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Spline-operational Calculus and Inverse Problem for Heat Equations

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The periodic splines of defect 1 with the equidistant nodes are of dual convolution nature. They are the semi-group in relation to the continuous convolution. On the other side, such spline, being written as the linear combination of the B-splines, is the discrete convolution. This observation makes it possible to use the periodic splines as the base for the peculiar operational calculus — we call it the spline-operational calculus (SOC). The methods of SOC are very universal. They have displayed themselves as very efficient ones for the solving a number of problems of one- and multi-dimensional numerical analysis. We enumerate some of these problems.

1. Constructing one- and multi-dimensional interpolating and smoothing splines of arbitrary degree without solving the systems of the algebraic equations.

2. The similar problem in the case, when the integral averages of the approximated function are known, may be with errors.

3. Asymptotic expansion the remainder of approximation functions and its derivatives by above-mentioned splines.

4. The stable, efficient algorithms for solving the one- and multi-dimensional convolution type integral equations of first and second kind, as well as the systems of such equations and integral-differential equations. We have as initial data the discrete indications with errors of the kernel and

right side of the equations. The convergence of algorithms is proved and the evaluation of the difference between the approximate solutions and the exact solutions is obtained.

5. Approximate solving the boundary value problem for the ordinary linear differential equations of arbitrary order with constant coefficients if we have as initial data the discrete indications with errors of the right side of the equations.

6. Approximate solving the Dirichlet and Neumann problems for the Laplace and Poisson equations in a ring and in a rectangle if we have as initial data the discrete indications with errors of the boundary conditions and the ring side of the equations. The approximate solving the ill-posed Cauchy problem for the Laplace equations in a ring.

7. Approximate solving the direct and inverse problems for the heat and wave equations in an interval, and in a rectangle.

8. The means are found which make it possible in some cases during the executing the fast Fourier transform to diminish essentially the size of the array processed and to increase the speed of the processing.

It is not an exhaustive list of problems at all.

It is possible to say, that SOC is an adequate mathematical apparatus for the solving a lot of linear problems, where a convolution appears in any form and where we have an information on the continuous object in discrete form and with errors. As a matter of fact SOC is a Fourier type analysis adapted for such problems. As the formulae obtained via SOC are explicit and contain the finite number of addends, they can be calculated simply and studied in details. SOC methods can be applied successfully for the solving a lot of ill-posed linear problems.

In the present paper we give the main formulae of SOC in one-dimensional case. As an example of application of the obtained formulae we consider the ill-posed inverse problem for heat equation in the thin ring. Another applications will be discussed in forthcoming papers of the author.

§1. Auxiliary information.

Let us introduce some notations. Let N be an even number, $\omega = e^{2\pi i/N}$, $\nu_n = 2\sin(\pi n/N)$, $V_n = \frac{\sin(\pi n/N)}{\pi n/N}$. The Discrete Fourier Transform (DFT) of the vector $\mathbf{a} = \{a_k\}_0^{N-1}$ is $T_n(\mathbf{a}) = \frac{1}{N}\sum_k \omega^{-nk} a_k$. Here and below the symbol $\sum_k$ denotes $\sum\limits_{k=0}^{n-1}$. The norm of vector $\mathbf{a}$ is

$$\|\mathbf{a}\| = \left(\frac{1}{N}\sum\nolimits_k a_k^2\right)^{\frac{1}{2}}. \tag{1.1}$$

We point out some well known properties of DFT,

$$a_k = \sum\nolimits_n \omega^{nk} T_n(\mathbf{a}), \tag{1.2}$$

$$\frac{1}{N}\sum\nolimits_k a_k b_k = \sum\nolimits_n T_n(\mathbf{a})\overline{T}_n(\mathbf{b}) \Rightarrow \|\mathbf{a}\|^2 = \sum\nolimits_n |T_n(\mathbf{a})|^2. \tag{1.3}$$

Discrete convolution of the vector $\mathbf{a}$ with the vector $\mathbf{b}$ and its DFT are

$$\mathbf{a} * \mathbf{b} = \left\{\frac{1}{N}\sum\nolimits_k a_{l-k} b_k\right\}, \qquad T_n(\mathbf{a} * \mathbf{b}) = T_n(\mathbf{a})T_n(\mathbf{b}). \tag{1.4}$$

Now we introduce two meshes on the x-axis: $\{x_k = k/N\}$, $\{x_k^p = \left(k + \frac{p}{2}\right)/N\}$. The symbol $\mathfrak{S}^p$ will denote the space of 1-periodic splines of degree $p-1$, of defect 1, with nodes in the points x_k^p. The symbol $M^p(x)$ will denote the central 1-periodic B-spline of degree $p-1$ [1]:

$$M^p(x) = \sum_{n=-\infty}^{\infty} V_n^p e^{2\pi i n x}. \tag{1.5}$$

Let us point out that the support of the B-spline can be described as

$$\operatorname{supp} M^p(x) \equiv \bigcup_{k=-\infty}^{\infty} \Omega_k^p, \qquad \Omega_k^p = \left(\left(k + \frac{p}{2}\right)\Big/ N, \left(k - \frac{p}{2}\right)\Big/ N\right).$$

On the interval Ω_k^p $M^p(x) = N^p \partial^p (x_+^{p-1}/(p-1)!)$, $x_+ = 0.5(x + |x|)$, where the symbol ∂ means the central difference with the step $\frac{1}{N}$.

Remark 1.1. The definition of the B-spline $M^p(x)$, given in formula (1.5), can be extended to arbitrary integer values of p. Here, if $p \le 0$, $M^p(x)$ is a distribution [2]. In particular, $M^0(x)$ is the 1-periodically extended δ-function of Dirac. ∎

Denote

$$(1.6) \qquad \mathbf{M^p} = \{M^p(x_k)\}_0^{N-1}, \qquad u_n^p = T_n(\mathbf{M^p}) = \frac{1}{N}\sum\nolimits_k \omega^{-nk} M^p(x_k).$$

The functions u_n^p were studied in [1], [3], [4]. The recurrent formulae and the explicit representations for the initial values of p are known. It's important for us, that

$$(1.7) \qquad 0 < \kappa_{p-1} = u_{n/2}^p \le u_n^p \le u_0^p = 1, \qquad \kappa_p = K_p\left(\frac{2}{\pi}\right)^p,$$

K_p is the Favard constant.

There holds the representation for an arbitrary value of p:

$$u_n^p = \sum_{k=0}^{l} \gamma_k^p \nu_n^{2k}, \qquad l = [(p+1)/2], \qquad \gamma_0^p = 1.$$

If $g(x)$ is any 1-periodic continuous function, $\mathbf{g} = \{g(x_k)\}_0^{N-1}$ then there exist the connection between the Fourier transform of the function g

$$c_n(g) = \int_0^1 e^{-2\pi i n y}\, g(y)\, dy,$$

and the DFT of the vector $\mathbf{g}$:

$$(1.8) \qquad T_n(\mathbf{g}) = \sum_{l=-\infty}^{\infty} c_{n+lN}(g).$$

It follows from the formulae (1.6), (1.8) that

$$(1.9) \qquad u_n^p = \sum_{l=-\infty}^{\infty} V_{n+lN}^p.$$

§2. The formulae of SOC.

