Periodic Splines, Harmonic Analysis and Wavelets

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Abstract. We discuss here wavelets constructed from periodic spline functions. Our approach is based on a new computational technique named Spline Harmonic Analysis (SHA). SHA to be presented is a version of harmonic analysis operating in the spaces of periodic splines of defect 1 with equidistant nodes. Discrete Fourier Transform is a special case of SHA. The continuous Fourier Analysis is the limit case of SHA as the degree of splines involved tends to infinity. Thus SHA bridges the gap between the discrete and the continuous versions of the Fourier Analysis. SHA can be regarded as a computational version of the harmonic analysis of continuous periodic functions from discrete noised data. SHA approach to wavelets yields a tool just as for constructing a diversity of spline wavelet bases, so for a fast implementation of the decomposition of a function into a fitting wavelet representation and its reconstruction. Via this approach we are able to construct wavelet packet bases for refined frequency resolution of signals. In the paper we present also algorithms for digital signal processing by means of spline wavelets and wavelet packets. The algorithms established are embodied in a flexible multitasking software for digital signal processing.

§1 Introduction

The objective of the paper is the presentation of techniques of adaptive signal processing based on the spline wavelet analysis.

At present most popular wavelet schemes are based on the compactly supported orthonormal wavelet bases invented by Ingrid Daubechies [9]. Exploiting the so called wavelet packets [12], [7], provides essential advantages because these generate a library of bases and provide opportunities for adaptive representation of signals. It should be pointed out that the compact support of basic wavelets and the orthonormality of corresponding wavelet bases are not compatible with the symmetry of the wavelets concerned. The lack of the symmetry is a noticeable handicap when using the wavelets by Daubechies for signal processing. To attain the symmetry one should sacrifice at least one of these properties. In [8] the authors had constructed biorthogonal bases of compactly supported symmetric wavelets. However, certain inconvenience of the construction lies in the fact that dual wavelets belong to different wavelet spaces.

Early examples of wavelets were based on spline functions [11], [1], [10]. Later spline wavelets were shadowed by the wavelets by Daubechies. However,

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in numerous situations spline wavelets offer advantages before these latter. In recent years spline-wavelet were subjected to the detailed study, mainly by C. Chui with collaborators [2] – [7]. In particular, in [3] the authors succeeded in constructing compactly supported wavelets in the spaces of cardinal splines. The significant property of such spline wavelets, in addition to the symmetry, is that their dual wavelets belong to the same spaces as the original ones.

We discuss in this paper periodic spline wavelets. Our approach to the spline wavelet analysis is based on an original computational technique – the so called Spline Harmonic Analysis (SHA) which is a version of the Harmonic Analysis (HA) in spline spaces. SHA in some sense bridges the gap between the continuous and discrete HA. It is rather universal technique applicable to a great variety of numerical problems, not necessarily to wavelet analysis [21]– [24]. Application of the SHA techniques to wavelet analysis is found to be remarkably fruitful. This approach has given a chance to construct a rich diversity of wavelet bases as well as wavelet packet ones. Moreover, we present an efficient scheme of decomposition into the wavelet representation and reconstruction of a signal, based on SHA.

The paper consists of Introduction and four sections. In Section 2 we outline the properties of splines with equidistant nodes which will be of use for wavelet analysis, especially the properties of $\mathcal{B}$-splines. Section 3 is devoted to the presentation of the SHA. In Section 4 we discuss the multiscale analysis of splines. We establish the two–scale relation, construct orthogonal bases in spline wavelet spaces and present a spectral algorithm for decomposition a spline into frequency multichannel representation and reconstruction from this representation. We introduce the high– and low–frequency wavelet spaces. In Section 5 we construct the families of *father* and *mother* wavelets as well as the spline wavelet packets. We discuss methods of the most informative digital representation of signals by means of spline wavelets. There established a useful quadrature formula.

Some results presented in the paper have been announced in the papers [16], [17], [18]. On thee base of algorithms established in the paper we have developed a flexible multitasking software for digital signal processing by means of spline wavelets and wavelet packets. The software allows to process periodic signals as well as non-periodic ones in the real time mode.

§2 Splines with equidistant nodes

This section is an introductory one. We outline here properties of polynomial splines with equidistant nodes most of which are known [16], [17]. A function ${}_pS(x)$ will be referred to as a spline of order p if

$$
\begin{aligned}
1)\, {}_pS(x) &\in \mathcal{C}^{p-2} \\
2)\, {}_pS(x) &= P_k(x) \quad \text{as}\, x \in (x_k, x_{k+1}), \quad P_k(x) \in \Pi_{p-1}
\end{aligned}
$$

Π_{p-1} is the space of polynomials whose degree doesn't exceed $p-1$. In what follows we deal exclusively with splines whose nodes $\{x_k\}$ are equidistant $x_k = hk$, $k = -\infty, ..., \infty$ and denote this space by ${}_p\mathcal{S}$. These splines are referred to as *cardinal* ones. The most significant advantages of these splines over others are originated from the fact that in the space ${}_p\mathcal{S}$ there exist bases each of which consists of translates of a unique spline. One of such bases is of most importance for our subject. We mean the basis of the $\mathcal{B}$-splines.

