On interpolation by discrete splines with equidistant nodes

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Abstract

In this paper we consider discrete splines $S(j)$, $j \in \mathbb{Z}$, with equidistant nodes which may grow as $O(|j|^s)$ as $|j| \to \infty$. Such splines are relevant for the purposes of digital signal processing. We give the definition of discrete B-splines and describe their properties. Discrete splines are defined as linear combinations of shifts of the B-splines. We present a solution to the problem of discrete spline cardinal interpolation of sequences of power growth and prove that the solution is unique within the class of discrete splines of a given order.

1 Introduction

The theory of cardinal interpolation is an essential topic in the spline studies, [8], [9]. The term *cardinal interpolation* means interpolation of a bi-infinite sequence by splines with equidistant nodes kh, $k \in \mathbb{Z}$. In the papers [8], [9], [12], [13] the authors studied cardinal interpolation by continuous polynomial splines. However, for the purposes of digital signal processing the discrete splines defined on the set $\mathbb{Z}$ of integers offer some advantages over the continuous ones. Discrete splines were studied in the early seventies ([10]), but recently reappeared as the subject of extensive investigations ([1] , [2, Chapter 6], [4], [5], [6]). We also mention the related work [7] which deals with wavelets of discrete argument. A large part of the investigations was devoted to the theory of periodic discrete splines. In this paper we develop the theory of non-periodic discrete splines of power growth. The subject and methods involved are related to those of the work [8] where the continuous splines of power growth were studied.

The paper is organized as follows. Section 2 is devoted to discrete B-splines. In Section 2.1 we give the definition of the B-spline B_p of order p, establish its structure and outline its properties.

In Section 2.2 we introduce the characteristic cosine polynomials corresponding to discrete B-splines and prove their positivity. This result is basic for the solution of the cardinal interpolation problem.

In Section 3 we handle the problem of cardinal interpolation. First we define in Section 3.1 the discrete spline $S(j)$ as a linear combination of shifts of the B-spline. In Section 3.2 we present a solution to the problem and establish its uniqueness.

2 Discrete B-splines

2.1 Definition and basic properties of the B-splines

The splines we deal with are defined on the set of integers $\mathbb{Z}$. We start with B-splines which are fundamental in almost any spline construction. Let p be a natural number. Throughout the paper we assume that n is an odd number.

The discrete B-spline of the first order is by definition the following sequence:

$$B_1(j) = \begin{cases} 1 & \text{if } j \in 0 : n-1, \\ 0, & \text{otherwise}, \quad j \in \mathbb{Z}. \end{cases} \tag{1}$$

Here and further we denote by $l : m$ the set of integers $\{l, l+1, \ldots, m\}$.

We define the higher order B-splines as the discrete convolutions by recurrence:

$$B_r = B_1 * B_{r-1}, \ r = 2, \ldots, p, \tag{2}$$

or, expressed differently,

$$B_r(j) = \sum_{k=0}^{n-1} B_{r-1}(j-k),\ j \in \mathbb{Z},\ r = 2, \dots, p. \tag{3}$$

It is readily seen that the B-spline of second order is a piecewise polynomial of first degree:

$$B_2(j) = \begin{cases} j+1, & \text{if } \ j \in 0 : n-1 \\ 2n-1-j, & \text{if } \ j \in n-1 : 2n-2, \\ 0, & \text{otherwise}, \quad j \in \mathbb{Z}. \end{cases} \tag{4}$$

In fact, any discrete B-spline is a piecewise polynomial. To prove this we use the z-transform [3].

Definition 2.1 *Let $f = \{f(k)\}_{k=-\infty}^{\infty}$ be a truncated sequence, that is, $f(k) = 0$ for all $k < 0$. The z-transform of f is the function of the complex variable z:*

$$\zeta[f] = F(z) = \sum_{k=0}^{\infty} f(k)\, z^k, \quad |z| < \rho, \tag{5}$$

where ρ is the radius of convergence of the series.