Each spline $S^p \in \mathfrak{S}^p$ can be represented in the following form:

$$S^p(x) = \frac{1}{N}\sum\nolimits_k q_k M^p(x - x_k). \tag{2.1}$$

Remark 2.1. Formula (2.1) can be extended to arbitrary integer values of p in conformity with Remark 1.1. If $p \leq 0$ this formula can be interpreted as the definition of the spline of non-positive order. ■

It can be seen from the formula (2.1), that the spline $S^p(x)$ is defined completely by its order p and the vector $\mathbf{q} = \{q_n\}_0^{N-1}$ of coefficients. Denote $\{T_n(\mathbf{q})\}_0^{N-1} = Q(S^p)$ and consider this vector as an image of the spline $S^p(x)$. The relation $S^p(x) \leftrightarrow Q(S^p)$ is a one-to-one mapping: the spline S^p can be restored simply via $Q(S^p)$ with the help of formulae (1.2) and (2.1). The mapping $\mathfrak{S}^p(x) \leftrightarrow Q(\mathfrak{S}^p)$ generates a peculiar operational calculus. Let us write the main formulae of this calculus.

Denote $\mathbf{S}^p = \{S^p(x_k)\}_0^{N-1}$. In accordance to the formula (2.1),

$$S^p(x_k) = \frac{1}{N}\sum\nolimits_l q_l M^p(x_k - x_l) = \frac{1}{N}\sum\nolimits_l q_l M^p(x_{k-l}).$$

This is a discrete convolution and from (1.4) we have

$$T_n(\mathbf{S}^p) = T_n(\mathbf{q})u_n^p. \tag{2.2}$$

Proposition 2.1. *If the spline $S^p \in \mathfrak{S}^p$, then its derivative $(S^p)^{(s)} \in \mathfrak{S}^{p-s}$:*

$$S^p(x)^{(s)} = \frac{1}{N}\sum\nolimits_k q_k^s M^{p-s}(x - x_k). \tag{2.3}$$

If we denote $\mathbf{S}^{s,p} = \left\{(S^p(x_k))^{(s)}\right\}_0^{N-1}$, then the following relations hold true:

$$T_n(\mathbf{q}^{\mathbf{S}}) = T_n(\mathbf{q})(\mathrm{i}N\nu)^s, \qquad T_n(\mathbf{S}^{s,p}) = T_n(\mathbf{q})(\mathrm{i}N\nu_n)^s u_n^{p-s}. \tag{2.4}$$

Proof. We obtain from the formulae (2.1), (1.5) the expressions for the Fourier coefficients:

$$\begin{aligned} c_n(S^p) &= \int_0^1 e^{-2\pi \mathrm{i} n y} S^p(y)\,dy = \\ &= \frac{1}{N}\sum\nolimits_k Q_k \int_0^1 e^{-2\pi \mathrm{i} n y} M^p(y - x_k)\,dy = T_n(\mathbf{q})V_n^p. \end{aligned} \tag{2.5}$$

For the derivatives:

$$c_n\left((S^p)^{(s)}\right) = (2\pi \mathbf{i} n)^s T_n(\mathbf{q})V_n^p = (\mathbf{i}N\nu_n)^s T_n(\mathbf{q})V_n^{p-s}. \tag{2.6}$$

Comparing (2.6) and (2.5), we see that $(S^p)^{(s)}$ is a spline from $\mathfrak{S}^{p-s}$, which can be written in the form (2.3) and $T_n(\mathbf{q}^s) = T_n(\mathbf{q})(\mathbf{i}N\nu_n)^s$. The second relation (2.4) follows from the formula (2.2). ∎

Consider now the convolution of two splines. The continuous $f * g$ we understand as the convolution of distributions ([2]). If f, g are integrable 1-periodic functions then $f * g(x) = \int_0^1 f(x-y)g(y)\,dy$. It is well known that for the convolution of two periodic functions

$$c_n(f * g) = c_n(f)c_n(g). \tag{2.7}$$

Let spline $S^l \in \mathfrak{S}^l$:

$$S^l(x) = \frac{1}{N}\sum\nolimits_k r_k M^l(x - x_k), \qquad \mathbf{r} = \{r_k\}_0^{N-1}. \tag{2.8}$$

Proposition 2.2. *The convolution of two splines $S^l \in \mathfrak{S}^l$ and $S^p \in \mathfrak{S}^p$ is a spline $S^{p+l} \in \mathfrak{S}^{p+l}$:*

$$S^{p+l}(x) = \frac{1}{N}\sum\nolimits_k j_k M^{l+p}(x - x_k), \qquad \mathbf{j} = \{j_k\}_0^{N-1}. \tag{2.9}$$

If we denote $\mathbf{S}^{l+p} = \{S^{l+p}(x_k)\}_0^{N-1}$, then there holds the relations

$$T_n(\mathbf{j}) = T_n(\mathbf{q})T_n(\mathbf{r}), \qquad T_n(\mathbf{S}^{p+l}) = T_n(\mathbf{q})T_n(\mathbf{r})u_n^{p+l}. \tag{2.10}$$

Proof. We obtain from the formulae (2.5), (2.7) the expressions for the Fourier coefficients:

$$c_n(S^p * S^l) = c_n(S^p)c_n(S^l) = T_n(\mathbf{q})T_n(\mathbf{r})V_n^{p+l}.$$

Comparing this formula and (2.5), we see that $S^p * S^l$ is a spline $S^{p+l} \in \mathfrak{S}^{p+l}$, which can be written in the form (2.9) and $T_n(\mathbf{j}) = T_n(\mathbf{q})T_n(\mathbf{r})$. The second of the relations (2.10) follows from the formula (2.2). ∎

Remark 2.2. On comparing the formula (2.6) with the formula (2.10), we see, that we can consider the differentiation of order s of the spline S^p as convolution with the spline

$$D^s(x) = \frac{1}{N}\sum\nolimits_k d_k^s M^{-s}(x - x_k) \in \mathfrak{S}^{-s}, \quad T_n(\mathbf{d}^S) = (\mathbf{i}N\nu_n)^s. \tag{2.11}$$ ∎

There also hold the equalities of Parseval type.