2.1 The $\mathcal{B}$-splines

We define the truncated powers as $x_+^k = \left(\frac{1}{2}(x+|x|)\right)^k$.

The following linear combination of truncated powers:

$$ {}_pB_h(x) = \frac{h^{-p}}{(p-1)!}\sum_{l=0}^{p}(-1)^l\binom{p}{l}(x-lh)_+^{p-1}. $$

is referred to as the $\mathcal{B}$-spline. It is a spline of order p with nodes in the points $\{hk\}_{-\infty}^{\infty}$.

Properties of $\mathcal{B}$-splines.

1. $\operatorname{supp} {}_pB_h(x) = (0, hp)$.

2. ${}_pB_h(x) > 0$ as $x \in (0, ph)$.

3. ${}_pB_h(x)$ *is symmetric about* $x = hp/2$ *where it attains its unique maximum.*

4. $\int_{-\infty}^{\infty} {}_pB_h(x)\,dx = \int_0^{ph} {}_pB_h(x)\,dx = 1.$

Point out a property concerned with discrete values of $\mathcal{B}$-splines. Define discrete and continuous moments of $\mathcal{B}$-splines

$$ \mu_s({}_pB_h)(t) = h\sum_{r=-\infty}^{\infty}\left(h\Big(t+r-\frac{p}{2}\Big)\right)^s {}_pB(h(t+r)) $$

$$ M_s({}_pB_h) = \int_0^{ph}(x-hp/2)^s\,{}_pB_h(x)\,dx. $$

Proposition 1. *[([17],[18]] Provided* $s \le p-1$*, the discrete moments* $\mu_s({}_pB_h)(t)$ *does not depend on* t *and coincide with the corresponding continuous moments* $M_s({}_pB_h)$*. The moment*

$$ \mu_p({}_pB_h)(t) = (-1)^{p-1}\beta_h(t)h^p + M_p({}_pB_h) \quad \text{as } t \in [0,1] $$

and 1-periodic with respect to t*;* $\beta_p(t)$ *is the Bernoulli polynomial of degree* p

This property will be of use in Section 5 to derive an important quadrature formula.

The $\mathcal{B}$-splines of any order can be computed immediately.

Define a function which will be of important concern in what follows

$$_pu_h(\omega) = h\sum_k e^{-i\omega hk}\, _pB((k+p/2)h). \tag{2.1.1}$$

Due to the symmetry of B-splines, the functions $_pu_h(\omega)$ are real-valued cosine polynomials. These functions were extensively studied in [14], [15]. They are related to the Euler-Frobenius polynomials [13].

Proposition 2. *The functions $_pu_h(\omega)$ are strictly positive, moreover*

$$0 < K_p = \, _pu_h(\pi/h) \le \, _pu_h(\omega) \le \, _pu_h(0) = 1.$$

The constants K_p do not depend on h and $\lim_{p\to\infty} K_p = 0$.

The Fourier Transform of the $\mathcal{B}$-spline is:

$$_p\widehat{B}_h(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\, _pB_h(t)\,dt \tag{2.1.2}$$

$$= \Big(\frac{1-e^{-i\omega h}}{i\omega h}\Big)^p = e^{-\frac{ip\omega h}{2}}\Big(\frac{\sin\omega h/2}{\omega h/2}\Big)^p. \tag{2.1.3}$$

2.2 The $\mathcal{B}$-spline representation of cardinal splines

Recall that $_p\mathcal{S}$ denotes the space of cardinal splines of order p with their nodes at the points $\{hk\}_{-\infty}^{\infty}$.

Proposition 3. *[13] Any spline $_pS(x) \in \, _p\mathcal{S}$ can be represented as follows*

$$_pS(x) = h\sum_k q_k\, _pB_h(x-hk). \tag{2.2.1}$$

Here and below $\sum_k$ stands for $\sum_{k=-\infty}^{\infty}$.

Remark 1. If x is any fixed value then the series(2.2.1) contains only p nonzero addends. So, given a set of coefficients $\{q_k\}$, values of the spline $_pS(x)$ can be computed immediately.

spline. Provided the coefficients $\{q_k\}_{k=-\infty}^{\infty} \in l_2$, the Fourier Transform of the spline $_pS(x)$ is

$$_p\widehat{S}(x) = \int_{-\infty}^{\infty} e^{-i\omega x} h \sum_{k=-\infty}^{\infty} q_k\, _pB_h(x-hk)$$

$$
\begin{aligned}
&= h\sum_k e^{-i\omega hk} q_k\ {}_p\widehat{B}_h(\omega) = \check{q}(\omega)e^{-\frac{ip\omega h}{2}}\Big(\frac{\sin\omega h/2}{\omega h/2}\Big)^p, \\
\check{q}(\omega) &= h\sum_k e^{-i\omega hk} q_k.
\end{aligned}
$$

So, we can say that the approximation of a signal by a spline is, as a matter of fact, a kind of low-pass filtering. Actually, the signal is being filtered into the band $[\frac{-1}{2h}, \frac{1}{2h}]$.

