which deals with wavelets of discrete argument. In [1] the authors operate with "quasi-discrete" splines which are defined through sampling of the splines of the continuous argument. Using these splines the authors build the multiresolution analysis of the space l_2.

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Much of the investigation of the discrete splines was concerned with the periodic case. In [18] we began developing the theory of non-periodic discrete splines of power growth. In the present paper we extend this theory.

We use the Zak transforms of the B-splines, which we call the Zak splines, as a base for the integral representation of discrete splines. This integral representation is similar to the Fourier integral, and the Zak splines play the part of the Fourier exponentials. Further we introduce the so-called TB-splines whose shifts form bases of the spline space. The complete characterization of the TB-splines, which we establish, enables construction of a variety of such splines. Moreover, we derive a Parseval type identity which leads us to the construction of biorthogonal (and, in particular, orthogonal) bases of the spline spaces. All these results are presented in Section 3, whereas Section 2 contains some preliminary information on the discrete B-splines.

The main results of the paper are contained in Sections 4, 5, 6. In Section 4 we proceed to the construction of the wavelet analysis. First we establish the refinement equations that link the B-splines and their Zak transforms that are defined on the coarse grid with those defined on the fine grid. The space of the discrete splines defined on the coarse grid $\mathbf{V}_{p,2n}$ is a subspace of the space $\mathbf{V}_{p,n}$ of the splines that are defined on the fine grid. We define the wavelet space $\mathbf{W}_{p,2n}$ as a s weak orthogonal complement of $\mathbf{V}_{p,2n}$ in the space $\mathbf{V}_{p,n}$. Further development in the wavelet space is similar to the constructions in the spline space. The refinement equations for the Zak splines lead us to the definition of the Zak wavelets which serves as a base for the integral representation of the elements of the wavelet space $\mathbf{W}_{p,2n}$.

In Section 5 we define and characterize the TB-wavelets whose shifts form bases of the space $\mathbf{W}_{p,2n}$ and their duals. As examples of such wavelets we present the compactly supported B-wavelets that are related to the continuous B-wavelets constructed independently by Unser-Aldroubi-Eden [24], [23] and by Chui and Wang [9]. Also we present the wavelets that are dual to the B-wavelets, and the TB-wavelets whose shifts form orthonormal bases of the space $\mathbf{W}_{p,2n}$. The latter are related to the well-known Battle-Lemarié wavelets of the continuous argument [6], [14].

In Section 6 we establish the multiscale representation of the space of growing signals and the set of bases for this space. Then we provide formulas for the signal transforms by discrete spline wavelets.

2 Preliminaries

In this section we give the definitions and a brief outlook of the properties of the discrete B-splines, and introduce the characteristic cosine polynomials that are of cardinal importance in the sequel. The subjects are presented comprehensively in [18].

2.1 B-splines

Let n be a natural number. A discrete B-spline of the first order is the following sequence:

$$B_{1,n}(j) = \begin{cases} 1 & \text{if } j \in 0 : n-1, \\ 0, & \text{otherwise}, \quad j \in \mathbb{Z}. \end{cases}$$

Here and further on $l : m$ means the set of integers $\{l, l+1, \ldots, m\}$.

We define the higher order B-splines as the discrete convolutions by recurrence:

$$B_{r,n} = B_{1,n} * B_{r-1,n}, \; r = 2, \ldots, p, \tag{1}$$

This definition is the same as the definition of the discrete B-splines given in [12]. In that work the authors also established that the staircase interpolation of the discrete B-splines converge to the continuous ones in the L_2 norm as $n \to \infty$. Discrete B-spline $B_{p,n}$ of order p is a piecewise polynomial of degree $p-1$. The breakpoints $\{kn\},\ k \in \mathbb{Z}$, are called the nodes of the B-spline. The B-spline $B_{p,n}$ of order p is the piecewise polynomial of degree $p-1$ [18]:

$$B_{p,n}(j) = \frac{1}{(p-1)!} \sum_{r=0}^{p} (-1)^r \binom{p}{r} (j+1-rn)_+^{(p-1)},$$

where $k_+^{(l)}$ denotes the truncated factorial polynomial:

$$k_+^{(l)} = \begin{cases} k(k+1)\ldots(k+l-1) & \text{if } k \in 0:\infty \\ 0, & k<0, \quad k \in \mathbb{Z}. \end{cases}$$

The following properties of the B-splines $B_{p,n}$ hold:

1. $B_{p,n}(p(n-1)-j) = B_{p,n}(j)$ for all integers j;
2. $B_{p,n}(j) > 0$ if $j \in 0:p(n-1)$, $\quad B_{p,n}(j) = 0$ otherwise.
3. $B_{p,n}(0) = B_{p,n}(p(n-1) = 1;$
4. The sequence $B_{p,n}(j)$ increases strictly monotonously as $0 \le j \le p(n-1)/2$ and decays as $p(n-1)/2 \le j \le p(n-1)$;
5. $$\sum_{j\in\mathbb{Z}} B_{p,n}(j-kn)B_{p,n}(j-qn) = B_{2p}(p(n-1)+(k-q)n), \quad k,q \in \mathbb{Z}. \tag{2}$$

The last assertion follows from Eq. (1).
Remark. We emphasize that the B-splines assume only integer non-negative values and their supports are compact (Property 2). It is worth noting that the discrete B-spline $B_{p,n}(j)$ is not a trace of a continuous B-spline .

2.2 Characteristic cosine polynomial

Now we introduce cosine polynomials that will be used repeatedly in the sequel. Suppose that the number $\nu = p(n-1)/2$ is integer. Let $b_{p,n}(k) = B_{p,n}(\nu + kn)$. Recall that $b_{p,n}(-k) = b_{p,n}(k)$ and $b_{p,n}(k)$ is nonzero only if $|k| \le \mu = [\nu/n] = [p(n-1)/2n]$. Here $[\alpha]$ means the integer part of the number α.

Definition 2.1 *We call the cosine polynomial*

$$T_{p,n}(x) \triangleq \sum_{k=-\mu}^{\mu} b_{p,n}(k)e^{ikx} = b_{p,n}(0) + 2\sum_{k=1}^{\mu} b_{p,n}(k)\cos kx \tag{3}$$

the characteristic cosine polynomial (CCP) of the B-spline $B_{p,n}$.

Using the convergence property of B-splines mentioned above, it can be proved that the CCP converges to the Euler-Frobenius polynomial ([20]) as $n \to \infty$. It is apparent that $T_{p,n}(x)$ is an even $2\pi-$periodic infinitely differentiable function. The basic property of the CCP is the following:

Theorem 2.1 ([18]) *The cosine polynomial $T_{p,n}(x)$ is strictly positive for all x.*

3 Discrete splines

3.1 Definition of the discrete spline

We say that a sequence $\{c(l)\}$ is of power growth as $|l| \to \infty$, if there exist positive constants M, s such that

$$|c(l)| \le M(1 + |l|^s) \quad \forall l \in \mathbb{Z}. \tag{4}$$

Definition 3.1 *Any linear combination of shifts of the discrete B-spline $B_{p,n}(j)$:*

$$S_{p,n}(j) \triangleq \sum_{l=-\infty}^{\infty} c(l)\, B_{p,n}(j - ln) \quad (j \in \mathbb{Z}), \tag{5}$$

we call the discrete spline of the order p.

Definition 3.2 *We denote by $\mathbf{V}_{p,n}$ the space of discrete splines $S_{p,n}$ such that the sequences $\{c(k)\}_{-\infty}^{\infty}$ in the representation (5) satisfy (4).*

If $S_{p,n} \in \mathbf{V}_{p,n}$ then representation (5) of the discrete spline is unique. This follows, for example, from the uniqueness of the solution of the cardinal interpolation problem [18]. Hence the shifts of B-spline are linearly independent in the space $\mathbf{V}_{p,n}$.

The B-spline is compactly supported. Therefore once j is fixed, the series in (5) actually comprises only p non-zero entries. To be specific, if $j \in kn : (k+1)n - 1$ then

$$S_{p,n}(j) = \sum_{l=k-p+1}^{k} c(l) B_{p,n}(j - ln). \tag{6}$$

Note that $S_{p,n}$ coincides with a discrete polynomial of degree $p-1$ as $j \in kn : (k+1)n$ ([18]).