Proposition 2.3. *Let $S^l \in \mathfrak{S}^l$, $S^p \in \mathfrak{S}^p$ be the splines defined in* (2.1), (2.8). *Then*

$$\int_0^1 S^p(x)S^l(x)\,dx = \sum\nolimits_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}u_n^{p+l}. \tag{2.12}$$

For the convolution we have

$$\int_0^1 \left(S^p(x) * S^l(x)\right)^2 dx = \sum\nolimits_n |T_n(\mathbf{q})T_n(\mathbf{r})|^2 u_n^{2(p+l)}. \tag{2.13}$$

Hence, in particular

$$\int_0^1 \left(S^p(x)^{(s)}\right)^2 dx = \sum\nolimits_n |T_n(\mathbf{q})|^2 u_n^{2(p-s)} \, (N\nu_n)^{2s}. \tag{2.14}$$

Proof. We can write, using the Parseval equality,

$$\int_0^1 S^p(x)S^l(x)\,dx = \sum_{n=-\infty}^{\infty} c_n(S^p)\bar{c}_n(S^l) = \sum_{n=-\infty}^{\infty} T_n(\mathbf{q})\overline{T_n(\mathbf{r})}V_n^{p+l} =$$
$$= \sum_{l=-\infty}^{\infty} \sum\nolimits_n T_{n+Nl}(\mathbf{q})\overline{T_{n+Nl}(\mathbf{r})}V_{n+Nl}^{p+l}.$$

But $T_{n+Nl}(\mathbf{q}) = T_n(\mathbf{q})$, $T_{n+Nl}(\mathbf{r}) = T_n(\mathbf{r})$. Therefore,

$$\int_0^1 S^p(x)S^l(x)\,dx = \sum\nolimits_n T_{n+Nl}(\mathbf{q})\overline{T_{n+Nl}(\mathbf{r})} \sum_{l=-\infty}^{\infty} V_{n+Nl}^{p+l} =$$
$$= \sum\nolimits_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}u_n^{p+l}$$

in accordance to (1.9). The formulae (2.14) and (2.13) can be obtained from the formulae (2.6) and (2.10). ∎

The formulae (1.3) and (2.2) imply the discrete "Parseval equalities."

Proposition 2.4. *Let $S^l \in \mathfrak{S}^l$ and $S^p \in \mathfrak{S}^p$ be the splines defined in the formulae* (2.1), (2.8). *Then*

$$\frac{1}{N}\sum\nolimits_k S^p(x_k)S^l(x_k) = \sum\nolimits_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}u_n^p u_n^l. \tag{2.15}$$

Hence, in particular:

$$\frac{1}{N}\sum\nolimits_k \left(S^p(x_k)^{(s)}\right)^2 = \sum\nolimits_n |T(\mathbf{q})|^2 (u_n^{p-s})^2 (N\nu)^{2s}. \tag{2.16}$$ ∎

Denote $\widehat{\mathbf{W}}_2^m$ the space consisting of periodic functions f so that $f^{(s)} \in \mathbf{L}_2(0,1)$, $s = 0, \dots, m$. The norm is defined as

$$\|f\|_m = \left(\sum_{i=0}^{m} \int_0^1 \left(f(x)^{(i)} \right)^2 dx \right)^{1/2}.$$

Lemma 2.1. *Let the spline $S^p(x) \in \mathfrak{S}^p$,*

$$S^p(x) = \frac{1}{N} \sum\nolimits_k q_k M^p(x - x_k), \qquad \mathbf{q} = \{q_k\}_0^{N_1}, \qquad \mathbf{S}^p = \{S^p(x_k)\}_0^{N_1}.$$

Then $\|S^p\|_0 \le (\kappa_{p-1})^{-1} \|\mathbf{S}^p\|$.

Proof. The formulae (2.14), (2.16) imply

$$\begin{aligned} \|S^p\|_0^2 &= \int_0^1 (S^p(x))^2 \, dx = \sum\nolimits_n |T_n(\mathbf{q})|^2 u_n^{2p} = \\ &= \sum\nolimits_n |T_n(\mathbf{q})|^2 (u_n^p)^2 \left[u_n^{2p} / (u_n^p)^2 \right] \le (\kappa_{p-1})^{-2} \sum\nolimits_n |T_n(\mathbf{q})|^2 (u_n^p)^2 = \\ &= (\kappa_{p-1})^{-2} \frac{1}{N} \sum\nolimits_k (S^p(x_k))^2 = (\kappa_{p-1})^{-2} \|\mathbf{S}^p\|^2. \qquad \blacksquare \end{aligned}$$

Consider now a problem which is a base for solving a great deal of linear problems, connected with the convolution.

§3. Basic problem.

Problem 3.1. *The splines are $S^m(x) \in \mathfrak{S}^m$,*

$$S^m(x) = \frac{1}{N} \sum\nolimits_k t_k M^m(x - x_k), \tag{3.1}$$

and $S^l(x) \in \mathfrak{S}^l$, which is defined in formula (2.8), as well as a vector $\mathbf{z} = \{z_k\}_0^{N_1}$. We need:

a) To find a spline $S^p(x) \in \mathfrak{S}^p$, which provides a minimum for the functional $I(S^p) = \int_0^1 (S^m(x) * S^p(X))^2 \, dx$ by the condition

$$E(S^p) = \frac{1}{N} \sum\nolimits_k \left[(S^l * S^p)(x_k) - z_k \right]^2 \le \varepsilon^2.$$

b) To find a spline $S^p(x) \in \mathfrak{S}^p$, which provides a minimum for the functional $J_\rho(S^p) = E(S^p) + \rho I(S^p)$. $J_\rho(S^p) = E(S^p)|\rho I(S^p)$.

c) To find a spline $S^p(x) \in \mathfrak{S}^p$, which satisfies the conditions $(S^l * S^p)(x_k) = z_k$, $k = 0, \dots, N-1$. ■

Solving Problem 3.1.b. Denote $\mathbf{r} = \{r_k\}_0^{N_1}$, $\mathbf{t} = \{t_k\}_0^{N_1}$. Let us write the unknown spline $S^p(x)$ in standard form:
$S^p(x) = \frac{1}{N}\sum_k q_k M^p(x - x_k)$. The formula (2.13) implies

$$I(S^p) = \sum_n |T_n(\mathbf{q})T_n(\mathbf{t})|^2 u_n^{2(p+m)}.$$

In view of formulae (1.3) and (2.10) we can write

$$E(S^p) = \sum_n |T_n(\mathbf{q})T_n(\mathbf{r})u_n^{p+l} - T_n(\mathbf{z})|^2. \tag{3.2}$$

The DFT of a vector $\mathbf{a}$ can be represented in the form $T_n(\mathbf{a}) = C_n(\mathbf{a}) - \mathbf{i}S_n(\mathbf{a})$, where $C_n(\mathbf{a})$ and $S_n(\mathbf{a})$ are the cosine- and sine-DFT, correspondingly. Now write the functional

$$\begin{aligned} J_\rho(S^p) &= E(S^p) + \rho I(S^p) = \\ &= \sum_n \left\{\rho\left(C_n(\mathbf{q})^2 + S_n(\mathbf{q})^2\right)\left(C_n(\mathbf{t})^2 + S_n(\mathbf{t})^2\right)u_n^{2(p+m)}\right\} + \\ &+ \left[(C_n(\mathbf{q})C_n(\mathbf{r}) - S_n(\mathbf{q})S_n(\mathbf{r}))u_n^{(p+l)} - C_n(\mathbf{z})\right]^2 + \\ &+ \left[(C_n(\mathbf{q})S_n(\mathbf{r}) + S_n(\mathbf{q})C_n(\mathbf{r}))u_n^{(p+l)} - S_n(\mathbf{z})\right]^2. \end{aligned}$$

It is easy to verify that the functional $J_\rho(S^p)$ reaches its minimum if

$$\begin{aligned} C_n(\mathbf{q}(\rho)) &= \frac{(C_n(\mathbf{r})C_n(\mathbf{z}) + S_n(\mathbf{r})S_n(\mathbf{z}))u_n^{p+l}}{A_n(\rho)}, \\ S_n(\mathbf{q}(\rho)) &= \frac{(S_n(\mathbf{r})C_n(\mathbf{z}) - C_n(\mathbf{r})S_n(\mathbf{z}))u_n^{p+l}}{A_n(\rho)}, \\ A_n(\rho) &= \rho|T_n(\mathbf{t})|^2 u_n^{2(p+m)} + |T_n(\mathbf{r})|^2 (U_n^{p+l})^2. \end{aligned} \tag{3.3}$$

These relations make it possible to formulate the proposition.