§3 Spline Harmonic Analysis

Harmonic Analysis (HA) is a powerful tool of mathematics for solving a great diversity of theoretical and computational problems. HA techniques are best suited for solving problems associated with the operators of convolution and of differentiation. It stems from the fact that the basic functions of the conventional HA – the exponential functions – are eigenvectors of these operators.

However, the conventional HA is not quite relevant for dealing with signals of finite order of smoothness determined by a finite set of functionals (may be noised) because of at least two reasons: 1) The basic functions of HA – the exponential functions – are infinitely differentiable. 2) Practical computing the coordinates – the Fourier coefficients or the Fourier integrals – put a lot of problems.

To circumvent these obstacles it would be attractive to have a version of HA which would deal with discrete issue data and would provide solutions of problems as immediately computable functions of the smoothness required.

We present here such version of HA. It is based on periodic splines and we name it the Spline Harmonic Analysis (SHA).

3.1 Periodic splines

We introduce some notations. In what follows we will assume the step of a mesh involved to be $h = 1/N$, $N = 2^j$. Throughout $\sum_k^j$ stands for $\sum_{k=-2^{j-1}}^{2^{j-1}-1}$. If a sequence is furnished with the upper index j it will imply that it is 2^j-periodic (e.g. $\{u_k^j\}$). Throughout we denote $\omega = e^{2\pi i/N}$. The direct and inverse Discrete Fourier Transform (DFT) of a vector $\vec{a} = \{a_k\}$ is

$$
\mathcal{T}_r^j(\vec{a}) = \frac{1}{N}\sum_k^j \omega^{-rk} a_k \qquad a_k = \sum_r^j \omega^{rk}\mathcal{T}_r^j(\vec{a}). \tag{3.1.1}
$$

The discrete convolution of the vector $\vec{a}$ with a vector $\vec{b} = \{b_k\}$ and its DFT are

$$
\vec{a} * \vec{b} = \Big\{2^{-j}\sum_l^j a_{k-l}b_l\Big\}_k^j \qquad \mathcal{T}_r^j(\vec{a} * \vec{b}) = \mathcal{T}_r^j(\vec{a})\cdot\mathcal{T}_r^j(\vec{b}).
$$

To unify the notations in what follows we will denote the $\mathcal{B}$-spline ${}_pB_{1/N}(x)$ as ${}_pB_j(x)$

Any 1-periodic cardinal spline of order p can be represented as follows

$$ {}_pS^j(x) = \frac{1}{N} \sum_k^j q_k^j \; {}_pM^j(x - k/N) \tag{3.1.2} $$

where

$$ {}_pM^j(x) =: \sum_l {}_pB_j(x - l) $$

is an 1-periodic spline. We name it the periodic $\mathcal{B}$-spline. The sequence of the coefficients $\{q_k^j\}$ is N-periodic.

The properties of $\mathcal{B}$-splines ${}_pM^j$ are being determined completely by these of $\mathcal{B}$-splines ${}_pB_j$. It should be noted only that if $p > N$ then the supports of adjacent splines ${}_pB_j(x + l)$ and ${}_pB_j(x + l + 1)$ overlap and, therefore, the support of the periodic spline ${}_pM^j$ has no gaps in this case.

As for spectral properties, due to the periodicity, the $\mathcal{B}$-spline ${}_pM^j$ is being expanded into the Fourier series:

$$ {}_pM^j(x) = \sum_n e^{2\pi inx} \, e^{-\pi inp/N} \Big(\frac{\sin \pi n/N}{\pi n/N} \Big)^p . $$

This relation implies that if a spline ${}_pS^j$ is represented as in (3.1.2) then its Fourier coefficients are

$$ C_n({}_pS^j) = \int_0^1 e^{-2\pi inx} \, {}_pS^j(x) \, dx \tag{3.1.3} $$

$$ = e^{-\pi inp/N} \Big(\frac{\sin \pi n/N}{\pi n/N} \Big)^p \mathcal{T}_n^j(\vec{q}^j) = \Big(\frac{1 - \omega^{-n}}{2\pi in/N} \Big)^p \mathcal{T}_n^j(\vec{q}^j). $$

Recall that DFT $\{\mathcal{T}_n^j(\vec{q}^j)\}$ form an N-periodic sequence.