We mention two important properties of the z-transform:

- The first is concerned with the discrete convolution:

$$\zeta[f * g] = \zeta[f]\, \zeta[g]. \tag{6}$$

- The second is the *shifting property*:

$$z^l\, \zeta[f(\cdot)] = \zeta[f(\cdot - l)]. \tag{7}$$

The symbol $k_+^{(l)}$ denotes the truncated factorial polynomial:

$$k_+^{(l)} = \begin{cases} k(k+1)\dots(k+l-1) & \text{if } \ k \in 0 : \infty \\ 0, & k < 0, \quad k \in \mathbb{Z}. \end{cases} \tag{8}$$

Let $k_+^{(0)} = 1$ for $k > 0$ and $k_+^{(0)} = 0$ for $k \le 0$. The z-transforms of the polynomials are:

$$\zeta[k_+^{(l)}] = \frac{l!z}{(1-z)^{(l+1)}}. \tag{9}$$

It is readily seen that

$$B_1(j) = (j+1)_+^{(0)} - (j+1-n)_+^{(0)}, \quad \zeta[B_1] = \frac{1-z^n}{1-z}.$$

This relation implies that the z-transform of the B-spline is

$$\zeta[B_p] = \sum_{j=0}^{p(n-1)} B_p(j) z^j = (1 + z + z^2 + \ldots + z^{n-1})^p. \tag{10}$$

So, $B_p(j)$ is the coefficient at z^j in the polynomial $(1 + z + z^2 + \ldots + z^{n-1})^p$.

Theorem 2.1 *The B-spline of order p is the piecewise polynomial of degree $p-1$:*

$$B_p(j) = \frac{1}{(p-1)!} \sum_{r=0}^{p} (-1)^r \binom{p}{r} (j+1-rn)_+^{(p-1)} = \Delta_n^p \Big(\frac{(j+1-pn)_+^{(p-1)}}{(p-1)!} \Big). \tag{11}$$

Proof: From (10) we have:

$$\zeta[B_p] = \frac{(1-z^n)^p}{(1-z)^p} = \frac{\sum_{r=0}^{p} (-1)^r \binom{p}{r} z^{rn}}{(1-z)^p} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} \frac{z^{rn-1}}{(p-1)!} \zeta[j_+^{(p-1)}]. \tag{12}$$

Hence, invoking (7) , we derive (11). ■

The breakpoints $\{kn\},\ k \in \mathbb{Z}$, are called the nodes of the B-spline.
The following properties of the B-splines B_p hold:

1.
$$B_p(p(n-1) - j) = B_p(j) \quad \text{for all integers } j; \tag{13}$$

2.
$$B_p(j) > 0 \text{ if } j \in 0 : p(n-1) \tag{14}$$
$$B_p(j) = 0 \text{ otherwise}; \tag{15}$$

3.
$$B_p(0) = B_p(p(n-1) = 1; \tag{16}$$

4. The sequence $B_p(j)$ increases strictly monotonically as $0 \le j \le p(n-1)/2$ and decays as $p(n-1)/2 \le j \le p(n-1)$;

5.
$$\sum_{j \in \mathbb{Z}} B_p(j) = n^p. \tag{17}$$

The last assertion follows from (10) when $z = 1$.
Remark. We emphasize that the B-splines assume only integer non-negative values and their supports are compact (Property 2). It is worth mentioning that the discrete B-spline $B_p(j)$ is not a trace of the continuous B-spline .

2.2 Characteristic cosine polynomial

Recall that n is an odd number: $n = 2\nu + 1$. Together with the B-spline $B_p(j)$ we introduce the central B-spline

$$M_p(j) := B_p(j + p\nu). \tag{18}$$

It is apparent that the central B-spline is an even sequence with its support at $-p\nu : p\nu$ and its maximum at zero. It is a piecewise polynomial of degree $p - 1$ with its nodes at $nk + p\nu$. We emphasize also that the convolution property holds:

$$M_r = M_1 * M_{r-1},\ r = 2, \ldots, p. \tag{19}$$

Lemma 2.1 *For all integers k, q the following relation holds:*

$$\sum_{j=-\infty}^{\infty} M_p(j - kn) M_p(j - qn) = M_{2p}((k - q)n). \tag{20}$$

Proof: The property (19) implies that $M_p * M_p = M_{2p}$ which, in turn, leads to (20) . ■

Now we define a cosine polynomial which is fundamental to the sequel. Denote $b_p(k) = M_p(kn)$. Recall, that $b_p(-k) = b_p(k)$ and $b_p(k)$ is nonzero only if $|k| \le \mu = \left[\frac{p\nu}{n}\right] = \left[\frac{p(n-1)}{2n}\right]$. Here $[\alpha]$ means the integer part of the number α.