3.2 Zak splines.

In this section we establish an integral representation of splines from $\mathbf{V}_{p,n}$ which is related to the Fourier integral. As a substitute for the Fourier exponentials we use the Zak transforms ([25, 5]) of the discrete B-spline:

$$E_{p,n}(x, j) \triangleq \sum_{l=-\infty}^{\infty} e^{-ilx} B_{p,n}(j - ln). \tag{7}$$

The functions $E_{p,n}(x, j)$ for any fixed x belong to the spline space $\mathbf{V}_{p,n}$. Therefore we call them the Zak splines. Since, for any fixed $j \in \mathbb{Z}$, the series (7) comprises only a finite number of entries, the function $E_{p,n}(x, j)$ is a trigonometric polynomial with respect to x. The similar splines of continuous argument appeared in [20, p.17] under the name the exponential splines and were extensively employed in [28], [27]. We mark the quasi-periodicity of the Zak splines with respect to the "time" variable j

$$E_{p,n}(x, j - kn) = e^{ikx} E_{p,n}(x, j) \tag{8}$$

and the relation between the Zak splines and CCP:

$$T_{p,n}(x) = \sum_{k \in \mathbb{Z}} e^{ikx}\, B_p(p(n-1)/2 + kn) = E_{p,n}(x, p(n-1)/2). \tag{9}$$

The values $B_{p,n}(j+ln)$ are the Fourier coefficients of the function $E_{p,n}(x,j)$ and therefore the following integral representation holds:

$$B_{p,n}(j-ln) = \frac{1}{2\pi}\int_0^{2\pi} e^{ilx} E_{p,n}(x,j)\,dx, \quad j,l\in\mathbb{Z}. \tag{10}$$

In particular, as $l=0$, we have

$$B_{p,n}(j) = \frac{1}{2\pi}\int_0^{2\pi} E_{p,n}(x,j)\,dx, \quad j\in\mathbb{Z}. \tag{11}$$

Using (10) we can represent any spline in the integral form. Namely, for $j\in kn:(k+1)n-1$ we may write

$$\begin{aligned} S_{p,n}(j) &= \sum_{l=k-p+1}^{k} c(l)B_{p,n}(j-ln) = \frac{1}{2\pi}\sum_{l=k-p+1}^{k} c(l)\int_0^{2\pi} e^{ilx}E_{p,n}(x,j)\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} E_{p,n}(x,j)\sum_{l=k-p+1}^{k} c(l)e^{ilx}\,dx. \end{aligned} \tag{12}$$

Eq. (12) could be presented in a more compact shape if the periodic distributions were employed. Before we do that we recall some facts about the subject.

Periodic distributions. Let $\{\eta(k)\}_{-\infty}^{\infty}$ be a sequence of power growth and define a 2π-periodic distribution ([26]) as the sum:

$$H(x) \triangleq \sum_{l\in\mathbb{Z}} \eta(l)e^{ilx}. \tag{13}$$

The distribution creates a functional over the space $\mathcal{D}$ of infinitely differentiable 2π-periodic functions in the sense that if a function $g\in\mathcal{D}$ and $\{\gamma(k)\}_{-\infty}^{\infty}$ is the sequence of its coefficients (which decreases faster than any power of $1/l$ as $|l|$ tends to infinity) then

$$\langle H,g\rangle \triangleq \sum_{l\in\mathbb{Z}} \eta(l)\overline{\gamma(l)}.$$

We will denote this functional as the integral with the central dot:

$$\frac{1}{2\pi}\int_0^{2\pi} H(x)\cdot\overline{g(x)}\,dx \triangleq \langle H,g\rangle = \sum_{l\in\mathbb{Z}} \eta(l)\overline{\gamma(l)}. \tag{14}$$

Of course, if H is an integrable periodic function the functional turns into a conventional integral. We denote the space of such distributions as $\mathcal{D}'$.

In the case of the integrable 2π-periodic function H the following identity is true:

$$\frac{1}{2\pi}\int_0^{2\pi} H(x)\overline{g(2x)}\,dx = \frac{1}{2\pi}\int_0^{2\pi}\frac{H(x/2)+H(x/2+\pi)}{2}\overline{g(x)}\,dx = \sum_{l\in\mathbb{Z}}\eta(2l)\overline{\gamma(l)}.$$

Therefore, if the distribution H is given as in (13) we define the distribution

$$\frac{H(x/2)+H(x/2+\pi)}{2} \triangleq \sum_{l\in\mathbb{Z}} \eta(2l)e^{ilx} \tag{15}$$

and, by definition,

$$\frac{1}{2\pi}\int_0^{2\pi} H(x)\cdot\overline{g(2x)}\,dx \triangleq \frac{1}{2\pi}\int_0^{2\pi}\frac{H(x/2)+H(x/2+\pi)}{2}\cdot\overline{g(x)}\,dx = \sum_{l\in\mathbb{Z}}\eta(2l)\overline{\gamma(l)}. \tag{16}$$

If $H \in \mathcal{D}'$ and the function

$$f \triangleq \sum_{l\in\mathbb{Z}}\nu(l)e^{ilx} \in \mathcal{D}$$

then the product $Hf \in \mathcal{D}'$ and

$$Hf(x) = \sum_{l\in\mathbb{Z}} e^{ilx}\sum_{k\in\mathbb{Z}}\nu(l-k)\eta(k). \tag{17}$$

Compact form of the integral representation We return to (5) and define the 2π-periodic distribution

$$C(x) \triangleq \sum_{l\in\mathbb{Z}} c(l)e^{ilx}, \quad C \in \mathcal{D}'.$$

We can now rewrite (12) in a global form

$$S_{p,n}(\cdot) = \frac{1}{2\pi}\int_0^{2\pi} C(x)\cdot E_{p,n}(x,\cdot)\,dx. \tag{18}$$

Of course, if the sequence $\{c_l\} \in l^1$ then $C(x)$ is a periodic integrable function and the integral in (18) should be understood in the conventional sense.

Conversely, it is readily seen that if $Q \in \mathcal{D}'$ then the function

$$\sigma(\cdot) \triangleq \frac{1}{2\pi}\int_0^{2\pi} Q(x)\cdot E_{p,n}(x,\cdot)\,dx$$

is a discrete spline of order p:

$$\sigma(j) = \sum_{l=-\infty}^{\infty} q(l)\,B_{p,n}(j-ln).$$

The coefficients $q(l)$ are the Fourier coefficients of the distribution $Q(x)$.

3.3 TB-splines and their duals

Let ξ be an infinitely differentiable 2π-periodic function and ξ_l be its Fourier coefficients. Let us introduce the function through the integral

$$\varphi(j) = \frac{1}{2\pi}\int_0^{2\pi}\xi(x)E_{p,n}(x,j)dx, \quad j \in \mathbb{Z}. \tag{19}$$

It is clear from previous considerations that φ may be written as the sum

$$\varphi(j) = \sum_{l=-\infty}^{\infty}\xi_l B_{p,n}(j-ln), \tag{20}$$

and, consequently, is a spline of order p. Recall that ξ_l are the Fourier coefficients of the function ξ.

Definition 3.3 *We will say that the spline φ is of type B (TB-spline) if its shifts $\{\varphi(\cdot - kn)\}_{k\in\mathbb{Z}}$ form a basis in the spline space $\mathbf{V}_{p,n}$, in the sense that any spline $S \in \mathbf{V}_{p,n}$ can be uniquely expanded into the series*

$$S(j) = \sum_{k=-\infty}^{\infty} c(k)\varphi(j - kn),$$

which converges pointwise with any $j \in \mathbb{Z}$.

We stress that the uniqueness of the expansion implies linear independence of the system $\{\varphi(\cdot - kn)\}$ in the spline space $\mathbf{V}_{p,n}$.

The following assertion characterizes TB-splines.

Theorem 3.1 *The spline φ is a TB-spline if and only if $\xi(x) \neq 0 \quad \forall x \in \mathbb{R}$. Moreover, if a spline $S \in \mathbf{V}_{p,n}$ is presented in two ways*

$$\frac{1}{2\pi}\int_0^{2\pi} C(x) \cdot E_{p,n}(x, j)\, dx = S(j) = \sum_{k\in\mathbb{Z}} c(k)\varphi(j - kn)$$

then

$$C(x) = \sum_{k\in\mathbb{Z}} c(k)\xi(x)e^{ikx}, \;\; c(k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{C(x)}{\xi(x)} \cdot e^{-ikx}\, dx. \tag{21}$$

Proof: Due to property (8) we have

$$\varphi(j - kn) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\xi(x)E_{p,n}(x, j)dx, \tag{22}$$

which means that $\varphi(j + kn)$ is the Fourier coefficient of the function $\xi(x)E_{p,n}(x, \cdot) \in \mathcal{D}$. Therefore

$$\xi(x)E_{p,n}(x, j) = \sum_{k=-\infty}^{\infty} e^{-ikx}\varphi(j - kn), \quad x \in \mathbb{R}. \tag{23}$$

1. Let $\xi(x) \neq 0 \quad \forall x \in \mathbb{R}$. For any spline S and $j \in \mathbb{Z}$ we have

$$\begin{aligned} S_{p,n}(j) &= \frac{1}{2\pi}\int_0^{2\pi} C(x) \cdot E_{p,n}(x, j)\, dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{C(x)}{\xi(x)} \cdot \xi(x)\, E_{p,n}(x, j)\, dx = \sum_{k=-\infty}^{\infty} c(k)\, \varphi(j - kn), \end{aligned}$$

due to (14). Here $c(k)$ are the Fourier coefficients of the distribution $C(x)/\xi(x) \in \mathcal{D}'$. Hence, any spline is represented uniquely through shifts of the spline φ and, therefore, φ is a TB-spline.