Denote by ${}_p\mathcal{V}^j$ the space of 1-periodic splines of order p with their nodes in the points $\{k2^{-j}\}_{-\infty}^{\infty}$. The relation (3.1.2) implies that the shifts $\{ {}_pM^j(x - k/2^j\}_k^j$ form a basis of ${}_p\mathcal{V}^j$

3.2 Periodic ortsplines

To start constructing SHA we carry out a simple transformation. Let a spline ${}_pS^j \in {}_p\mathcal{V}^j$ be represented as follows

$$ {}_pS^j(x) = \frac{1}{N} \sum_k^j q_k \; {}_pM^j(x - k/N). \tag{3.2.1} $$

Due to (3.1.1) we can write

$$q_k = \sum_r^j \omega^{rk} \mathcal{T}_r^j(\vec{q})$$

where $\vec{q} = \{q_k\}_k^j$. Substituting it to (3.2.1) we come to the relation

$$\begin{aligned} {}_pS^j(x) &= \frac{1}{N}\sum_k^j {}_pM^j(x-k/N)\sum_r^j \omega^{rk}\mathcal{T}_r^j(\vec{q}) \\ &= \sum_r^j \mathcal{T}_r^j(\vec{q})\frac{1}{N}\sum_k^j \omega^{rk}\,{}_pM^j(x-k/N). \end{aligned}$$

Setting

$$\xi_r = \mathcal{T}_r^j(\vec{q}) = \frac{1}{N}\sum_k^j \omega^{-rk} q_k, \tag{3.2.2}$$

$${}_pm_r^j(x) = \frac{1}{N}\sum_k^j \omega^{rk}\,{}_pM^j(x-k/N), \quad r = 0, ..., 2^j - 1, \tag{3.2.3}$$

we write finally

$${}_pS^j(x) = \sum_r^j \xi_r\ {}_pm_r^j(x).$$

Point out at once the reciprocal relations

$$q_k = \sum_r^j \omega^{rk}\xi_r \tag{3.2.4}$$

$${}_pM^j(x-k/N) = \sum_r^j \omega^{-rk}\,{}_pm_r^j(x), \quad ,_p M^j(x) = \sum_r^j {}_pm_r^j(x). \tag{3.2.5}$$

We take some time to discuss properties of the splines ${}_pm_r^j$ which are basic for our constructions.

Proposition 4. *The sequence $\{{}_pm_r^j(x)\}$ is N-periodic with respect to r.*

It follows immediately from 1-periodicity of the $\mathcal{B}$-splines ${}_pM^j(x)$.

Eq. (3.2.5) implies

Proposition 5. *The splines* $\{{}_pm_r^j(x)\}_r^j$ *form a basis of the space* ${}_p\mathcal{V}^j$.

It will be of use to compute the Fourier coefficients. Keeping in mind (3.1.3) we write

$$C_n({}_pm_r^j) = e^{-\pi inp/N}\Big(\frac{\sin \pi n/N}{\pi n/N}\Big)^p \cdot \frac{1}{N}\sum_k \omega^{-k(n-r)} \tag{3.2.6}$$

$$= \delta_n^r(\mathrm{mod}N)\cdot e^{-\pi inp/N}\Big(\frac{\sin \pi n/N}{\pi n/N}\Big)^p$$

$$= \delta_n^r(\mathrm{mod}N)(1-\omega^{-r})^p\frac{1}{(2\pi in/N)^p}.$$

So,

$${}_pm_r^j(x) = (1-\omega^{-r})^p\sum_l e^{2\pi i(r+lN)x}\frac{1}{(2\pi i(r+lN)/N)^p} \tag{3.2.7}$$

$$= \Big(\frac{N(1-\omega^{-r})}{2\pi i}\Big)^p e^{2\pi irx}\sum_l e^{2\pi ilNx}\frac{1}{(r+lN)^p}$$

$$= \sum_l e^{2\pi i(r+lN)(x-p/2N)}\Big(\frac{\sin \pi(r+lN)/N}{(\pi(r+lN))/N}\Big)^p$$

$$= e^{2\pi i(x-p/2N)r}(\sin \pi r/N)^p\sum_l e^{2\pi ilNx}(\pi(r+lN)/N)^{-p}.$$

Let us derive some consequences from (3.2.6), (3.2.7). First we denote

$${}_pu_r^j =: {}_pm_r^j(p/2N) = \frac{1}{N}\sum_k^j \omega^{-rk}\,{}_pM((p/2+k)/N).$$

Substituting the identity ${}_pM^j(x) = \sum_l {}_pB_j(x-l)$ into the latter relation we obtain

$$\begin{aligned} {}_pu_r^j &= \frac{1}{N}\sum_k^j e^{-2\pi irk/N}\sum_{l=-\infty}^{\infty} {}_pB_j((p/2+k+lN)/N) \\ &= \sum_{n=-\infty}^{\infty} e^{-2\pi irn/N}\,{}_pB_j\Big(\Big(\frac{p}{2}+n\Big)/N\Big) = {}_pu_{1/N}(2\pi r). \end{aligned}$$

Recall that the function ${}_pu_h(\omega)$ was defined in (2.1.1). So, dealing with the N-periodic sequence ${}_pu_r^j$ we can compute it immediately and may refer to the properties ${}_pu_h(\omega)$ marked in Section 2. This sequence is of importance for us. Substitution $x = p/2N$ into (3.2.7) results in the identity

$${}_pu_r^j = \sum_l\Big(\frac{\sin \pi(r+lN)/N}{(\pi(r+lN)/N)}\Big)^p = (N\sin \pi r/N)^p\sum_l\frac{(-1)^{lp}}{(\pi(r+lN))^p}.$$