Theorem 3.1. *The solution of Problem 3.1.b is the spline*

$$S_\rho^p(\mathbf{z}, x) = \frac{1}{N}\sum_k q_k(\rho)M^p(x - x_k), \qquad \mathbf{q}(\rho) = \{Q_k(\rho)\}_0^{N_1}, \tag{3.4}$$

$$T_n(\mathbf{q}(\rho)) = \frac{\overline{T_n(\mathbf{r})} T_n(\mathbf{z}) u_n^{p+l}}{A_n(\rho)}, \tag{3.5}$$

where the function $A_n(\rho)$ is defined by the formula (3.3). ■

Solving Problem 3.1.a. Denote $e(\rho) = E(S_\rho^p)$. Let o_N be the set of numbers n so that $T_n(\mathbf{r}) = 0$, $\mu_N(\mathbf{z}) = \sum_{n \in o_N} |T_n(\mathbf{z})|^2$, and q_N be the set of numbers n so that $T_n(\mathbf{t}) = 0$.

Lemma 3.1. *If the set q_N is empty then the function*

$$e(\rho) = \sum_n \frac{\rho^2 \left|T_n(\mathbf{t}) T_n(\mathbf{z}) u_n^{2(p+m)}\right|^2}{A_n(\rho)^2} = \sum_{n \notin o_N} + \mu_N(\mathbf{z}) \tag{3.6}$$

increases strictly monotonously, moreover

$$e(o) = \mu_N(\mathbf{z}), \qquad \lim_{\rho \to \infty} e(\rho) = \|\mathbf{z}\|^2. \tag{3.7}$$

Proof. The relations (3.7) follow in an obvious way from (3.6). It is easy to calculate the derivative

$$e'(\rho) = \sum_{n \notin o_N} \frac{2\rho \left|T_n(\mathbf{t}) T_n(\mathbf{r}) T_n(\mathbf{z}) u_n^{2(p+m)}\ u_n^{p+l}\right|^2}{A_n(\rho)^3} > 0, \quad \forall \rho > 0. \tag{3.8}$$ ■

Denote by the symbol $\mathfrak{N}^p$ the set of splines S from $\mathfrak{S}^p$, so that $(S * S^l)(x) \equiv 0$.

Theorem 3.2. *If the set q_N is empty then Problem 3.1.a has a unique solution for every value of ε satisfying the inequalities*

$$\mu_N(\mathbf{z}) \le \varepsilon^2 < \|\mathbf{z}\|^2. \tag{3.9}$$

This solution is the spline $S_P^p(\mathbf{z}, x)$, which is built by the formulae (3.3)–(3.5). The value of the parameter P can be found from the equation $e(P) = \varepsilon^2$.

Proof. Denote by the symbol $\mathfrak{U}_\varepsilon$ the set of splines belonging to $\mathfrak{S}^p$ so that the inequality (3.9) is satisfied. If $S \in \mathfrak{N}^p$ then $E(S) = \|\mathbf{z}\|^2$. Therefore $S \notin \mathfrak{U}_\varepsilon$. In particular, $S(x) \equiv 0 \notin \mathfrak{U}_\varepsilon$. Suppose, that the spline σ

provides $\inf I(S)$ on the space $\mathfrak{S}^p$ and $E(\sigma) = h^2 < \varepsilon^2$. There exists such vicinity $C(\sigma)$ in the N-dimensional space $\mathfrak{S}^p$, that $\forall S \in C(\sigma)$ the inequality $\frac{1}{N}\sum_k \left[((S-\sigma)*S^l)(x_k)\right]^2 < (\varepsilon^2 - h^2)/2$ holds. In view of this it is easy to verify that $S \in \mathfrak{U}_\varepsilon$. Since $0 \notin \mathfrak{U}_\varepsilon$, there exists in this vicinity a spline S_0 so that $I(S_0) < I(\sigma)$. Therefore, $\inf I(S)$ in the space $\mathfrak{S}^p$ cannot be reached on a spline for which $E(S) < \varepsilon^2$. This implies $E(\sigma) = \varepsilon^2$. By the condition (3.9), the equation $e(P) = \varepsilon^2$ has a unique solution and for the corresponding spline $E(S_P^p) = \varepsilon^2$. Therefore, $S_P^p(\mathbf{z}, x)$ is a solution of Problem 3.1.a. ∎

Remark 3.1. Let $\tau = 1/\rho$, $b(\tau) = e(1/\tau)$. It is easy to see, that $b(\tau)$ decreases monotonously and it is convex from below. Therefore in order to find $T = 1/P$ from the equation $b(T) = \varepsilon^2$, the Newton method can be used. ∎

Solving Problem 3.1.c. Suppose, that the set $o_N = \emptyset$ and let us choose $\rho = 0$ in the formulae (3.3)–(3.5). Then we obtain the spline

$$S_0^p(\mathbf{z}, x) = \frac{1}{N}\sum_k q_k(0) M^p(x - x_k), \tag{3.10}$$

$$T_n(\mathbf{q}(0)) = \frac{T_n(\mathbf{z})}{u_n^{p+l}\, T_n(\mathbf{r})}.$$

This spline provides a solution of Problem 3.1.c with arbitrary data vector $\mathbf{z}$. If $\mathfrak{R}^p = \emptyset$ then Problem 3.1.c has unique solution. If $\exists \nu : T_\nu(\mathbf{r}) = 0$, then Problem 3.1.c has a solution only for vectors $\mathbf{z}$ such that $T_\nu(\mathbf{z}) = 0$. This solution is not a unique one. ∎

Point out one particular case. If we choose the spline $S^l(x) = \delta(x)$ (δ-function) then $E(S^p) = \frac{1}{N}\sum_k (S^p(x_k) - z_k)^2$ and the solution of the problem 3.1.c is the spline

$$S_0^p(\mathbf{z}, x) = \frac{1}{N}\sum_k q_k(0) M^p(x - x_k), \qquad T_n(\mathbf{q}(0)) = \frac{T_n(\mathbf{z})}{u_n^p}, \tag{3.11}$$

which interpolates the vector $\mathbf{z}$ at the points $\{x_k\}$.