2.Conversely, let the spline φ, which is given by (19), be a TB-spline. Then for any $x \in \mathbb{R}$ the Zak spline

$$E_{p,n}(x, j) = \sum_{k=-\infty}^{\infty} \mu_k(x)\varphi(j - kn), \quad j \in \mathbb{Z},$$

with some coefficients $\mu_k(x)$. Invoking (23), we obtain:

$$\sum_{k=-\infty}^{\infty} \xi(x)\mu_k(x)\varphi(j-kn) = \xi(x)E_{p,n}(x,j) = \sum_{k=-\infty}^{\infty} e^{-ikx}\varphi(j-kn), \quad j \in \mathbb{Z}. \tag{24}$$

Since the shifts $\{\varphi(\cdot - kn)\}_{k\in\mathbb{Z}}$ of the TB-spline φ are linearly independent in the spline space, (24) implies that $\xi(x)\mu_k(x) = e^{-ikx}$. Hence it follows that $\xi(x) \neq 0 \quad \forall x \in \mathbb{R}$. ■

Remark 3.3. Eq. (22) implies that for any j, $\varphi(j-kn)$ decreases faster than any power of $1/k$ as $k \to \infty$. Hence it can be derived that the spline $\varphi(j)$ decreases faster than any power of $1/j$ as $j \to \infty$. It is clear that the shifts $\varphi(\cdot - kn)$ of a TB-spline form a Riesz basis for the space $\mathbf{V}_{p,n} \bigcap l_2$. Therefore Theorem 3.1 characterises also Riesz bases for the space $\mathbf{V}_{p,n} \bigcap l_2$ (cf. Theorem 1 in [2]).

We now introduce a TB-spline whose shifts form a basis biorthogonal to the basis $\{\varphi(\cdot - kn)\}$. Let

$$\chi(j) \triangleq \frac{1}{2\pi}\int_0^{2\pi} \eta(x)E_{p,n}(x,j)dx, \quad \eta \in \mathcal{D},\ \eta(x) \neq 0\, \forall x, \tag{25}$$

be a TB-spline as well as φ.

Definition 3.4 *The TB-splines φ and χ are said to be dual to each other if the following relation holds:*

$$\sum_{j=-\infty}^{\infty} \varphi(j-kn)\overline{\chi(j-qn)} = \delta(k-q), \quad k,q \in \mathbb{Z}, \tag{26}$$

where $\delta(k)$ denotes the Kroneker delta.

To find the conditions of duality, we first establish a relation similar to the Parseval identity.

Theorem 3.2 *For the TB-splines given by (19) and (25) the following relation is true:*

$$\sum_{j=-\infty}^{\infty} \varphi(j)\overline{\chi(j)} = \frac{1}{2\pi}\int_0^{2\pi} \xi(x)\overline{\eta(x)}T_{2p,n}(x)dx. \tag{27}$$

Proof: From Eq. (25) we derive

$$\sum_{j=-\infty}^{\infty} \varphi(j)\overline{\chi(j)} = \frac{1}{2\pi}\int_0^{2\pi} \overline{\eta(x)}\,\sigma(x)\,dx \text{ where } \sigma(x) \triangleq \sum_{j=-\infty}^{\infty} \overline{E_{p,n}(x,j)}\varphi(j). \tag{28}$$

We transform the product using (20)

$$\overline{E_{p,n}(x,j)}\varphi(j) = \sum_{l=-\infty}^{\infty} \xi_l \overline{E_{p,n}(x,j)}B_{p,n}(j-ln) = \sum_{l=-\infty}^{\infty} \xi_l \sum_{k=l-p}^{l+p} e^{ikx}B_{p,n}(j-kn)B_{p,n}(j-ln).$$

But Property (2) of the B-splines implies that

$$\sum_{j=-\infty}^{\infty} B_{p,n}(j-kn)\,B_{p,n}(j-ln) = b_{2p}(k-l)$$

and, consequently, we may write

$$\sigma(x) = \sum_{l=-\infty}^{\infty} \xi_l \sum_{k=l-p}^{l+p} e^{ikx} b_{2p}(k-l) = \sum_{l=-\infty}^{\infty} \xi_l e^{ilx} T_{2p,n}(x) = T_{2p,n}(x)\xi(x).$$

Substituting the relation into (28), we arrive at (27). ■

Now we are able to characterize dual pairs of TB-splines.

Theorem 3.3 *The TB-splines that are given by (19) and (25) are dual to each other if and only if*

$$\xi(x)\overline{\eta(x)}T_{2p,n}(x) = 1 \ \forall x \in \mathbb{R}. \tag{29}$$

Proof: It follows from Eq. (22) and Theorem 3.2 that

$$\sum_{j=-\infty}^{\infty} \varphi(j-kn)\overline{\chi(j-qn)} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-q)x}\xi(x)\overline{\eta(x)}T_{2p,n}(x)dx. \tag{30}$$

If (29) is true then (30) implies (26). Conversely, if (26) holds then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\xi(x)\overline{\eta(x)}T_{2p,n}(x)dx = \delta(k),$$

and (29) follows.■

The theorem allows us to create dual pairs of TB-splines.

Examples:

1. Self-dual TB-splines. Let us choose $\xi(x) = \eta(x) = 1/\sqrt{T_{2p,n}(x)}$. Then

$$\varphi(j) = \chi(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{E_{p,n}(x,j)}{\sqrt{T_{2p,n}(x)}}dx.$$

   We recall that CCP $T_{2p,n}(x)$ is strictly positive for all x. The splines obtained are related to the continuous splines by Battle [6] and Lemarié [14].

2. TB-splines dual to B-splines. Accordingly to (11), when $\xi(x) \equiv 1$ we have $\varphi(j) = B_{p,n}(j)$. The dual spline χ is constructed immediately as follows:

$$\chi(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{E_{p,n}(x,j)}{T_{2p,n}(x)}dx.$$

4 Basics of the spline-wavelet analysis

The integral representation of splines presented above allows construction of wavelet bases in the spline space $\mathbf{V}_{p,n}$ and, by this means, of the spline-wavelet analysis of the discrete functions of power growth.

4.1 Refinement equations

We start with the so-called refinement equations which are fundamental to any wavelet construction as well as to subdivision schemes [11]. These equations link basic splines of the spaces $\mathbf{V}_{p,n}$ and $\mathbf{V}_{p,2n}$.

First we establish the relation between B-splines.

Theorem 4.1 *The following relation holds:*

$$B_{p,2n}(j) = \sum_{r=0}^{p} \binom{p}{r} B_{p,n}(j - rn), \;\; j \in \mathbb{Z}. \tag{31}$$

Proof: We derive the desired relation from the z-transform of $B_{p,n}(j)$:

$$\zeta[B_{p,n}] \triangleq \sum_{j=0}^{p(n-1)} B_{p,n}(j) z^j.$$

It is apparent for $p = 1$ that $\zeta[B_{1,n}] = \sum_0^{n-1} z^k$. Hence it follows that $\zeta[B_{p,n}] = \left(\sum_0^{n-1} z^k\right)^p$ and

$$\zeta[B_{p,n}(\cdot - rn)] = z^{rn}(1 + z + \ldots + z^{n-1})^p.$$

For $B_{p,2n}$ we have:

$$\begin{aligned} \zeta[B_{p,2n}] &= (1 + z + \ldots + z^{2n-1})^p = \frac{(1 - z^n)^p (1 + z^n)^p}{(1 - z)^p} \\ &= \sum_{r=0}^{p} \binom{p}{r} z^{rn} (1 + z + \ldots z^{n-1})^p = \sum_{r=0}^{p} \binom{p}{r} \zeta[B_{p,n}(\cdot - rn)]. \end{aligned}$$

The latter equation implies (31). ■

The following theorem which is fundamental to the sequel, relates the Zak splines $E_{p,n}(x, j)$ to $E_{p,2n}(x, j)$.