The Parseval equality for the Fourier series entails the important relation

Proposition 6. *The inner product is*

$$\langle\, {}_p m_r^j,\, {}_p m_s^j \rangle = \delta_r^s \; {}_{2p} u_r^j.$$

Corollary 1. *The splines* $\{\, {}_p m_r^j \}_r^j$ *form an orthogonal basis of* ${}_p\mathcal{V}^j$ *and the splines* $\left\{ {}_p^1 m_r^j = {}_p m_r^j / \sqrt{{}_{2p} u_r^j} \right\}_r^j$ *form an orthonormal one.*

Proposition 7. *The splines* ${}_p m_r^j$ *are eigenvectors to the shift operator. To be specific*

$$ {}_p m_r^j (x + l/N) = \omega^{rl} \; {}_p m_r^j (x). \tag{3.2.8}$$

Proof: Eq. (3.2.3) implies

$$\begin{aligned} {}_p m_r^j (x + l/N) &= \frac{1}{N} \sum_k^j \omega^{rk} \; {}_p M^j \Big(x - \frac{(k-l)}{N} \Big) \\ &= \omega^{rl} \frac{1}{N} \sum_k^j \omega^{rk} \; {}_p M^j (x - k/N) = \omega^{rl} \; {}_p m_r^j (x). \end{aligned}$$

■

Corollary 2. *The splines* ${}_p^2 m_r^j (x) = {}_p m_r^j (x + p/2N) / {}_p u_r^j$ *interpolate the exponential functions* $\mu_r(x) = e^{2\pi i r x}$*. Namely,*

$${}_p^2 m_r^j (l/N) = \mu_r (l/N).$$

Indeed, (3.2.8) implies

$${}_p m_r^j ((p/2 + l)/N) = e^{2\pi i r l/N} \; {}_p m_r^j (p/2N) = e^{2\pi i r l/N} \; {}_p u_r^j.$$

Let $\sigma_n^j (x) = \sum_r^j {}^n\xi_r \; {}_p m_r^j (x)$ be an orthogonal projection of the exponential function $\mu_n(x)$ onto the space ${}_p\mathcal{V}^j$. Then

$$\begin{aligned} {}^n\xi_r \langle\, {}_p m_r^j,\, {}_p m_s^j \rangle &= \langle \mu_n,\, {}_p m_s^j \rangle \quad \Longleftrightarrow \\ {}^n\xi_r \delta_r^s \; {}_{2p} u_r^j &= \overline{C_n} (\, {}_p m_s^j) \quad \Longrightarrow \\ {}^n\xi_r &= \delta_n^r (\mathrm{mod} N) \cdot e^{\pi i n p/N} \Big(\frac{\sin \pi n/N}{\pi n/N} \Big)^p \frac{1}{{}_{2p} u_r^j}. \end{aligned}$$

Hence it follows

Proposition 8. *The spline*

$$_p\sigma_n^j(x) = \delta_n^r(\mathrm{mod}N) \cdot e^{\pi inp/N} \left(\frac{\sin \pi n/N}{\pi n/N}\right)^p \frac{_pm_r^j(x)}{_{2p}u_r^j}.$$

is an orthogonal projection of the exponential function $\mu_n(x)$ onto the space $_p\mathcal{V}^j$ provided $n = r(\mathrm{mod}N)$.

Denote $^3_pm_r^j(x) = {}_p\sigma_r^j(x)$.

The latter properties relate the splines $_pm_r^j$ to the orthogonal exponential functions $\mu_n(x) = e^{2\pi inx}$. We will see further that this relation is much more intimate, but for the moment make a terminology remark.

As the splines $\{_pm_r^j\}$ form an orthogonal basis, it is pertinent to call these *Ortsplines* (OS). The connection of OS with the operators of convolution and of differentiation is related to this of $\mu_n(x)$. To be specific

Proposition 9. *There holds the relation*

$$_pm_r^j(x)^{(s)} = (N(1-\omega^{-r}))^s \, _{p-s}m_r^j(x).$$

Proof: In accordance with (3.2.7)

$$\begin{aligned} _pm_r^j(x)^{(s)} &= (1-\omega^{-r})^s(1-\omega^{-r})^{p-s} \sum_l e^{2\pi i(r+lN)x} \frac{(2\pi i(r+lN))^s}{(2\pi i(r+lN)/N)^p} \\ &= [N(1-\omega^{-r})]^s \, _{p-s}m_r^j(x). \end{aligned}$$

■

Remark 2. Emphasize that OS $_{p-s}m_r^j(x)$ is a replica of OS $_pm_r^j(x)$ in the space $_{p-s}\mathcal{V}^j$.

Let us turn now to the convolution. Provided f, g are square integrable 1-periodic functions we mean under the convolution $f * g$ the following integral

$$f * g(x) = \int_0^1 f(x-y)\, g(y)\, dy.$$

Recall that the Fourier coefficients

$$C_n(f * g) = C_n(f) \cdot C_n(g). \tag{3.2.9}$$

Proposition 10. *The convolution of the ortsplines is*

$$_qm_s^j * {}_pm_r^j(x) = {}_{q+p}m_r^j(x) \cdot \delta_s^r.$$

The relation results immediately from (3.2.6) and (3.2.9).