On the base of Problem 3.1 it is possible to built up the numerical solutions of the series of problems from the list given in the Introduction, especially of the ill-posed problems. Now we discuss one of these problems.

§4. Numerical solution of the inverse problem for the heat equation.

Let us consider the inverse problems for the heat equation in a thin homogeneous ring of radius $R = 1/2\pi$, provided the discrete data with stochastic errors. It is well known, that this problem is ill-posed and demand a regularization. We present now one of such regularizing algorithms, which is based on the SOC methods. Let $x \in [0,1]$ be a distance from any initial point of a ring. Then we can consider all functions involved as 1-periodic by x.

Definition 4.1. The symbol $\mathbb{U}_t$ will denote the linear operator which is defined on the set of 1-periodic functions and $\mathbb{U}_t f(x) = u(x,t)$, where $u(x,t)$ is a function 1-periodic in x, and is a solution of the heat equation $u'_t = u''_{xx}$ with the initial condition $u(x,0) = f(x)$. ∎

Problem 4.1. *Let the function $f \in \widehat{\mathbf{W}}_2^m$ $m \geq 3$. Denote $g(x) = \mathbb{U}_\tau f(x)$, $\tau > 0$ be some given value. We have in our disposal the vector of indications on the interval* $[0,1]$*:*

$$\mathbf{z} = \{z_k = g(x_k) + e_k\}_0^{N-1}, \qquad x_k = k/N; \qquad \mathbf{e} = \{e_k\}_0^{N-1}$$

is a vector of stochastic errors. It is known that $\|\mathbf{e}\| \leq \varepsilon$. Construct a family of functions $f_\varepsilon(N,x) \in \widehat{\mathbf{W}}_2^m$ so that if $\varepsilon \to 0$, $N \to \infty$, then $f_\varepsilon(N,x) \to f(x)$ in the metric of the space $\mathbb{C}^{m-1}$. ∎

The problems of such kind were studied in [5].

Definition 4.2. The symbol $\mathbb{S}_t$ will denote the linear operator which is defined on the set $\mathfrak{S}^p$ and $\mathbb{S}_t(x) = S^p(x,t)$, where the spline $S^p(x,t) \in \mathfrak{S}^p(x)$ satisfies the collocation conditions

$$S^p(x_k,t)'_t = S^p(x_k,t)''_{xx}, \qquad k = 0,\dots,N-1 \tag{4..1}$$

and the initial condition $S^p(x,0) = S^p(x)$. ∎

Let us formulate now an auxiliary problem.

Problem 4.2. *Find a spline*

$$S^p(x,t) = \mathbb{S}_t S^p(x) \in \mathfrak{S}^p(x),$$

$$S^p(x,t) = \frac{1}{N}\sum_k q_k(t) M^p(x-x_k), \qquad \mathbf{q}(t) = \{q_k(t)\}_0^{N-1}, \tag{4..2}$$

if $$S^p(x) = \frac{1}{N}\sum_k q_k M^p(x-x_k), \qquad \mathbf{q} = \{q_k\}_0^{N-1}.$$

Solution of Problem 4.2. Denote $\mathbf{S}(t) = \{S^p(x_k,t)\}_0^{N-1}$, $\mathbf{S}_x(t) = \{S^p(x_x,t)''_{xx}\}_0^{N-1}$. The equality (4.1) is equivalent to $T_n(\mathbf{S}(t))'_t = T_n(\mathbf{S}_x(t))$. From the last one we obtain according to the formula (2.4)

$$T_n(\mathbf{q}(t))'_t = -G_n T_n(\mathbf{q}(t)), \qquad G_n = (N\nu_n)^2 u_n^{p-2}/u_n^p. \tag{4..3}$$

The formula (4.3) implies

$$T_n(\mathbf{q}(t)) = T_n(\mathbf{q})\exp(-G_n t), \qquad n = 0,\dots,N-1. \tag{4.4}$$ ■

Problem 4.3. *Find a spline $S_\rho^p(N,\mathbf{z},x) \in \mathfrak{S}^p(x)$,*

$$S_\rho^p(N,\mathbf{z},x) = \frac{1}{N}\sum\nolimits_k q_k(\rho)M^p(x-x_k), \qquad \mathbf{q}(\rho) = \{q_k(\rho)\}_0^{N-1}, \tag{4..5}$$

which provides a minimum for the functional $J_\rho(S^p) = E(S^p) + \rho I(S^p)$, where $I(S^p) = \sum\limits_{i=0}^{m} \int_0^1 \left(S^p(x)^{(i)}\right)^2 dx = \|S^p\|_m^2$,
$E(S_\rho^p) = \frac{1}{N}\sum_k \left(\mathbb{S}_\tau S^p(x_k) - z_k\right)^2$. ■

Solution of Problem 4.3. Denote $S^0(x) = \frac{1}{N}\sum_k r_k M^0(x-x_k)$, $T_n(\mathbf{r}) = \exp(-G_n\tau)$. Then, as it follows from solving the former problem, we can write $E(S_\rho^p) = \frac{1}{N}\sum_k \left[(S^0 * S^p)(x_k) - z_k\right]^2$. Now we see that this problem is a particular case of the basic problem 3.1.b. Therefore it's possible to write the solution:

$$T_n(\mathbf{q}(\rho)) = \frac{T_n(\mathbf{z})u_n^p\exp(-\tau G_n)}{A_n(\rho)}, \tag{4..6}$$

$$A_n(\rho) = \rho\sum_{i=0}^{m} u_n^{2(p-i)}\left(N\nu_n\right)^{2i} + \left(u_n^p\right)^2\exp(-2G_n\tau).$$ ■

Proposition 4.1. *There is a constructive algorithm for the choice of the parameter $\rho = \rho(\varepsilon, N)$ so that the family of splines $S_\rho^p(\mathbf{z},x,T) \in \mathfrak{S}^p$, $p = 2m$, which is built by the formulae (4.5), (4.6), provides a solution for Problem 4.1.*

The convergence $f_\varepsilon(N,x) \to f(x)$ will be discussed in details at the next paragraph.

§5. Convergence of the approximate solutions.

At this paragraph we show that the approximate solutions of the inverse problem converge to the exact solution, provided the level of the errors diminishes as well as the step of the mesh. A special algorithm will be presented for choosing the regularization parameter ρ. To find the value of the parameter ρ we shall use the generalized discrepancy principle [6], [7]. In order to apply it we need some auxiliary propositions.