Definition 2.2 *The cosine polynomial*

$$T_p(x) = \sum_{k=-\mu}^{\mu} b_p(k) e^{ikx} = b_p(0) + 2\sum_{k=1}^{\mu} b_p(k) \cos kx \tag{21}$$

is called the characteristic cosine polynomial (CCP) of the B-spline M_p. It is related to the Euler-Frobenius polynomial ([9]).

It is apparent that $T_p(x)$ is an even $2\pi-$periodic infinitely differentiable function. The basic property of the CCP is that it is strictly positive for all x. To establish this we first prove the following assertion.

Lemma 2.2 *Let m be an even positive number. Then for all $\lambda \in 1 : m/2$ and natural p the function*

$$G_m(\lambda, p) := \sum_{s=0}^{n-1} \left(\frac{(-1)^s}{\sin \frac{\pi(sm+\lambda)}{mn}} \right)^p \tag{22}$$

is strictly positive and the following inequalities hold:

$$G_m(\lambda, p) \ge \begin{cases} 1, & p \text{ is odd} \\ \left(\sin \frac{\pi\lambda}{mn}\right)^{-p}, & p \text{ is even} \end{cases} \tag{23}$$

Proof: The estimate for the even exponents p is readily seen, since in this case all terms of the sum $G_m(\lambda, p)$ are positive and, therefore, the value of the sum exceeds its first term which is $\left(\sin\frac{\pi\lambda}{mn}\right)^{-p}$. For odd p the situation is more complicated.

The function $q_\lambda(x) = \sin\frac{\pi(xm+\lambda)}{mn}$ has its only maximum on the interval $[0, n-1]$ at the point $x_0 = n/2 - \lambda/m$. On the intervals $[0, x_0]$ and $[x_0, n-1]$ the function is strictly monotone. This fact implies that the minimal term of the positive sequence

$$h_\lambda(s) = \left(\sin\frac{\pi(sm+\lambda)}{mn}\right)^{-1}, \quad s \in 0 : n-1$$

is $h_\lambda(\nu)$, where $\nu = \frac{n-1}{2}$ and the subsequences $\{h_\lambda(s)\}_{s=0}^{\nu}$ and $\{h_\lambda(s)\}_{s=\nu+1}^{n-1}$ are strictly monotone.

Let us return to the sum $G_m(\lambda, p)$. The cases for ν even or odd require slightly different considerations.

1. In the case of even ν we write the sum as follows:

$$\begin{aligned} G_m(\lambda, p) &= \sum_{s=0}^{n-1}\left((-1)^s h_\lambda(s)\right)^p \\ &= \sum_{s=0}^{\nu-1}\left((-1)^s h_\lambda(s)\right)^p + h_\lambda(\nu)^p + \sum_{s=\nu+1}^{n-1}\left((-1)^s h_\lambda(s)\right)^p. \end{aligned} \quad (24)$$

Due to monotonicity the sums in (24) are positive and we have

$$G_m(\lambda, p) > h_\lambda(\nu)^p \geq 1. \quad (25)$$

2.When ν is odd we write the sum as

$$G_m(\lambda, p) = \sum_{s=0}^{\nu}\left((-1)^s h_\lambda(s)\right)^p + h_\lambda(\nu+1)^p + \sum_{s=\nu+2}^{n-1}\left((-1)^s h_\lambda(s)\right)^p.$$

Hence we derive the inequality

$$G_m(\lambda, p) > h_\lambda(\nu+1)^p > h_\lambda(\nu)^p \geq 1. \quad (26)$$

■

Now we proceed to establishing the basic property of the CCP.

Theorem 2.2 *The cosine polynomial $T_p(x)$ is strictly positive for all x.*

Proof: Let us choose some even integer m subject to the inequality $m \geq 2\mu + 2$. Denote $\omega_m = e^{2\pi i/m}$. Then

$$T_p\left(\frac{2\pi l}{m}\right) = \sum_{k=-\mu}^{\mu} b_p(k)\omega_m^{-kl} = \sum_{k=-m/2}^{m/2-1} b_p(k)\omega_m^{-kl} = F_m(b_p)(l).$$

Here $F_m(b_p)$ denotes the m-point discrete Fourier transform (DFT) of the sequence b_p. We represent the function in an explicit form. To do that, we denote $N = mn$ and find the N-point DFT of the sequence $\{M_p(j)\}_{j=-N/2}^{N/2-1}$.