Corollary 3. *Let a spline* ${}_q\sigma^j(x)$ *be given as*

$$ {}_q\sigma^j(x) = \sum_s^j \eta_s \ {}_q m_s^j(x). $$

Then the convolution is

$$ {}_q\sigma^j * {}_p m_r^j(x) = \eta_r \cdot {}_{p+q} m_r^j(x). $$

Emphasize that OS ${}_{p+s} m_r^j(x)$ is a replica of OS ${}_p m_r^j(x)$ in the space ${}_{p-s}\mathcal{V}^j$.

Remark 3. We may interpret Proposition 9 and Corollary 3 as follows: OS $\{{}_p m_r^j\}$ are *generalized* eigenvectors of the operators of *convolution* and *differentiation* unlike the exponential functions which are the conventional eigenvectors of these operators.

The properties of OS established give rise to formulas related to corresponding ones of HA.

Proposition 11. *Given two splines of* ${}_p\mathcal{V}^j$

$$ {}_p S^j(x) = \sum_r^j \xi_r \ {}_p m_r^j(x) \qquad (3.2.10) $$

$$ {}_p \widetilde{S}^j(x) = \sum_r^j \chi_r \ {}_p m_r^j(x), \qquad (3.2.11) $$

there holds the Parseval equality

$$ \langle {}_p S^j, \ {}_p \widetilde{S}^j \rangle = \sum_r^j \xi_r \overline{\chi_r} \ {}_{2p} u_r^j. $$

Proposition 12. *Let a spline* ${}_p S^j$ *be given by (3.2.10) and* ${}_q\sigma^j(x) = \sum_s^j \eta_s \ {}_q m_r^j(x) \in {}_q\mathcal{V}^j$*. Then the convolution is*

$$ {}_p S^j * {}_q\sigma^j(x) = \sum_r^j \xi_r \eta_r \ {}_{p+q} m_r^j(x) \in {}_{p+q}\mathcal{V}^j. $$

Proposition 13. *Let a spline ${}_pS^j$ be given by (3.2.10). Then the derivative is*

$$ {}_pS^j(x)^{(s)} = \sum_r^j (N(1-\omega^r))^s \xi_r \ {}_{p-s}m_r^j(x) \in {}_{p-s}\mathcal{V}^j. $$

Remark 4. Proposition 11, Proposition 12 and Proposition 13 enable us to look upon the expansion of a spline ${}_pS^j \in {}_p\mathcal{V}^j$ with respect to OS basis (3.2.10) as upon a peculiar HA of the spline. Moreover, OS $\{{}_pm_r^j\}$ acts as harmonics, and the coordinates $\{\xi_r\}$ as the Fourier coefficients or a *spectrum* of the spline.

We stress that, given the $\mathcal{B}$-spline representation of the spline (3.1.2) the OS representation can be derived at once by means of DFT (3.2.2) as well as the reciprocal change (3.2.4). It is natural to employ in the process the Fast Fourier Transform (FFT) algorithms.

We call this HA in spline spaces the Spline Harmonic Analysis (SHA). A great deal of operations with splines rises to remarkable simplicity by means of SHA. It provides a powerful and flexible tool to dealing with splines and, moreover, with functions of finite order of smoothness when discrete (may be noised) samples of these functions are available. Recently a promising field of application of SHA has appeared – the *spline wavelet analysis.* We discuss it further in details.

Point out that in [17] we have presented relations which allow to assert that DFT is a special case of SHA whereas the continuous Fourier Analysis is a limit case of SHA. In some sense SHA bridges the gap between the discrete and the continuous versions of HA.

Remarks on SHA applications. SHA techniques can be applied successfully just as for solving problems of spline functions (interpolation, smoothing, approximation) so also for problems associated with the operators of convolution and of differentiation (integral equations of convolution type, differential equations with constant coefficients). When solving inverse problems it occurs frequently the phenomena of ill-posedness. The SHA approach allows the implementation of efficient regularizing algorithms. These applications of SHA techniques are presented in [21] – [24].

§4 Multiscale analysis of a spline

4.1 Decomposition of a spline space into wavelet spaces

First we point out that the space ${}_p\mathcal{V}^{j-1}$ is the subspace of ${}_p\mathcal{V}^j$. The space ${}_p\mathcal{W}^{j-1}$ is the orthogonal complement of ${}_p\mathcal{V}^{j-1}$ in the space ${}_p\mathcal{V}^j$. So

$$ {}_p\mathcal{V}^j = {}_p\mathcal{V}^{j-1} \oplus {}_p\mathcal{W}^{j-1}. $$

The space ${}_p\mathcal{W}^{j-1}$ is called usually the *wavelet space.*

The space ${}_p\mathcal{V}^{j-1}$ can be in turn decomposed as

$$ {}_p\mathcal{V}^{j-1} = {}_p\mathcal{V}^{j-2} \oplus {}_p\mathcal{W}^{j-2}. $$