Theorem 4.2 *The following identity is true*

$$E_{p,2n}(x, j) = c\left(\frac{x}{2}\right) E_{p,n}\left(\frac{x}{2}, j\right) + c\left(\frac{x}{2} + \pi\right) E_{p,n}\left(\frac{x}{2} + \pi, j\right), \quad c(x) \triangleq \frac{1}{2}(1 + e^{ix})^p. \tag{32}$$

Proof: Using (31) we may write

$$c\left(\frac{x}{2}\right) E_{p,n}\left(\frac{x}{2}, j\right) = \frac{1}{2} \sum_{r=0}^{p} \binom{p}{r} e^{irx/2} \sum_{k \in \mathbb{Z}} e^{-ikx/2} B_{p,n}(j - kn)$$

$$= \frac{1}{2} \sum_{r=0}^{p} \binom{p}{r} \sum_{k \in \mathbb{Z}} e^{-i(k-r)x/2} B_{p,n}(j - kn)$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z}} e^{-ikx/2} \sum_{r=0}^{p} \binom{p}{r} B_{p,n}(j - (k + r)n) = \frac{1}{2} \sum_{k \in \mathbb{Z}} e^{-ikx/2} B_{p,2n}(j - kn), \tag{33}$$

$$c\left(\frac{x}{2} + \pi\right) E_{p,n}\left(\frac{x + 2\pi}{2}, j\right) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (-1)^k e^{-ikx/2} B_{p,2n}(j - kn). \tag{34}$$

Combining (33) and (34) we get

$$c\left(\frac{x}{2}\right)E_{p,n}\left(\frac{x}{2},j\right)+c\left(\frac{x}{2}+\pi\right)E_{p,n}\left(\frac{x+2\pi}{2},j\right)=\sum_{k\in\mathbb{Z}}e^{-ikx}B_{p,2n}(j-2kn)=E_{p,2n}(x,j).$$

■

We can derive from the theorem an important identity for CCP.

Corollary 4.1 *The following identity for CCP is true*

$$T_{2p,2n}(x)=2\Big|c\left(\frac{x}{2}\right)\Big|^2T_{2p,n}\left(\frac{x}{2}\right)+2\Big|c\left(\frac{x}{2}+\pi\right)\Big|^2T_{2p,n}\left(\frac{x}{2}+\pi\right). \tag{35}$$

Proof: Invoking (9), we derive from Theorem 4.2 that

$$\begin{aligned}T_{2p,2n}(x) &= E_{2p,2n}(x,p(2n-1))\\ &= c_{2p}\left(\frac{x}{2}\right)E_{2p,n}\left(\frac{x}{2},p(2n-1)\right)+c_{2p}\left(\frac{x}{2}+\pi\right)E_{2p,n}\left(\frac{x}{2}+\pi,p(2n-1)\right).\end{aligned}$$

Property (8) of the Zak splines implies that

$$E_{2p,n}\left(\frac{x}{2},p(n-1)+pn\right)=e^{-ipx/2}E_{2p,n}\left(\frac{x}{2},p(n-1)\right)=e^{-ipx/2}T_{2p,n}\left(\frac{x}{2}\right).$$

Hence it follows that

$$\begin{aligned}T_{2p,2n}(x) &= c_{2p}\left(\frac{x}{2}\right)e^{-ipx/2}T_{2p,n}\left(\frac{x}{2}\right)+c_{2p}\left(\frac{x}{2}+\pi\right)e^{-ip(x/2+\pi)}T_{2p,n}\left(\frac{x}{2}+\pi\right)\\ &= \frac{1}{2}\left(2\cos\frac{x}{4}\right)^{2p}T_{2p,n}\left(\frac{x}{2}\right)+\frac{1}{2}\left(2\sin\frac{x}{4}\right)^{2p}T_{2p,n}\left(\frac{x}{2}+\pi\right).\end{aligned}$$

The latter equation is equivalent to (63). ■

4.2 Definition of wavelet spaces

Now we should define wavelet spaces. Since we cannot use the conventional definition of such spaces as orthogonal complements of coarse-grid spaces in fine-grid ones, we introduce wavelet spaces by proceeding as follows.

Since the B-splines $B_{p,2n}(j)$ could be expressed through $B_{p,n}(j-rn)$, the space $\mathbf{V}_{p,2n}$ is a subspace of the spline space $\mathbf{V}_{p,n}$. We define the wavelet space $\mathbf{W}_{p,2n}$ as a weak orthogonal complement of $\mathbf{V}_{p,2n}$ in the space $\mathbf{V}_{p,n}$.

Definition 4.1 *The space $\mathbf{W}_{p,2n}$ of all splines $R\in\mathbf{V}_{p,n}$ subject to the conditions*

$$\sum_{j\in\mathbb{Z}}R(j)B_{p,2n}(j-2kn)=0\quad\forall k\in\mathbb{Z} \tag{36}$$

is called a wavelet space.

This definition does not depend on this special choice of basis of the space $\mathbf{V}_{p,2n}$.

Proposition 4.1 *Let $R(j)$ belongs to the wavelet space $\mathbf{W}_{p,2n}$. Then for any TB-spline $\varphi \in \mathbf{V}_{p,2n}$ the following relations are true:*

$$\sum_{j\in\mathbb{Z}} R(j)\varphi(j-2kn) = 0 \quad \forall k \in \mathbb{Z} \tag{37}$$

Proof: Since the TB-spline $\varphi(j) \in \mathbf{V}_{p,2n}$ decays exponentially as $j \to \infty$, the series in (37) converges uniformly while k belongs to any finite interval. Therefore we can substitute into (37) the expansion of φ through B-splines: $\varphi(j) = \sum_{l=l_1}^{l_2} q(l)B_{p,2n}(j-2ln)$. Hence we obtain

$$\sum_{j\in\mathbb{Z}} R(j)\varphi(j-2kn) = \sum_{l=l_1}^{l_2} q(l) \sum_{j\in\mathbb{Z}} R(j)B_{p,2n}(j-2(l+k)n) = 0 \quad \forall k \in \mathbb{Z}.$$

■

Our immediate purpose is to build basis elements of the wavelet space $\mathbf{W}_{p,2n}$ .similar to the Zak splines and to derive a representation of type (18) in the wavelet space.

4.3 Zak wavelets

Let us write a spline with a shape similar to (32)

$$W_{p,2n}(x,j) = a\left(\frac{x}{2}\right) E_{p,n}\left(\frac{x}{2},j\right) + a\left(\frac{x+2\pi}{2},j\right) E_{p,n}\left(\frac{x+2\pi}{2},j\right), \tag{38}$$

and use the yet undefined function $a(x)$ to place the spline $W_{p,2n}(x,\cdot)$ into the space $\mathbf{W}_{p,2n}$ for all x, that is, to subject the spline to the conditions

$$\sum_{j\in\mathbb{Z}} W_{p,2n}(x,j)B_{p,2n}(j-2kn) = 0 \quad \forall k \in \mathbb{Z}. \tag{39}$$

Using (8) we may transform (39) into the relation

$$a\left(\frac{x}{2}\right) A(x) + a\left(\frac{x}{2}+\pi\right) A(x+2\pi) = 0 \quad \forall x \in \mathbb{R} \tag{40}$$

where

$$A(x) \triangleq \sum_{j\in\mathbb{Z}} E_{p,n}(x/2,j)B_{p,2n}(j)$$

is a 4π-periodic function. We could satisfy (40) by the choosing $a(x) \triangleq e^{ix}A(2x+2\pi)$ but it is left to express A in an explicit form. Keeping in mind (31), we write

$$\begin{aligned} E_{p,n}(x/2,j)B_{p,2n}(j) &= \sum_{l\in\mathbb{Z}} e^{-ilx/2}B_{p,n}(j-ln) \sum_{r=0}^{p} \binom{p}{r} B_{p,n}(j-rn) \\ &= \sum_{r=0}^{p} \binom{p}{r} \sum_{l\in\mathbb{Z}} e^{-ilx/2} B_{p,n}(j-ln)B_{p,n}(j-rn). \end{aligned}$$

Summing the expression over the variable $j \in \mathbb{Z}$ and using Property (2) of B-splines we get

$$\begin{aligned} A(x) &= \sum_{r=0}^{p} \binom{p}{r} \sum_{l\in\mathbb{Z}} e^{-ilx/2} b_{2p,n}(l-r) = \sum_{r=0}^{p} \binom{p}{r} e^{-irx/2} \sum_{s\in\mathbb{Z}} b_{2p,n}(s) e^{-isx/2} \\ &= (1+e^{-ix/2})^p T_{2p,n}(x/2) = 2\overline{c(x/2)} T_{2p,n}(x/2). \end{aligned}$$

The final expression for $a(x)$ is

$$a(x) = 2e^{ix}\overline{c(x+\pi)} T_{2p,n}(x+\pi) = e^{ix}(1-e^{-ix})^p \, T_{2p,n}(x+\pi). \tag{41}$$

Let us outline some properties of the spline $W_{p,2n}(x,\cdot)$ which resemble the properties of the Zak spline $E_{p,n}(x,\cdot)$:

1. For any x, the spline $W_{p,2n}(x,\cdot)$ belongs to the space $\mathbf{W}_{p,2n}$.
2. For all j, the function $W_{p,2n}(\cdot,j)$ belongs to $\mathcal{D}$.
3. $$W_{p,2n}(x, j-2kn) = e^{ikx} W_{p,2n}(x,j) \quad \forall x, j, k. \tag{42}$$

These properties justify the following definition.