For the first order B-splines we have with $l \in -N/2 : N/2 - 1$ that:

$$u(l) := F_N(M_p)(l) = \sum_{j=-N/2}^{N/2-1} M_1(j)\omega_N^{-jl} = \sum_{j=-\nu}^{\nu} 1 \cdot \omega_N^{-jl} = \begin{cases} 2\nu + 1 = n, & l = 0, \\ \dfrac{\sin \pi l/m}{\sin \pi l/N}, & l \neq 0. \end{cases}$$

Due to the convolution property (19),

$$F_N(M_p)(l) = [F_N(M_1)(l)]^p = u^p(l).$$

Let us extend periodically the sequence $u(l)$ with the period N. Then $u(sm) = 0$ when $s \in 1 : n - 1$ and

$$M_p(j) = \frac{1}{N}\sum_{l=0}^{N-1} u^p(l)\omega_N^{lj}, \quad j \in -N/2 : N/2 - 1. \tag{27}$$

Hence we have for $k \in -\mu : \mu$:

$$b_p(k) = M_p(kn) = \frac{1}{N}\sum_{l=0}^{N-1} u^p(l)\omega_m^{lk}.$$

Representing l as $l = sm + r$, $s \in 0 : n - 1$, $r \in 0 : m - 1$, we come to the relation:

$$b_p(k) = \frac{1}{m}\sum_{r=0}^{m-1}\left[\frac{1}{n}\sum_{s=0}^{n-1} u^p(sm + r)\right]\omega_m^{rk}. \tag{28}$$

For even integers p, Eq. (28) was established in [5]. Eq. (28) implies that

$$\begin{aligned} T_p\left(\frac{2\pi\lambda}{m}\right) &= F_m(b_p)(\lambda) = \frac{1}{n}\sum_{s=0}^{n-1} u^p(sm + \lambda) \\ &= \begin{cases} \dfrac{1}{n}(\sin \lambda\pi/m)^p\, G_m(\lambda, p), & \lambda \in 1 : m - 1 \\ n^{p-1}, & \lambda = 0. \end{cases} \end{aligned} \tag{29}$$

The function $G_m(\lambda, p)$ was defined in (22).

It suffice to evaluate $T_p(2\pi\lambda/m)$ when $\lambda \in 1 : m/2$. On the interval $(0, \frac{\pi}{2})$ the inequalities $\frac{2}{\pi}x < \sin x < x$ are true. They result in the estimates

$$\Big(\frac{2\lambda}{m}\Big)^p < \Big(\sin\frac{\lambda\pi}{m}\Big)^p, \quad \Big(\frac{mn}{\lambda\pi}\Big)^p < \Big(\sin\frac{\lambda\pi}{mn}\Big)^{-p}. \tag{30}$$

We again need to distinguish between the cases when p is even or odd.

1. In the case of even p the estimates (25) and (23) lead us straightforwardly to the following inequality

$$T_p\left(\frac{2\pi\lambda}{m}\right) \geq \frac{1}{n}\left(\frac{2n}{\pi}\right)^p > 0. \tag{31}$$

2.In the case of odd p only the estimate $G_m(\lambda, p) \geq 1$ for G is available. Then we have

$$T_p\left(\frac{2\pi\lambda}{m}\right) \geq \frac{1}{n}\left(\frac{2\lambda}{m}\right)^p > 0. \tag{32}$$

Increasing m we come to the estimate

$$T_p(x) \geq \frac{1}{n}\left(\frac{2n}{\pi}\right)^p, \ x \in (-\infty, \infty)$$

when p is even and see that $T_p(x) \geq 0 \ \forall x$ when p is odd. To make sure that $T_p(x) > a > 0 \ \forall x$ with some a we recall that $T_p(x)$ is a continuous function and $T_p(0) = n^{p-1}$.■