Correspondingly

$$_pS^{j-1}(x) = {_pS^{j-2}}(x) \oplus {_pW^{j-2}}(x).$$

Iterating this procedure we obtain

$$_p\mathcal{V}^j = {_p\mathcal{V}^{j-m}} \oplus {_p\mathcal{W}^{j-m}} \oplus {_p\mathcal{W}^{j-m+1}} \oplus \cdots \oplus {_p\mathcal{W}^{j-1}} \tag{4.1.1}$$

$$_pS^j(x) = {_pS^{j-m}}(x) \oplus {_pW^{j-m}}(x) \oplus {_pW^{j-m+1}}(x) \oplus \cdots \oplus {_pW^{j-1}}(x). \tag{4.1.2}$$

The relation (4.1.2) represents a spline as the sum of its smoothed out, "blurred" version $_pS^{j-m}$ and "details" $\{{_pW^{j-\nu}}\}_{\nu=1}$. Emphasize that all addends in (4.1.2) are mutually orthogonal.

We will call the decomposition of a spline $_pS^j(x)$ the *multiscale analysis* (MSA) of this spline and, provided, the spline $_pS^j(x) = {_pS^j}(f, x)$ is a spline approximating a signal f, as MSA of the signal f.

Now, provided we are able to project a signal f onto the appropriate spline space $_p\mathcal{V}^j$, $f \to {_pS^j}(f)$ and to decompose the spline $_pS^j(f)$, in accordance with (4.1.1) we get an opportunity to process the signal in several frequency channels simultaneously. If need be, the channels obtained which bandwidths arranged accordingly to the logarithmic scale can be subdivided into more narrow channels by means of the so called *wavelet packets*. This subject will be discussed later.

After a multichannel processing one needs to reconstruct the spline processed from its multiscale representation of type (4.1.2) into the conventional $\lfloor$-spline representation where its values can be computed immediately. It is being implemented in accordance with the pyramidal diagram

$$\begin{array}{llllll} _pS^{j-m} \longrightarrow & S^{j-m+1} \longrightarrow & \cdots \longrightarrow & S^{j-2} \longrightarrow & S^{j-1} \longrightarrow & S^j \\ _pW^{j-m} \nearrow & W^{j-m+1} \nearrow & \cdots \nearrow & W^{j-2} \nearrow & W^{j-1} \nearrow & \end{array} \tag{4.1.3}$$

The algorithms of such reconstruction to be established are of high-rate efficiency as well.

SHA provides a powerful tool for developing the wavelet analysis. We start with MSA, i.e., we establish algorithms to decompose a spline into an orthogonal sum of type (4.1.2) and to reconstruct it.

4.2 Two-scale relations

We discuss first projecting a spline $_pS^j \in {_p\mathcal{V}^j}$ onto the subspace $_p\mathcal{V}^{j-1}$. Corresponding algorithms result from the so called two-scale relations which correlate ortsplines of the spaces $_p\mathcal{V}^j$ and $_p\mathcal{V}^{j-1}$.

As a rule in what follows we will omit the index $_p\cdot$ for splines belonging to the spaces $_p\mathcal{V}^\nu$. The term u_r^ν will stand for $_{2p}u_r^\nu$.

Theorem 1. *There hold the two-scale relations for* $r = 0, 1, ..., 2^{j-1} - 1$*:*

$$m_r^{j-1}(x) = b_r^j \, m_r^j(x) + b_{r-N/2}^j \, m_{r-N/2}^j(x), \qquad b_r^j = 2^{-p}(1 + \omega^{-r})^p. \tag{4.2.1}$$

Proof: In accordance with (3.2.7) we have

$$\begin{aligned}
m_r^{j-1}(x) &= \Big(\frac{N(1-\omega^{-2r})}{4\pi i}\Big)^p e^{2\pi irx} \sum_{l=-\infty}^{\infty} e^{\pi ilNx} \frac{1}{(r+lN/2)^p} \\
&= 2^{-p}(1+\omega^{-r})^p \Big(\frac{N(1-\omega^{-r})}{2\pi i}\Big)^p e^{2\pi irx} \sum_{\nu=-\infty}^{\infty} e^{2\pi i\nu Nx} \frac{1}{(r+\nu N)^p} \\
&+ 2^{-p}(1+\omega^{-(r-N/2)}) \Big(\frac{N(1-\omega^{-(r-N/2)})}{2\pi i}\Big)^p e^{2\pi i(r-N/2)x} \\
&\cdot \sum_{\nu=-\infty}^{\infty} e^{2\pi i\nu Nx} \frac{1}{(r-N/2+\nu N)^p} \\
&= 2^{-p}(1+\omega^{-r})^p m_r^j(x) + 2^{-p}(1+\omega^{-(r-N/2)})^p m_{r-N/2}^j(x).
\end{aligned}$$