Definition 4.2 *We call the spline $W_{p,2n}(x,\cdot)$ defined in (38) with $a(x)$ given by (41) the Zak wavelet.*

Similarly to the Zak spline, the Zak wavelet generates a representation of elements of the space $\mathbf{W}_{p,2n}$.

Proposition 4.2 *Let D be a distribution from $\mathcal{D}'$. Then the function*

$$R(j) \triangleq \frac{1}{2\pi} \int_0^{2\pi} D(x) \cdot W_{p,2n}(x,j)\, dx,$$

is a spline belonging to the wavelet space $\mathbf{W}_{p,2n}$.

Proof: We attain the result via the following chain of relations

$$\begin{aligned} \sum_{j\in\mathbb{Z}} R(j) B_{p,2n}(j-2kn) &= \sum_{j=2kn}^{2kn+p(2n-1)} \frac{1}{2\pi} \int_0^{2\pi} D(x) \cdot W_{p,2n}(x,j)\, dx \, B_{p,2n}(j-2kn) \\ &= \frac{1}{2\pi} \int_0^{2\pi} D(x) \cdot \left(\sum_{j=2kn}^{2kn+p(2n-1)} W_{p,2n}(x,j) B_{p,2n}(j-2kn) \right) dx = 0 \quad \forall k \in \mathbb{Z} \end{aligned}$$

due to (39). Hence $R \in \mathbf{W}_{p,2n}$. ■

Now we turn to the main result of the section. Let us define

$$\Omega(x) \triangleq 4T_{2p,n}(x/2) T_{2p,n}(x/2+\pi) T_{2p,2n}(x) \in \mathcal{D}. \tag{43}$$

Theorem 4.3 *Let a spline $S_n \in \mathbf{V}_{p,n}$ be presented as*

$$S_n(j) = \frac{1}{2\pi}\int_0^{2\pi} C_n(x) \cdot E_{p,n}(x,j)\,dx \text{ with } C_n \in \mathcal{D}'. \tag{44}$$

Then it can be decomposed uniquely into the sum

$$S_n(j) = S_{2n}(j) + R_{2n}(j), \tag{45}$$

where

$$S_{2n}(j) = \frac{1}{2\pi}\int_0^{2\pi} C_{2n}(x) \cdot E_{p,2n}(x,j)\,dx \in \mathbf{V}_{p,2n}, \tag{46}$$

$$C_{2n}(x) = \frac{C_n(\frac{x}{2})\overline{c(\frac{x}{2})}T_{2p,n}(\frac{x}{2}) + C_n(\frac{x}{2}+\pi)\overline{c(\frac{x}{2}+\pi)}T_{2p,n}(\frac{x}{2}+\pi)}{T_{2p,2n}(x)}, \tag{47}$$

$$R_{2n}(j) = \frac{1}{2\pi}\int_0^{2\pi} D_{2n}(x) \cdot W_{p,2n}(x,j)\,dx \in \mathbf{W}_{p,2n}, \tag{48}$$

$$D_{2n}(x) = \frac{C_n(\frac{x}{2})\overline{a(\frac{x}{2})}T_{2p,n}(\frac{x}{2}) + C_n(\frac{x}{2}+\pi)\overline{a(\frac{x}{2}+\pi)}T_{2p,n}(\frac{x}{2}+\pi)}{\Omega(x)}. \tag{49}$$

The function $a(x)$ is given in (41), and $c(x)$ – in (32). Distributions in (47) and (49) should be understood in the sense of (16).

Conversely, given $C_{2n}(x)$ and $D_{2n}(x)$, the distribution $C_n(x)$ can be found as follows:

$$C_n(x) = 2(c(x)C_{2n}(2x) + a(x)D_{2n}(2x)). \tag{50}$$

Preparatory to proving the theorem we present some facts concerning the B-spline.

Proposition 4.3 ([18]) *Let $b_{2p}(s) \triangleq B_{2p,n}(p(n-1)+sn)$. If the relation*

$$\sum_{l\in\mathbb{Z}} c(l)b_{2p}(k-l) = 0 \quad \forall k \in \mathbb{Z}$$

holds, then $c(l) = 0 \quad \forall l \in \mathbb{Z}$.

Lemma 4.1 *If a spline $S \triangleq \sum_{l\in\mathbb{Z}} c(l)B_{p,n}(j-ln) \in \mathbf{V}_{p,n}$ is subject to the conditions*

$$\sum_{j\in\mathbb{Z}} S(j)B_{p,n}(j-kn) = 0 \quad \forall k \in \mathbb{Z},$$

then $S \equiv 0$.

Proof: For any fixed $k \in \mathbb{Z}$ we have

$$\sum_{j\in\mathbb{Z}}\sum_{l=k-p}^{k+p} c(l)B_{p,n}(j-ln)B_{p,n}(j-kn) = 0$$

(we recall that $B_{p,n}(j-ln)B_{p,n}(j-kn) = 0 \quad \forall j \in \mathbb{Z}$ when $|l-k| > p$). Reversing the order of the summations and using (2) we arrive at

$$\sum_{l=k-p}^{k+p} c(l)\sum_{j\in\mathbb{Z}} B_{p,n}(j-ln)B_{p,n}(j-kn) = \sum_{l=k-p}^{k+p} c(l)b_{2p}(k-l) = 0 \quad \forall k \in \mathbb{Z}.$$

But Proposition 4.3 implies that $c(l) = 0 \quad \forall l \in \mathbb{Z}$. This results in $S \equiv 0$.■

Proof of Theorem: First, we express $E_{p,n}$ through $E_{p,2n}$ and $W_{p,2n}$. We find the expression from the identities

$$E_{p,2n}(2x,j) = c(x)E_{p,n}(x,j) + c(x+\pi)E_{p,n}(x+\pi,j),$$
$$W_{p,2n}(2x,j) = a(x)E_{p,n}(x,j) + a(x+\pi)E_{p,n}(x+\pi,j),$$

which are consequences of (32) and (38). Using (41) and (35), we calculate the determinant of this system:

$$\begin{aligned}\Delta(x) &= c(x)a(x+\pi) - a(x)c(x+\pi) \\ &= -e^{ix}[2|c(x)|^2 T_{2p,n}(x) + 2|c(x+\pi)|^2 T_{2p,n}(x+\pi)] = -e^{ix}T_{2p,2n}(2x) \neq 0\end{aligned}$$

for all x. Hence

$$E_{p,n}(x,j) = \frac{a(x+\pi)}{\Delta(x)}E_{p,2n}(2x,j) - \frac{c(x+\pi)}{\Delta(x)}W_{p,2n}(2x,j).$$

Let us substitute the latter identity into (44):

$$S_n(j) = \frac{1}{2\pi}\int_0^{2\pi} C_n(x)\cdot\frac{a(x+\pi)}{\Delta(x)}E_{p,2n}(2x,j)\,dx - \frac{1}{2\pi}\int_0^{2\pi} C_n(x)\cdot\frac{c(x+\pi)}{\Delta(x)}W_{p,2n}(2x,j)\,dx.$$

Consider the first addend of the sum. From (41) and (16) we have

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} C_n(x)\cdot\frac{a(x+\pi)}{\Delta(x)}E_{p,2n}(2x,j)\,dx = \frac{1}{2\pi}\int_0^{2\pi} C_n(x)\frac{2\overline{c(x)}T_{2p,n}(x)}{T_{2p,2n}(2x)}\cdot E_{p,2n}(2x,j)\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{C_n(\frac{x}{2})\overline{c(\frac{x}{2})}T_{2p,n}(\frac{x}{2}) + C_n(\frac{x}{2}+\pi)\overline{c(\frac{x}{2}+\pi)}T_{2p,n}(\frac{x}{2}+\pi)}{T_{2p,2n}(x)}\cdot E_{p,2n}(x,j)\,dx.\end{aligned}$$

The relation implies (46). To derive (48) we note that

$$-\frac{c(x+\pi)}{\Delta(x)} = \frac{\overline{a(x)}}{2T_{2p,n}(x+\pi))T_{2p,2n}(2x)} = \frac{2T_{2p,n}(x))\overline{a(x)}}{\Omega(2x)}.$$

are similarly.