Corollary 2.1 *The function $V(x) = 1/T_p(x)$ is even, $2\pi-$periodic, and infinitely differentiable. It could be expanded into the Fourier series*

$$V(x) = \sum_{k=-\infty}^{\infty} v(k)e^{ikx}, \tag{33}$$

with the coefficients

$$v(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(x)e^{-ikx}\,dx = \frac{1}{\pi}\int_0^{\pi} V(x)\cos kx\,dx \tag{34}$$

decaying faster then any power of $1/k$ as $k \to \infty$. Namely, for any $\beta > 0$ there exists a constant $C(\beta)$ such that

$$|v(k)| \leq \frac{C(\beta)}{(1+|k|)^\beta}, \qquad k \in \mathbb{Z}. \tag{35}$$

Remark. Note that Eq. (29) implies the identity:

$$\sum_{l=-\infty}^{\infty} M_p(ln) = n^{p-1}. \tag{36}$$

3 Discrete splines and cardinal interpolation

3.1 Definition of the discrete spline and some preliminaries

Definition 3.1 *Any linear combination of shifts of the central discrete B-spline $M_p(j)$:*

$$S_p(j) = \sum_{l=-\infty}^{\infty} c(l)\, M_p(j - ln) \quad (j \in \mathbb{Z}) \tag{37}$$

is called a discrete spline of order p.

The B-spline is compactly supported. Hence, once j is fixed, the series in (37) actually comprises only a few non-zero entries. To be specific, if $j \in kn : (k+1)n - 1$ then

$$S_p(j) = \sum_{l=k-\mu}^{k+\mu+1} c(l) M_p(j - ln), \quad \mu = \left[\frac{p\nu}{n}\right] = \left[\frac{p(n-1)}{n}\right]. \tag{38}$$

Here $[\alpha]$ means the integer part of the number α. Therefore the series in (37) converges with any coefficients $c(l)$. Moreover, due to Eq. (36), if $j \in kn : (k+1)n - 1$ then

$$\left|S_p(j)\right| \le n^{p-1} \max\{|c(l)|\}, \quad l = k - \mu : k + \mu + 1. \tag{39}$$

Note that S_p coincides with a polynomial of degree $p - 1$ on the set $kn - p\nu : (k+1)n - p\nu$. The points $\{kn - p\nu\}$, $k \in \mathbb{Z}$, are called the nodes of the spline S_p. We will handle the interpolation problem within a somewhat restricted class of discrete splines. Before proceeding with it we state some definitions and auxiliary facts.

Definition 3.2 *We denote by $\mathbf{G}^s$ the space of sequences $\vec{a} = \{a(k)\}_{-\infty}^{\infty}$ which satisfy the requirement $|a(k)| \le M(1 + |k|^s)\, \forall k \in \mathbb{Z}$ with a fixed integer s and a positive constant M. The space $\mathbf{G} := \bigcup_{s=-\infty}^{\infty} \mathbf{G}^s$ is said to be the space of sequences of power growth.*

Definition 3.3 *We denote by $\mathbf{V}_p^s$ the space of discrete splines S_p such that the sequences $\{c(k)\}_{-\infty}^{\infty}$ in the representation (37) belong to $\mathbf{G}^s$ and the space $\mathbf{V}_p$ we define as $\mathbf{V}_p = \bigcup_{s=-\infty}^{\infty} \mathbf{V}_p^s$.*

Remark. We stress that any spline $S(j) \in \mathbf{V}_p^s$ belongs to the space $\mathbf{G}^s$ with respect to $j \in \mathbb{Z}$. This follows directly from (39).

Some remarks on periodic distributions. Let $\vec{a} = \{a(k)\}_{-\infty}^{\infty} \in \mathbf{G}$. Denote

$$\mathcal{F}(\vec{a}, x) = \sum_k e^{ikx} a(k). \tag{40}$$

This series is a 2π-periodic distribution [11, page 331]. The numbers

$$a(k) = \frac{1}{2\pi} \langle \mathcal{F}(\vec{a}, x), e^{-ikx} \rangle$$

are called the Fourier coefficients of the distribution.

Definition 3.4 *We denote by* $\mathbf{D}^s$ *the space of* 2π*-periodic distributions given by (40) with* $\vec{a} \in \mathbf{G}^s$*, and* $\mathbf{D} := \bigcup_{s=-\infty}^{\infty} \mathbf{D}^s$*. The space of* 2π*-periodic complex-valued infinitely differentiable functions we denote by* $\mathbf{C}^\infty$*.*

Under the discrete convolution of two sequences $\vec{q}$ and $\vec{r}$ we mean the sum:

$$\vec{q} * \vec{r} = \{s(k)\} = \{\sum_l q(k-l)\, r(l)\}.$$

The following assertion is readily verified.