As approximate solutions of Problem 4.1 we shall use, according to §4, the splines

$$(5.1)\quad S_\rho^{2m}(N,\mathbf{z},x)=\frac{1}{N}\sum\nolimits_k q_k(\rho)M^{2m}(x-x_k),\qquad \mathbf{q}(\rho)=\{Q_k(\rho)\}_0^{N-1},$$

$$(5.2)\quad T_n(\mathbf{q}(\rho))=\frac{T_n(\mathbf{z})u_n^{2m}\exp(-\tau G_n)}{A_n(\rho)},\qquad G_n=(N\nu_n)^2u_n^{2m-2}/u_n^{2m},$$

$$A_n(\rho)=\rho\sum_{i=0}^{m}u_n^{2(2m-i)}(N\nu_n)^{2i}+(u_n^{2m})^2\exp(-2G_n\tau).$$

Let $S_0^{2m}(\mathbf{f},x)$ be the spline which interpolates the function f at the points $\{x_k\}$, $u(x,t)=\mathbb{U}_tf(x)$, and

$$(5.3)\quad S_0^{2m}(u,x,t)=\frac{1}{N}\sum\nolimits_k Q_k(t)M^{2m}(x-x_k),\qquad \mathbf{Q}(t)=\{Q_k(t)\}_0^{N-1},$$

be the spline which interpolates the function $u(x,t)$ at the points $\{x_k\}$ for $\forall t\geq 0$. If

$$(5.4)\qquad S_0^{2m}(\mathbf{f},x)=\frac{1}{N}\sum\nolimits_k h_kM^{2m}(x-x_k),\qquad \mathbf{h}=\{h_k\}_0^{N-1},$$
$$\mathbb{S}_tS_0^{2m}(\mathbf{f},x)=\frac{1}{N}\sum\nolimits_k h_k(t)M^{2m}(x-x_k),\qquad \mathbf{h}(t)=\{h_k(t)\}_0^{N-1},$$

then, in view of Problem 4.2, we can write

$$(5.5)\qquad T_n(\mathbf{h}(t))=T_n(\mathbf{h})\exp(-G_nt),\qquad T_n(\mathbf{h})=T_n(\mathbf{f})/u_n^{2m},$$

where $\mathbf{f}=\{f(x_k)\}_0^{N-1}$.

Lemma 5.1. *Let $f \in \widehat{\mathbf{W}}_2^m$. Then*

$$\left\| S_0^{2m}(\mathbf{f},x)^{(i)} \right\|_0 \le (\kappa_{2m-1})^{-1} \left\| f^{(i)} \right\|_0, \tag{5.6}$$

$i = 0, \dots, m$, where the constants κ_{2m-1} are defined by the formula (1.7).

Proof. In view of formula (1.9), $u_n^{2i} = \sum_{l=-\infty}^{\infty} V_{n+lN}^{2i}$. Since $V_n^2 \le 1$, it follows from the relation $j > i$, that $u_n^{2j} \le u_n^{2i}$. Recall the well known extremal property of interpolating splines (see, for example, [8]): if $f \in \widehat{\mathbf{W}}_2^m$, then $\left\| S_0^{2m}(\mathbf{f},x)^{(m)} \right\|_0 \le \| f^{(m)} \|_0$. Hence we obtain the statement of our lemma in the case $i = m$. Let $0 < i < m$. Then, in accordance to formulae (1.7), (2.14) and (4.6),

$$\begin{aligned}\left\| S_0^{2m}(\mathbf{f},x)^{(i)} \right\|_0^2 &= \sum_n |T_n(\mathbf{f})|^2 u_n^{2(2m-i)} (N\nu_n)^{2i}/(u_n^{2m})^2 \le \\ &\le (\kappa_{2m-1})^{-2} \sum_n |T_n(\mathbf{f})|^2 u_n^{2i} (N\nu_n)^{2i}/(u_n^{2i})^2 = \\ &= (\kappa_{2m-1})^{-2} \left\| S_0^{2i}(\mathbf{f},x)^{(i)} \right\|_0^2 \le (\kappa_{2m-1})^{-2} \left\| f^{(i)} \right\|_0^2 .\end{aligned}$$

Proposition 5.1 ([8]). *Let $f \in \widehat{\mathbf{W}}_2^m$. Then*

$$S_0^{2m}(\mathbf{f},x)^{(i)} = f^{(i)}(x) + N^{i-m+1/2}\, d_i(x), \qquad i = 0, \dots, m-1, \tag{5.7}$$

$$|d_i(x)| \le d_i \left\| f^{(m)} \right\|_0, \qquad d_i = m^{1/2}(m-1)\dots(i+1)2^{i-m+0.5} . \qquad \blacksquare$$

Lemma 5.2. *Let $f \in \widehat{\mathbf{W}}_2^m$. Then there exist constants $K(N)$, K so that*

$$\| S_0^{2m}(\mathbf{f},x) \|_m^2 \le K(N) \| f \|_m^2 \le K \| f \|_m^2, \quad \lim_{N\to\infty} K(N) = (\kappa_{2m-1})^{-2}. \tag{5.8}$$

As for the constant, one can choose

$$K(N) = (\kappa_{2m-1})^{-2} + \left[(\kappa_{2m-1})^2 / \left(1 - (\kappa_{2m-1})^2 \right) \right] d_0^2 N^{1-2m} .$$

Proof. In accordance to Proposition 5.1

$$\| S_0^{2m}(\mathbf{f},x) \|_0 \le \| f \|_0 + N^{-m+1/2} \| d_0(x) \|_0 \le \| f \|_0 + N^{-m+1/2}\, d_0 \left\| f^{(m)} \right\|_0 .$$

Hence we have, in view of the inequality $2ab \le ca^2 + c^{-1}b^2$,

$$\|S_0^{2m}(\mathbf{f},x)\|_0^2 \le (\kappa_{2m-1})^{-2}\|f\|_0^2 + \\ + \left[(\kappa_{2m-1})^2 \Big/ \left(1-(\kappa_{2m-1})^2\right)\right] d_0^2 N^{1-2m} \left\|f^{(m)}\right\|_0^2 .$$

From the last relation and Lemma 5.1 we obtain

$$\|S_0^{2m}(\mathbf{f},x)\|_m^2 \le (\kappa_{2m-1})^{-2}\|f\|_m^2 + \\ + \left[(\kappa_{2m-1})^2 \Big/ \left(1-(\kappa_{2m-1})^2\right)\right] d_0^2 N^{1-2m} \left\|f^{(m)}\right\|_0^2 \le \\ \le \left[(\kappa_{2m-1})^{-2} + (\kappa_{2m-1})^2 \Big/ \left(1-(\kappa_{2m-1})^2\right)\right] d_0^2 N^{1-2m} \|f\|_m^2 . \qquad \blacksquare$$

Denote

$$B(N,x) = \mathbb{S}_\tau S_0^{2m}(\mathbf{f},x) - S_0^{2m}(u,x,\tau), \qquad \mathbf{B}(N) = \{B(N,x_k\}_0^{N_1} . \tag{5.9}$$