To prove (50) we first consider the spline $R_{2n}(j)$ and invoke (38):

$$\begin{aligned}R_{2n}(j) &= \frac{1}{2\pi}\int_0^{2\pi} D_{2n}(x)\cdot W_{p,2n}(x,j)\,dx \in \mathbf{W}_{p,2n} \\ &= \frac{1}{2\pi}\int_0^{2\pi} D_{2n}(x)\cdot\left(a\left(\frac{x}{2}\right)E_{p,n}\left(\frac{x}{2},j\right) + a\left(\frac{x+2\pi}{2},j\right)E_{p,n}\left(\frac{x+2\pi}{2},j\right)\right)\,dx \\ &= 2\frac{1}{2\pi}\int_0^{2\pi} D_{2n}(2x)a(x)\cdot E_{p,n}(x,j)\,dx. \qquad (51)\end{aligned}$$

Similarly we obtain

$$S_{2n}(j) = 2\frac{1}{2\pi}\int_0^{2\pi} C_{2n}(2x)c(x)\cdot E_{p,n}(x,j)\,dx. \qquad (52)$$

The latter two equations imply (50).

Suppose now that there exists another decomposition of the spline $S = S'_{2n} + W'_{2n}$, with $S'_{2n} \in \mathbf{V}_{p,2n}$, $W'_{2n} \in \mathbf{W}_{p,2n}$. Then $(S_{2n} - S'_{2n}) + (W_{2n} - W'_{2n}) = 0$. Hence

$$\sum_{j\in\mathbb{Z}} [S_{2n}(j) - S'_{2n}(j)] B_{p,2n}(j - 2kn) = 0 \quad \forall k \in \mathbb{Z}.$$

Lemma 4.1 implies then that $S_{2n} - S'_{2n} \equiv 0$. Similarly $W_{2n} - W'_{2n} \equiv 0$.■

The theorem implies the integral representation of elements of the wavelet space $\mathbf{W}_{p,2n}$.

Corollary 4.2 *A spline $R \in \mathbf{V}_{p,n}$ belongs to $\mathbf{W}_{p,2n}$ if and only if it is representable as follows:*

$$R(j) = \frac{1}{2\pi}\int_0^{2\pi} D(x) \cdot W_{p,2n}(x, j)\, dx, \tag{53}$$

where $D \in \mathcal{D}'$.

5 TB-wavelets and their duals

5.1 Definition and characterization of TB-wavelets

Now we introduce basic elements of the wavelet space that are similar to TB-splines. The considerations here are very similar to those in Section 3.3 and we briefly outline the scheme.

Let a function τ be in the space $\mathcal{D}$, that is, τ is 2π-periodic and infinitely differentiable, and define the spline

$$\psi(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x) W_{p,2n}(x, j)\, dx, \tag{54}$$

which belongs to $\mathbf{W}_{p,2n}$ due to Proposition 4.2.

Definition 5.1 *We call the spline ψ a TB-wavelet if its shifts $\{\psi(\cdot - 2kn) | k \in \mathbb{Z}\}$ form a basis of the space $\mathbf{W}_{p,2n}$ in the sense that any $R \in \mathbf{W}_{p,2n}$ can be uniquely expanded into the series*

$$R(j) = \sum_{k\in\mathbb{Z}} d(k)\psi(j - 2kn),$$

which converges for any $j \in \mathbb{Z}$. The function τ we call the density of the TB-wavelet ψ.

Following is the Parseval identity for TB-wavelets.

Theorem 5.1 *Let ψ_1 and ψ_2 be the TB-wavelets generated by densities τ_1 and τ_2, respectively, as in (54). Then the following identity is true:*

$$\sum_{j\in\mathbb{Z}} \psi_1(j)\overline{\psi_2(j)} = \frac{1}{2\pi}\int_0^{2\pi} \tau_1(x)\overline{\tau_2(x)}\Omega(x)\, dx, \tag{55}$$

where $\Omega(x)$ is defined in (43)

Proof: Similarly to (57), we write

$$\psi_k(j) = \frac{1}{\pi}\int_0^{2\pi} \tau_k(2x)a(x)E_{p,n}(x,j)\,dx, \quad k = 1,2.$$

This allows application of the Parseval identity (27):

$$S \triangleq \sum_{j\in\mathbb{Z}} \psi_1(j)\overline{\psi_2(j)} = \frac{1}{2\pi}\int_0^{2\pi} 4\tau_1(2x)\overline{\tau_2(2x)}|a(x)|^2 T_{2p,n}(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \tau_1(x)\overline{\tau_2(x)}\{|a(x/2)|^2 T_{2p,n}(x/2) + |a(x/2+\pi)|^2 T_{2p,n}(x/2+\pi)\}\,dx.$$

But $|a(x/2)|^2 = 4|c(x/2+\pi)|^2 T^2_{2p,n}(x/2+\pi)$. Substituting this into the integral and invoking (35), we arrive at (55).■

Definition 5.2 *Two TB-wavelets* ψ_1 *and* ψ_2 *are said to be dual to each other if*

$$\sum_{j\in\mathbb{Z}} \psi_1(j-2kn)\overline{\psi_2(j-2ln)} = \delta(k-l) \quad \forall k,l \in \mathbb{Z}.$$

Theorem 5.2 *The spline* ψ *given by (54) is a TB-wavelet if and only if* $\tau(x) \neq 0 \ \forall x \in \mathbb{R}$. *Moreover, if a spline* $R \in \mathbf{W}_{p,2n}$ *is presented in two ways*

$$\frac{1}{2\pi}\int_0^{2\pi} D(x)\cdot W_{p,2n}(x,j)\,dx = R(j) = \sum_{k\in\mathbb{Z}} d(k)\psi(j-2kn)$$

then

$$D(x) = \sum_{k\in\mathbb{Z}} d(k)\tau(x)e^{ikx}, \ \ d(k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{D(x)}{\tau(x)}\cdot e^{-ikx}\,dx. \tag{56}$$

There exists a unique TB-wavelet ψ_1 *dual to* ψ*:*

$$\psi_1(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau_1(x)W_{p,2n}(x,j)\,dx, \quad \tau_1(x) = (\overline{\tau(x)}\Omega(x))^{-1}.$$

Proof: The properties of the Zak wavelet s $W_{p,2n}(x,j)$ are similar to those of the Zak spline s $E_{p,n}(x,j)$. Therefore, to prove the first assertion, one should repeat the considerations of Theorem 3.1 with obvious modifications. The last assertion is an immediate consequence of (55). ■

Proposition 5.1 *Each TB-wavelet decays faster than any degree of* $1/j$ *as* $|j| \to \infty$.

Proof: From (51) we have

$$\psi(j) = \frac{1}{\pi}\int_0^{2\pi} \tau(2t)a(t)E_{p,n}(t,j)\,dt \tag{57}$$

$$= \sum_{k\in\mathbb{Z}} q(k)B_{p,n}(j-kn), \text{ where } q(k) = \frac{1}{\pi}\int_0^{2\pi} \tau(2t)a(t)e^{-ikt}\,dt. \tag{58}$$

This means that $q(k)$ are doubled Fourier coefficients of the infinitely differentiable periodic function $\tau(2t)a(t)$. They decay faster than any degree of $1/k$ as $|k| \to \infty$. Hence the assertion follows. ■

5.2 Examples

Characterization of TB-wavelets established in the previous section allows us to build a library of such wavelets and their duals. We present three examples which may be useful in applications.

B-wavelet We start by building the compactly supported TB-wavelet which is related to the B-wavelet by Unser-Aldroubi-Eden [24], [23] and Chui and Wang [9]. Putting $\tau(x) \equiv 1$ in (54), we get the TB-wavelet:

$$\begin{aligned}
\psi(j) &= \frac{1}{2\pi}\int_0^{2\pi} W_{p,2n}(x,j)\,dx = \frac{1}{\pi}\int_0^{2\pi} a(x)E_{p,n}(x,j)\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} a(x)\sum_{l\in\mathbb{Z}} e^{-ilx}B_{p,n}(j-ln)\,dx = \sum_{l\in\mathbb{Z}} G(l)B_{p,n}(j-ln), \\
G(l) &\triangleq \frac{1}{\pi}\int_0^{2\pi} a(x)e^{-ilx}\,dx.
\end{aligned}$$

The coefficients $G(l)$ are doubled Fourier coefficients of the trigonometric polynomial $a(x)$ and, thus, the set $\{G(l)\}$ is finite and, correspondingly, the TB-wavelet $\psi(j)$ is compactly supported.