Proposition 3.1 *The discrete convolution with a sequence from* $\mathbf{G}^{-\infty} = \bigcap_{s=-\infty}^{\infty} G^s$ *maps the space* $\mathbf{G}^s$ *into itself.*

The proposition implies that, provided

$$\vec{q} \in \mathbf{G}^{-\infty}, \ \vec{r} \in \mathbf{G}^s, \ \text{and } \vec{s} = \vec{q} * \vec{r},$$

the series

$$\sigma(x) := \sum_k e^{ikx} s(k) = \mathcal{F}(\vec{s}, x)$$

is the distribution from the space $\mathbf{D}^s$ as well as $\mathcal{F}(\vec{r}, x)$.

This fact justifies the following

Definition 3.5 *The product of a distribution* $\rho = \mathcal{F}(\vec{r}, \cdot)$ *from* $\mathbf{D}^s$ *with a function* $Q = \mathcal{F}(\vec{q}, \cdot)$ *from* $\mathbf{C}^\infty$ *is understood as follows:*

$$Q(x)\, \mathcal{F}(\vec{r}, x) = \mathcal{F}(\vec{r} * \vec{q}, x) \in \mathbf{D}^s. \tag{41}$$

It corresponds with the conventional definition of the multiplication of a distribution by a function.

3.2 Cardinal interpolation problem

Let us formulate the problem.

Cardinal discrete spline interpolation problem (CDSIP) of order p**.** *Given a sequence* $\vec{z} = \{z(k)\}$ *of power growth, find a discrete spline of order* p $S_p \in \mathbf{V}_p$ *subject to the equations:*

$$S_p(kn) = z(k), \ k \in \mathbb{Z}. \tag{42}$$

To obtain the solution of the CDSIP, we will generally follow the classical scheme by Schoenberg [8],[9].

Fundamental splines. Let us define the spline of order p:

$$L_p(j) := \sum_{l=-\infty}^{\infty} v(l) M_p(j - ln), \tag{43}$$

where $v(l)$ are the Fourier coefficients of the function $V = 1/T$, (see (34)).

Proposition 3.2 *The spline L_p has the following properties:*

1.
$$L_p(kn) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

2. *The B-spline M_p can be represented uniquely through the spline L_p:*
$$M_p(j) := \sum_{l=-\mu}^{\mu} b_p(l) L_p(j - ln). \tag{45}$$

3. *For any $\beta > 0$ there exists a constant $D(\beta)$ such that the inequality*
$$|L_p(j - kn)| \le \frac{D(\beta)}{(1 + |k|)^{\beta}}, \qquad k \in \mathbb{Z} \tag{46}$$
holds uniformly for $j \in \mathbb{Z}$.

Proof: 1. The spline L_p at the points kn is:

$$L_p(kn) = \sum_{l=-\infty}^{\infty} v(l) M_p((k - l)n) = \sum_{l=-\mu}^{\mu} v(k - l) b_p(l).$$

Invoking Eq. (34) we get:

$$\begin{aligned} L_p(kn) &= \frac{1}{2\pi} \sum_{l=-\mu}^{\mu} b_p(l) \int_{-\pi}^{\pi} V(x)\, e^{-i(k-l)x}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, V(x)\, e^{-ikx} \sum_{l=-\mu}^{\mu} b_p(l)\, e^{ilx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, V(x)\, T(x)\, e^{-ikx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, e^{-ikx} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2. Using (43) we may write

$$\begin{aligned}
\sum_{l=-\mu}^{\mu} b_p(l)L_p(j-ln) &= \sum_{l=-\mu}^{\mu} b_p(l) \sum_{k=-\infty}^{\infty} v(k)M_p(j-(k+l)n) \\
= \sum_{r=-\infty}^{\infty} M_p(j-rn) \sum_{l=-\mu}^{\mu} b_p(l)v(r-l) \\
= \sum_{r=-\infty}^{\infty} M_p(j-rn)\, L_p(rn) = M_p(j).
\end{aligned}$$

To verify the uniqueness of the representation (45), suppose that there exists another representation

$$M_p(j) := \sum_{l=-\infty}^{\infty} q(l)L_p(j-ln).$$

But Eq. (44) implies that

$$q(l) = M_p(ln) = b_p(l).$$

3. The inequality (46) is an immediate consequence of the estimate (35).■

The spline L_p is called the fundamental spline. Now we are in a position to establish the main result of the paper.