Lemma 5.3. *Let $f \in \widehat{\mathbf{W}}_2^m$, $m \ge 3$. Then there exist constants $L(N)$ so, that*

$$\|\mathbf{B}(N)\|^2 \le N^{5-2m} K(N) L(N) \|f\|_m^2 .$$

Here the constant $K(N)$ is defined in Lemma 5.2,

$$L(N) = (\tau d_2)^2/K(N), \qquad d_2 = m^{1/2}(m-1)\cdots 3\cdot 2^{2.5-m} .$$

Proof. Let us consider the difference $D_k(t) = S_0^{2m}(u,x_k,t)_t' - S_0^{2m}(u,x_k,t)_{xx}''$. It is obvious that $S_0^{2m}(u,x_k,t)_t' = u(x_k,t)_t'$. In accordance to Proposition 5.1,

$$S_0^{2m}(u,x_k,t)_{xx}'' = u(x_k,t)_{xx}'' + N^{2.5-m} d_2(x_k,t), \\ |d_2(x,t)| \le d_2 \left\|u(.,t)^{(m)}\right\|_0 . \tag{5.10}$$

It is well known that the Fourier coefficients of the function $u(x,t)$ are $c_n(u) = c_n(f)\exp(-4\pi^2 n^2 t)$. Therefore

$$\left\|\mathbb{U}_t f^{(m)}\right\|_0 \le \left\|f^{(m)}\right\|_0 \ \forall\, t \ge 0, \tag{5.11}$$

and we obtain finally

$$|d_2(x,t)| \le d_2 \left\|f^{(m)}\right\|_0 . \tag{5.12}$$

Thus,

$$D_k(t) = u(x_k, t)'_t - u(x_k, t)''_{xx} - N^{2.5-m}\, d_2(x_k, t) = \\ = -N^{2.5-m}\, d_2(x_k, t).$$

From here we obtain the relation

$$T_n(\mathbf{Q}(t))'_t + G_n T_n(\mathbf{Q}(t)) = N^{2.5-m}\, \phi_n(t), \tag{5.13}$$

$$\phi_n(t) = -T_n(\mathbf{d}_2(t)), \qquad \mathbf{d}_2(t) = \{d_2(x_k, t)\}_0^{N-1}.$$

Let us remark the inequality

$$\sum\nolimits_n |\phi_n(t)|^2 = \|\mathbf{d}_2(t)\|^2 \le \left(d_2 \left\|f^{(m)}\right\|_0\right)^2. \tag{5.14}$$

Solving the equation (5.13) with the initial condition $T_n(\mathbf{Q}(o)) = T_n(\mathbf{h})$, we obtain

$$T_n(-\mathbf{Q}(t)) = T_n(\mathbf{h}(t)) + N^{2.5-m}\, \Phi_n(t), \\ \Phi_n(t) = \exp(-G_n t) \int_0^t \exp(G_n\theta)\phi_n(\theta)\, d\theta. \tag{5.15}$$

We estimate the value

$$\sum\nolimits_n |\Phi_n(\tau)|^2 = \sum\nolimits_n \left|\exp(-G_n\tau) \int_0^\tau \exp(G_n\theta)\phi_n(\theta)\, d\theta\right|^2 \le \\ \le \sum\nolimits_n \left|\int_0^\tau \phi_n(\theta)\, d\theta\right|^2 \le \tau \sum\nolimits_n \int_0^\tau |\phi_n(\theta)|^2\, d\theta = \\ = \tau \int_0^\tau \sum\nolimits_n |\phi_n(\theta)|^2\, d\theta \le \left(\tau d_2 \left\|f^{(m)}\right\|_0\right)^2. \tag{5.16}$$

We can write now, in view of formula (2.14),

$$\|\mathbf{B}(N)\|^2 = \sum\nolimits_n |T_n(\mathbf{Q}(\tau)) - T_n(\mathbf{h}(\tau)))(u_n^{2m})|^2 = \\ = N^{5-2m} \sum\nolimits_n |\Phi_n(\tau) u_n^{2m}|^2 \le N^{5-2m} \left(\tau d_2 \left\|f^{(m)}\right\|_0\right)^2 \le \\ \le N^{5-2m}\, (\tau d_2)^2 \|f\|_m^2.$$

Hence the assertion of the lemma follows. ∎

Lemma 5.4. *If there holds the inequality* $\|\mathbf{e}\| \le \varepsilon$, *then*

$$E(S_0^{2m}(\mathbf{f},x)) = \frac{1}{N}\sum\nolimits_k \left(S_0^{2m}(\mathbf{f},x_k,\tau) - z_k\right)^2 \le$$
$$\le \left[\varepsilon + \left(N^{5-2m} K(N)L(N)\|f\|_m^2\right)^{1/2}\right]^2.$$

Proof. By the condition of the Lemma, $z_k = g(x_k)+e_k = S_0^{2m}(u, x_k, \tau)+e_k$, therefore

$$\sqrt{E(S_0^{2m}(\mathbf{f},x))} \le \|\mathbf{B}(N)\| + \|\mathbf{e}\|.$$

Hence the assertion of the lemma follows. ∎

Let $e(\rho) = E(S_\rho^{2m})$, $a(\rho) = I(S_\rho^{2m}) = \|S_\rho^{2m}\|_m^2$,

$$a(\rho) = \sum_{i=0}^{m} \int_0^1 \left(S_\rho^{2m}(x)^{(i)}\right)^2 dx =$$
$$= \sum\nolimits_n \sum_{i=0}^{m} u_n^{2(2m-i)} (N\nu_n)^{2i} \frac{|T_n(\mathbf{r})T_n(\mathbf{z})u_n^{2m+l}|^2}{A_n(\rho)^2}.$$

Point out that $T_n(\mathbf{r}) = \exp(-G_n\tau) > 0$, therefore $\mu_n(\mathbf{z}) = 0\ \forall N$. As it was remarked above, the function $e(\rho)$ increases strictly monotonously and

$$e(0) = 0, \qquad \lim_{\rho\to\infty} e(\rho) = \|\mathbf{z}\|^2.$$

It is straightforward to verify that the function $a(\rho)$ decreases strictly monotonously and

$$\lim_{\rho\to 0} \rho a(\rho) = 0, \qquad \lim_{\rho\to\infty} a(\rho) = 0. \tag{5.19}$$

Denote

$$Q(N,\varepsilon,\rho) = e(\rho) - \left[\varepsilon + (N^{5-2m} L(N)a(\rho))^{1/2}\right]^2.$$

Theorem 5.1. *If for a given vector* $\mathbf{z}$ *the condition*

$$\varepsilon^2 < \|\mathbf{z}\|^2 \tag{5.20}$$

is satisfied, then the equation $Q(N,\varepsilon,\rho)=0$ has the unique solution $\rho(N,\varepsilon)$ with respect to the argument ρ.

Proof. The function $Q(N,\varepsilon,\rho)$ increases monotonously, and $Q(N,\varepsilon,0)<-\varepsilon^2$, $\lim_{\rho\to\infty} Q(N,\varepsilon,\rho)=\|\mathbf{z}\|^2-\varepsilon^2$. Hence we obtain the statement of our theorem. ■

Let N_n be a sequence of natural numbers so that $\lim_{n\to\infty} N_n=\infty$, and ε_n be a sequence of positive numbers so that $\lim_{n\to\infty}\varepsilon_n=0$. Denote $f_\rho(x)=S_\rho^{2m}(N,\mathbf{z},x)$, where the spline S_ρ^{2m} is defined by formula (5.1).