Let us find an explicit expression for the coefficients $G(l)$ and establish some properties of the wavelet $\psi(j)$. Recall that

$$\begin{aligned}
a(x) &= 2e^{ix}\overline{c(x+\pi)}T_{2p,n}(x+\pi) = e^{ix}(1-e^{-ix})^p\,T_{2p,n}(x+\pi), \\
T_{2p,n}(x+\pi) &= \sum_{k=-\mu}^{\mu}(-1)^k b_{2p}(k)e^{ikx}, \quad b_{2p}(k) > 0, \quad \mu = \left[\frac{p(n-1)}{n}\right], \\
(1-e^{-ix})^p &= (-1)^p e^{-ipx}\sum_{r=0}^{p}\binom{p}{r}(-1)^r e^{irx}.
\end{aligned}$$

Hence

$$a(x) = (-1)^p e^{-i(p-1)x}\sum_{q=-\mu}^{\mu+p} M(q)e^{iqx}, \text{ where } M(q) = (-1)^q\sum_{r=0}^{p}\binom{p}{r}b_{2p}(q-r).$$

From these relations we immediately derive that $G(l)$ differs from zero only as
$l \in -\mu-p+1 : \mu+1$, and, provided it holds,

$$G(l) = 2(-1)^p M(l+p-1) = 2(-1)^{l-1}\sum_{r=0}^{p}\binom{p}{r}b_{2p}(l+p-1-r). \tag{59}$$

Note that $G(l)$ is an alternating sequence $(-1)^{l-1}G(l) > 0$ as $l \in -\mu - p + 1 : \mu + 1$.

Proposition 5.2 *The wavelet ψ of order p is symmetric about $n - p/2$ when p is even and anti-symmetric when p is odd:*

$$\psi(j) = (-1)^p \psi(2n - p - j), \quad j \in \mathbb{Z}.$$

Proof: Since $\operatorname{supp} B_{p,n} = 0 : p(n-1)$, the support of the wavelet $\psi(j)$ occupies the set $(-\mu-p+1)n : (\mu+1)n + p(n-1)$ which is symmetric about $n - p/2$. Note that

$$G(-p+2-l) = 2(-1)^{p+l-1} \sum_{r=0}^{p} \binom{p}{p-r} b_{2p}(1-l+r) = (-1)^p G(l). \tag{60}$$

Let us consider

$$\psi(2n-p-j) = \sum_{l=-\mu-p+1}^{\mu+1} G(l) B_{p,n}(2n-p-j-ln).$$

Putting $l = -p + 2 - m$ and using (60), we obtain

$$\psi(2n-p-j) = (-1)^p \sum_{m=-\mu-p+1}^{\mu+1} G(m) B_{p,n}(2n-p-j+pn-2n+mn).$$

To accomplish the proof it suffices to recall that $B_{p,n}(p(n-1)-j) = B_{p,n}(j)$.■

Remark Since the TB-wavelet $\psi(j)$ has similar properties to the B-wavelet by Unser-Aldroubi-Eden and Chui and Wang, it is pertinent to call $\psi(j)$ the discrete B-wavelet.

Self-dual TB-wavelets Theorem 5.2 allows the construction of a wavelet $\theta(j)$ which is dual to itself:

$$\theta(j) \triangleq \frac{1}{2\pi} \int_0^{2\pi} \frac{W_{p,2n}(x,j)}{\sqrt{\Omega(x)}}\, dx,$$

where $\Omega(x)$ is defined in (55). The system $\{\theta(\cdot - 2kn) | k \in \mathbb{Z}\}$ forms an orthonormal (in the sense of $\ell^2(\mathbb{Z})$) basis of the space $\mathbf{W}_{p,2n}$.

TB-wavelet dual to the B-wavelet and Galerkin projections The TB-wavelet dual to the B-wavelet is

$$\psi^d(j) = \frac{1}{2\pi} \int_0^{2\pi} \frac{W_{p,2n}(x,j)}{\Omega(x)}\, dx.$$

This wavelet is useful in the construction of the so-called Galerkin projection of a discrete signal onto the wavelet space. We will write $f \in \mathbf{F}$ if $|f(j)| \le M(1+|j|^s)$ $\forall j \in \mathbb{Z}$ for some M, s.

Definition 5.3 *The spline $R(f;j) \in \mathbf{W}_{p,2n}$ is called the Galerkin projection of a signal $f \in \mathbf{F}$ onto the wavelet space $\mathbf{W}_{p,2n}$ if for all $k \in \mathbb{Z}$ the following relation holds:*

$$\sum_{j\in\mathbb{Z}} R(f;j)\psi(j-2kn) = \sum_{j\in\mathbb{Z}} f(j)\psi(j-2kn).$$

We emphasize that the above definition does not depend on a special choice of basis wavelets.

Proposition 5.3 *Let $R(f;j) \in \mathbf{W}_{p,2n}$ be the Galerkin projection of a signal $f \in \mathbf{F}$ onto the wavelet space $\mathbf{W}_{p,2n}$. Then for any TB-wavelet $\psi^1 \in \mathbf{W}_{p,2n}$ the following relations are true:*

$$\sum_{j\in\mathbb{Z}} R(f;j)\psi^1(j-2kn) = \sum_{j\in\mathbb{Z}} f(j)\psi^1(j-2kn).$$

Proof: is similar to the proof of Proposition 4.1.■

Any spline $R(f;j) \in \mathbf{W}_{p,2n}$ can be expanded with respect to the basis $\psi^d(j-2kn)$:

$$R(f;j) = \sum_{l\in\mathbb{Z}} p(l)\psi^d(j-2ln). \tag{61}$$

Multiplying (61) by $\psi(\cdot - 2kn)$, we derive

$$p(k) = \sum_{j\in\mathbb{Z}} R(f;j)\psi(j-2kn) = \sum_{j\in\mathbb{Z}} f(j)\psi(j-2kn), \quad k \in \mathbb{Z}.$$

Since the support of ψ is compact, we express the coefficients $p(k)$ through finite sums:

$$p(k) = \sum_{j=(-\mu-p+1)n}^{(\mu+p+1)n-p} \psi(j)f(2kn+j), \quad k \in \mathbb{Z}, \quad \mu = \left[\frac{p(n-1)}{n}\right].$$

6 On signal transforms by discrete spline wavelets. Bases for the signal space.

We note first that by $n=1$ the B-spline of any order is the Kroneker delta: $B_{p,1}(j) = \delta(j)$, and the Zak spline $E_{p,1}(x,j) = e^{-ijx}$. Therefore $\mathbf{V}_{p,1} = \mathbf{F}$ and any signal $f = \{f(j)\}_{j\in\mathbb{Z}} \in \mathbf{F}$ can be regarded as the spline

$$f(j) = S_{p,1}(j) = \sum_{l=-\infty}^{\infty} c_1(l)\, B_{p,1}(j-l) \;\; c_1(l) = f(l),\ (j \in \mathbb{Z}). \tag{62}$$

The transform is started by doubling n and decomposing the signal into the sum

$$f(j) = S_{p,1}(j) = S_2(j) + R_2(j), \quad S_2 \in \mathbf{V}_{p,2}, \quad R_2 \in \mathbf{W}_{p,2}.$$

We emphasize that at that moment we may choose the parameter p – the order of splines to be involved.

Then the procedure is repeated m times as follows:

$$S_n(j) = S_{2n}(j) + R_{2n}(j), \quad S_{2n} \in \mathbf{V}_{p,2n}, \quad R_n \in \mathbf{W}_{p,n}, \;\; n = 2,\, 4,\, 8 \ldots 2^m. \tag{63}$$

As a result we have the signal transformed into the multiscale representation:

$$f(j) = \sum_{r=1}^{m} R_{2^r}(j) + S_{2^m}(j). \tag{64}$$

Remark. Equation (64) means that actually we designed a wide library of bases for the space of signals of power growth. Let φ_{2^m} be a TB-spline from $\mathbf{V}_{p,2^m}$ and ψ_n be TB-wavelets from the spaces $\mathbf{W}_{p,n}, \quad n = 2, 4, 8 \dots 2^m$. Equation (64) implies that the set $\{\varphi_{2^m}(\cdot - k2^m), \psi_{2^m}(\cdot - kn)\}, \quad k \in \mathbb{Z}, \quad n = 2, 4, 8 \dots 2^m$ form a basis of the signal space. Basis vectors of different levels are orthogonal to each other. The vectors φ are orthogonal to vectors φ. This basis is linked to the dual one which can be explicitly constructed. In addition, this set yields a Riesz basis for the space l_2. We emphasize that the depth of the decomposition m, the order p of basis elements are variable. Moreover, having m and p chosen, one may choose an appropriate TB-spline y φ_{2^m} and TB-wavelets ψ_n separately at each level.