Theorem 3.1 *The CDSIP of order p has a unique solution with any set of data $\{z(k)\}$ of power growth. The solution is given by the formulas*

$$S_p^i(j) = \sum_{-\infty}^{\infty} z(k)\, L_p(j-kn) \tag{47}$$

$$= \sum_{-\infty}^{\infty} c(k)\, M_p(j-kn), \quad c(k) = \sum_{-\infty}^{\infty} v(l)z(k-l). \tag{48}$$

Moreover, if the sequence $\vec{z} = \{z(k)\}$ belongs to $\mathbf{G}^s$ then the discrete spline S_p^i belongs to the space $\mathbf{V}^s$.

Proof: Let the data sequence $\vec{z} = \{z(k)\}$ be from $\mathbf{G}^s$. Then, due to the estimate (46), the series

$$J(j) := \sum_{-\infty}^{\infty} z(k)\, L_p(j-kn)$$

converges absolutely and locally uniformly with respect to j. The property (44) implies that

$$J(kn) = z(k).$$

Substituting (43) we may write

$$\begin{aligned} J(j) &= \sum_{-\infty}^{\infty} z(k) \sum_{l=-\infty}^{\infty} v(l) M_p(j-(l+k)n) \\ &= \sum_{-\infty}^{\infty} c(k)\, M_p(j-kn), \quad c(k) = \sum_{-\infty}^{\infty} v(l) z(k-l). \end{aligned}$$

Proposition 3.1 guarantees that the sequence $\{c(k)\}$ belongs to $\mathbf{G}^s$, from which we conclude that J is a discrete spline of order p from the space $\mathbf{V}_p^s$ which provides a solution to the CDSIP. We redenote

$$S_p^i(j) := J(j).$$

It remains to prove the unicity of the solution S_p^i within the class $\mathbf{V}_p$. Suppose, that a discrete spline

$$R(j) = \sum_{l=-\infty}^{\infty} d(l)\, M_p(j-ln) \ \in \mathbf{V}_p$$

interpolates the zero sequence:

$$R(kn) = 0, \ k \in \mathbb{Z}. \tag{49}$$

Using (45) we rewrite the spline R:

$$R(j) = \sum_{l=-\infty}^{\infty} f(l)\, L_p(j-ln), \quad f(l) = \sum_{k=-\mu}^{\mu} b_p(k) d(l-k).$$

The relation (49) is equivalent to the following one:

$$f(k) = 0, \ k \in \mathbb{Z}. \tag{50}$$

We need to prove that Eq. (50) implies that

$$d(k) = 0, \ k \in \mathbb{Z}. \tag{51}$$

Actually, the array $\vec{f} = \{f(l)\}_{-\infty}^{\infty}$ is the convolution:

$$\vec{f} = \vec{d} * \vec{b}_p$$

where $\vec{d} = \{d(l)\}_{-\infty}^{\infty}$, $\vec{b}_p = \{b_p(l)\}_{-\infty}^{\infty}$. Note, that $\{b_p(l)\}_{-\infty}^{\infty}$ are the Fourier coefficients of the cosine polynomial T_p. Denote by $P(x) = \mathcal{F}(\vec{d}, x) = \sum_k e^{ikx} d(k)$ the distribution from $\mathbf{D}$. Then $\{f(l)\}$ are the Fourier coefficients of the distribution PT_p from $\mathbf{D}$. Eq. (50) implies that $P(x)T_p(x) \equiv 0$. But the cosine polynomial T_p is strictly positive. Hence we have $P(x) \equiv 0$, which, in turn, leads us to Eq. (51) ■

Corollary 3.1 *Any discrete spline $S_p \in \mathbf{V}_p$ can be uniquely represented through the series*

$$S_p(j) = \sum_{-\infty}^{\infty} S_p(k)\, L_p(j-kn)$$

which converges locally uniformly.

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