We can prove the main result of this paragraph.

Theorem 5.2. *Let the function $f\in\widehat{\mathbf{W}}_2^m$ $m\geq 3$, the periodic function $u(x,t)$ be a solution of the equation $u'_t=u''_{xx}$ with the initial condition $u(x,0)=f(x)$, $g(x)=u(x,\tau)$, and $x_k^n=\frac{k}{N_n}$, $\mathbf{z}^n=\{z_k^n\}_0^{N_n-1}$, $n=1,2,\ldots$, be two sequence of vectors so that $\frac{1}{N_n}\sum_k\left(g(x_k^n)-z_k^n\right)^2\leq\varepsilon_n^2$. If for an arbitrary value of n the relations*

$$\varepsilon_n^2\leq\|\mathbf{z}^n\|^2 \tag{5.21}$$

are true, and ρ_n is a root of the equation

$$e(\rho)=\left[\varepsilon_n+\left(N_n^{-2m+5}\,L(N_n)a(\rho)\right)^{1/2}\right]^2, \tag{5.22}$$

then $f_{\rho_n}\to f$ in the metric of the space $\mathbb{C}^{m-1}$.

Proof. Since the spline $f_\rho(x)$ minimizes the functional $J_\rho=\rho I+E$ on the space $\mathfrak{S}^{2m}$,

$$\begin{aligned}J_\rho(f_\rho)&\leq J_\rho(S_0^{2m}(\mathbf{f},x))=\\&=\rho\|S_0^{2m}(\mathbf{f},x)\|_m^2+\frac{1}{N}\sum\nolimits_k\left(\mathbb{S}_\tau S_0^{2m}(bff,x_k)-z_k\right)^2\leq\\&\leq\rho K(N)\|f\|_m^2+\left[\varepsilon+\|f\|_m\left(N^{-2m+5}\,K(N)L(N)\right)^{1/2}\right]^2.\end{aligned}$$

But, if $\rho=\rho_n$, and ρ_n is a root of the equation (5.22), we have

$$\begin{aligned}&\rho_n\|f_{\rho_n}\|_m^2+\left[\varepsilon_n+\|f_{\rho_n}\|_m\left(N_n^{-2m+5}\,L(N_n)\right)^{1/2}\right]^2\leq\\&\leq\rho_nK(N_n)\|f\|_m^2+\left[\varepsilon_n+\|f\|_m\left(N_n^{-2m+5}\,K(N_n)L(N_n)\right)^{1/2}\right]^2.\end{aligned}$$

Hence it follows directly

$$\|f_{\rho_n}\|_m^2 \leq K(N_n)\|f\|_m^2 \leq K\|f\|_m^2.$$

Therefore the set of functions $\{f_{\rho_n}\}$, $n = 1, 2, \ldots$, belongs to a sphere in the space $\widehat{\mathbf{W}}_2^m$ and, consequently, is compact in the space $\mathbb{C}^{m-1}$. Therefore we can choose from the sequence $\{f_{\rho_n}\}$ any subsequence $\{f_{\rho_n}^k\}$ which converges in the space $\mathbb{C}^{m-1}$ to any element $\tilde{f}$.

Let us show that $\tilde{f} = f$. Suppose $\tilde{u}(x,t) = \mathbb{U}_t\tilde{f}(x)$, $\tilde{g}(x) = \tilde{u}(x,\tau)$, $u_n(x,t) = \mathbb{U}_t(f_{\rho_n}(x)$, $g_n(x) = u_n(x,\tau)$. Let, as above, $S_0^{2m}(u,x,t)$ be the spline interpolating the function $u(x,t)$ on the mesh $\{x_k^n\}$. We may write

$$\|\mathbb{U}_\tau\tilde{f} - \mathbb{U}_\tau f\|_0 \leq \sum_{k=1}^{5} I_k,$$

$$I_1 = \|\mathbb{U}_\tau\tilde{f} - \mathbb{U}_\tau f_{\rho_n}\|_0, \qquad I_2 = \|\mathbb{U}_\tau f_{\rho_n} - S_0^{2m}(u_n,x,\tau)\|_0,$$
$$I_3 = \|\mathbb{S}_\tau f_{\rho_n} - S_0^{2m}(u_n,x,\tau)\|_0, \qquad I_4 = \|\mathbb{S}_\tau f_{\rho_n} - S_0^{2m}(u,x,\tau)\|_0,$$
$$I_5 = \|\mathbb{U}_\tau f - S_0^{2m}(u_n,x,\tau)\|_0.$$

Consider the addends I_k separately. The relations $I_2 \underset{n\to\infty}{\longrightarrow} 0$, $I_5 \underset{n\to\infty}{\longrightarrow} 0$ are obvious. There holds the inequality $I_1 \leq \|\mathbb{U}_\tau\| \, \|\tilde{f} - f_{\rho_n}^k\|_0 \leq \|\tilde{f} - f_{\rho_n}^k\|_0$ which follows directly from the formula (5.11). Therefore $I_1 \underset{n\to\infty}{\longrightarrow} 0$. It follows from Lemma 2.1 that

$$\begin{aligned}
I_4^2 &\leq (\kappa_{p-1})^{-2}\frac{1}{N}\sum\nolimits_k \left(\mathbb{S}_\tau f_{\rho_n}(x_k^n) - S_0^{2m}(u,x_k^n,\tau)\right)^2 = \\
&= (\kappa_{p-1})^{-2}\frac{1}{N}\sum\nolimits_k \left(\mathbb{S}_\tau f_{\rho_n}(x_k^n) - u(x_k^n,\tau)\right)^2 = \\
&= (\kappa_{p-1})^{-2}\frac{1}{N}\sum\nolimits_k \left(\mathbb{S}_\tau f_{\rho_n}(x_k^n) - z_k^n + e_k^n)\right)^2 \leq \\
&\leq (\kappa_{p-1})^{-2}\left(e(\rho_n)^{1/2} + \varepsilon_n\right)^2 = \\
&= (\kappa_{p-1})^{-2}\left[2\varepsilon_n + \left(N_n^{2m+5}L(N_n)a(\rho_n)\right)^{1/2}\right]^2 \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

As for the addend I_3 — repeating the speculations of Lemma 5.3, we can obtain the inequality $I_3^2 \leq N^{5-2m}K(N)L(N)\|f_{\rho_n}\|_m^2 \underset{n\to\infty}{\longrightarrow} 0$, since $\|f_{\rho_n}\|_m^2$ are bounded.

Thus $\mathbb{U}_\tau\tilde{f} = \mathbb{U}_\tau f$ and, since the kernel of the operator $\mathbb{U}_\tau$ consist of zero only, $\tilde{f} = f$. This fact holds for each subsequence of the sequence $\{f_{\rho_n}\}$. Therefore the entire sequence $\{f_{\rho_n}\}$ converges to the function f. ■

Thus, if we choose the values of the parameter ρ according to the scheme given above, then the family of splines f_{ρ_n} yields the solution of Problem 4.1.

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