The inverse transform or the reconstruction of the signal consists in changing from the representation (64) to the δ-representation (62).

Theorem 4.3 provides formulas for the decomposition and reconstruction but to practically implement the transform we need to choose some bases of the corresponding spaces and to operate in the time domain. For this purpose we must convert the formulas of Theorem 4.3 into convolution type relations. We describe a single step of the transform.

Theorem 6.1 *Let φ_n and φ_{2n} be the TB-splines*

$$\varphi_\nu(j) = \frac{1}{2\pi}\int_0^{2\pi} \rho_\nu(x) E_{p,\nu}(x,j)\,dx, \quad \rho_\nu(x) \neq 0\,\forall x, \quad \nu = n, 2n, \tag{65}$$

and ψ_{2n} be a TB-wavelet

$$\psi(j) = \frac{1}{2\pi}\int_0^{2\pi} \tau(x) W_{p,2n}(x,j)\,dx. \tag{66}$$

Let the splines involved in (63) be expanded as follows:

$$S_n(j) = \sum_{k\in\mathbb{Z}} c_n(k)\varphi_n(j - kn), \quad S_{2n}(j) = \sum_{k\in\mathbb{Z}} c_{2n}(k)\varphi_{2n}(j - 2kn),$$

$$R_{2n}(j) = \sum_{k\in\mathbb{Z}} d(k)\psi(j - 2kn).$$

Then

$$c_{2n}(k) = \sum_{l\in\mathbb{Z}} c_n(2k - l)A(l), \text{ where} \tag{67}$$

$$A_n(l) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 + e^{-ix})^p\, T_{2p,n}(x)\,\rho_n(x)}{T_{2p,2n}(2x)\,\rho_{2n}(2x)} e^{-ikx}\,dx \tag{68}$$

$$d_{2n}(k) = \sum_{l\in\mathbb{Z}} c_n(2k - l)Q_n(l), \text{ where} \tag{69}$$

$$Q_n(l) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - e^{ix})^p\,\rho_n(x)}{T_{2p,2n}(2x)\tau_{2n}(2x)} e^{-i(l+1)x}\,dx, \tag{70}$$

$$c_n(k) = \sum_{l\in\mathbb{Z}} c_{2n}(l)G(k - 2l) + d_{2n}(l)H(k - 2l), \text{ where} \tag{71}$$

$$G_n(r) \triangleq \frac{1}{2\pi}\int_0^{2\pi} \frac{c(x)\,\rho_{2n}(2x)}{\rho_n(x)} e^{-irx}\,dx, \quad H_n(r) \triangleq \frac{1}{2\pi}\int_0^{2\pi} \frac{a(x)\,\tau_{2n}(2x)}{\rho_n(x)} e^{-irx}\,dx.$$

Proof: Theorem 4.3 asserts that if the splines $S_n(j),\ S_{2n}(j),\ R_{2n}(j)$ are given in the integral form then their densities are linked as in (47), (49), (50). Eq. (47) implies the relation:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{C_{2n}(x)}{\rho_{2n}(x)}\cdot e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{C_n\left(\frac{x}{2}\right)}{\rho_n(x/2)}\alpha\left(\frac{x}{2}\right)+\frac{C_n\left(\frac{x}{2}+\pi\right)}{\rho_n(x/2+\pi)}\alpha\left(\frac{x}{2}+\pi\right)\right)\cdot e^{-ikx}\,dx$$

$$= \quad 2\frac{1}{2\pi}\int_0^{2\pi}\frac{C_n(x)}{\rho_n(x)}\alpha(x)\cdot e^{-i2kx}\,dx,\ \text{where}\ \alpha(x)\triangleq\frac{\overline{c(x)}\,T_{2p,n}(x)\,\rho_n(x)}{T_{2p,2n}(2x)\,\rho_{2n}(2x)}.$$

Due to Theorem 3.1, the left-hand term is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{C_{2n}(x)}{\rho_{2n}(x)}\cdot e^{-ikx}\,dx = c_{2n}(k).$$

As for the right-hand term, it is equal to the convolution

$$2\frac{1}{2\pi}\int_0^{2\pi}\frac{C_n(x)}{\rho_n(x)}\alpha(x)\cdot e^{-i2kx}\,dx = \sum_{l\in\mathbb{Z}} c_n(2k-l)A_n(l),\ \text{where}\ A_n(l)=\frac{1}{2\pi}\int_0^{2\pi}2\alpha(x)e^{-ikx}\,dx.$$

Hence, invoking (32) we obtain (67) and (68). Using (41) we derive Eqs. (69), (70) similarly.

To establish the reconstruction formula, we deduce from (50) the following relation:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{C_n(x)}{\rho_n(x)}\cdot e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{C_{2n}(2x)}{\rho_{2n}(2x)}\beta\left(x\right)+\frac{D_{2n}(2x)}{\tau_{2n}(2x)}\gamma\left(x\right)\right)\cdot e^{-ikx}\,dx,$$

$$\text{where}\qquad \beta(x)\triangleq 2\frac{c(x)\,\rho_{2n}(2x)}{\rho_n(x)},\quad \gamma(x)\triangleq 2\frac{a(x)\,\tau_{2n}(2x)}{\rho_n(x)}.$$

The left-hand term is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{C_n(x)}{\rho_n(x)}\cdot e^{-ikx}\,dx = c_n(k).$$

The terms on the right-hand are:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{C_{2n}(2x)}{\rho_{2n}(2x)}\beta\left(x\right)e^{-ikx}\,dx \;=\; \sum_{l\in\mathbb{Z}} c_{2n}(l)G_n(k-2l),\quad G_n(r)\triangleq\frac{1}{2\pi}\int_0^{2\pi}\beta(x)e^{-irx}\,dx;$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{D_{2n}(2x)}{\tau_{2n}(2x)}\beta\left(x\right)e^{-ikx}\,dx \;=\; \sum_{l\in\mathbb{Z}} d_{2n}(l)H_n(k-2l),\quad H_n(r)\triangleq\frac{1}{2\pi}\int_0^{2\pi}\gamma(x)e^{-irx}\,dx.$$

Hence (71) follows. ■

Remark We stress that the cosine polynomials $T_{p,n}$, and weight functions $\rho,\ \tau$ depend on the step n of the grid. Therefore, unlike similar constructions with continuous splines, the filters $A_n,\ Q_n,\ G_n,\ H$ used for processing a signal depend of the current decomposition (reconstruction) level.

7 Discussion

Although, generally, the operations of decomposition and reconstruction of signals in our scheme are implemented as filtering with infinite impulse response (IIR) filters, the terms of these filters $\{F(k)\}$ decay faster than any degree of $1/k$ as $k \to \infty$ and could be canceled when k is large. One may obtain finite impulse response (FIR) or rational filters by appropriate choice of the basic TB-splines and wavelets. The latter filters could be implemented through fast recursive procedures. For example, in the case where basic splines are B-splines and wavelets are B-wavelets, the reconstruction is implemented as FIR filtering and the decomposition as recursive filtering. The idea to use recursive filtering for similar transforms based on continuous splines and wavelets was introduced in [24]. In [3] we applied recursive filtering for lifting implementation of wavelet transforms derived from discrete interpolatory splines.

Summarizing, the contributions of our construction are the following:

- Using a discrete version of the Zak transform we devised a distributional integral representation of discrete splines of power growth.
- The integral representation enabled us to construct the multiresolution analysis in the space of discrete-time signals of power growth.
- We designed a wide diversity of wavelet transforms and of bases for the space of signals of power growth, which, at the same time, are the Riesz bases for the space l_2.
- To some extent, the paper presents an extension of the works [1, 2] on wavelet bases in l_2 to the signals of power growth. A remarkable difference is that the presented transforms do not require pre- and post-filtering of the signals.
- The transforms are conducted by filtering with FIR and IIR filters which are *scale-dependent*. The ş of IIR filters decay exponentially. These filters can be well approximated by FIR filters or implemented in a recursive manner.

An obvious field of our further research is application of the designed transforms to signal and image processing.

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