■

Remark 5. The relation (4.2.1) implies an identity which will be employed repeatedly in what follows. Namely, writing this relation for the splines of order $2p$ with $x = 2p/N$, we have

$$\begin{aligned}
{}_{2p}u_r^{j-1} &= {}_{2p}m_r^{j-1}(2p/N) = 4^{-p}(1+\omega^{-r})^{2p}\, {}_{2p}m_r^j(2p/N) \\
&+ 4^{-p}(1-\omega^{-r})^{2p}\, {}_{2p}m_{r-N/2}^j(2p/N).
\end{aligned}$$

But

$$ {}_{2p}m_r^j(2p/N) = {}_{2p}m_r^j(p/N)\omega^{pr} = \omega^{rp}\, {}_{2p}u_r^j $$

and we obtain finally

$$\begin{aligned}
{}_{2p}u_r^{j-1} &= 4^{-p}\omega^{rp}\big[(1+\omega^{-r})^{2p}\, {}_{2p}u_r^j + (-1)^p(1-\omega^{-r})^{2p}\, {}_{2p}u_{r-N/2}^j\big] \\
&= |b_r^j|^2\, {}_{2p}u_r^j + |b_{r-N/2}^j|^2\, {}_{2p}u_{r-N/2}^j. \qquad (4.2.2)
\end{aligned}$$

Theorem 1 enables us to implement projecting a spline $S^j(x) \in {}_p\mathcal{V}^j$ onto the space ${}_p\mathcal{V}^{j-1}$.

Theorem 2. *Let the spline*

$$ S^{j-1}(x) = \sum_r^{j-1} \xi_r^{j-1}\, m_r^{j-1}(x) $$

be the orthogonal projection of a spline

$$ S^j(x) = \sum_l^j \xi_l^j\, m_l^j(x) \in {}_p\mathcal{V}^j $$

onto the space ${}_p\mathcal{V}^{j-1}$*. Then the coordinates are*

$$\xi_r^{j-1} = \left\langle S^j, m_r^{j-1} \right\rangle / u_r^{j-1} = \left\langle S^j, (b_r^j\, m_r^j + b_{r-N/2}^j\, m_{r-N/2}^j) \right\rangle / u_r^{j-1}$$

$$= \frac{1}{u_r^{j-1}} \left(\xi_r^j\, u_r^j\, \overline{b_r^j} + \xi_{r-N/2}^j\, u_{r-N/2}^j\, \overline{b}_{r-N/2}^j \right). \qquad (4.2.3)$$

4.3 Ortwavelets

Now, we proceed to projecting a spline of ${}_p\mathcal{V}^j$ onto the space ${}_p\mathcal{W}^{j-1}$. To do it we need first of all a basis of the space ${}_p\mathcal{W}^{j-1}$. We construct now an orthogonal basis of ${}_p\mathcal{W}^{j-1}$.

Theorem 3. *There exists an orthogonal basis* $\{w_r^{j-1}(x)\}_r^{j-1}$ *of* ${}_p\mathcal{W}^{j-1} \subset {}_p\mathcal{V}^j$

$$w_r^{j-1}(x) = a_r^j\, m_r^j(x) + a_{r-N/2}^j\, m_{r-N/2}^j(x), \qquad (4.3.1)$$

$$a_r^j = \omega^r\, \overline{b}_{r-N/2}^j\, u_{r-N/2}^j = 2^{-p}\omega^r(1-\omega^r)^p\, u_{r-N/2}^j, \qquad (4.3.2)$$

moreover

$$\|w_r^{j-1}\|^2 = \left\langle w_r^{j-1}, w_r^{j-1} \right\rangle = v_r^{j-1}$$

where $v_r^{j-1} = u_r^j\, u_{r-N/2}^j\, u_r^{j-1}$ *is a* 2^{j-1}*-periodic sequence.*

Proof: The orthogonality of a spline w_r^{j-1} to any w_l^{j-1}, m_l^{j-1}, $l \neq r$ is readily apparent from the orthogonality of the splines $\{m_r^j\}_r^j$ each to another. We should establish the orthogonality w_r^{j-1} to m_r^{j-1}. Due to (4.2.1) we write

$$\begin{aligned} \langle m_r^{j-1}, w_r^{j-1} \rangle &= b_r^j\, \overline{a}_r^j\, u_r^j + b_{r-N/2}\, \overline{a}_{r-N/2}^j\, u_{r-N/2}^j \\ &= \omega^{-r} b_r^j\, b_{r-N/2}^j\, u_{r-N/2}^j\, u_r^j - \omega^{-r} b_{r-N/2}^j\, b_r^j\, u_{r-N/2}^j\, u_r^j. \end{aligned}$$

We have employed here the periodicity of the sequence u_r^j, namely, the relation $u_{r-N/2-N/2}^j = u_r^j$. Moreover, $\omega^{-r-N/2} = \omega^{-r} \cdot e^{\pi i} = -\omega^{-r}$. Therefore

$$\langle m_r^{j-1}, w_r^{j-1} \rangle = 0.$$

Similarly, in view of (4.2.2) we can write

$$\begin{aligned} \langle w_r^{j-1}, w_r^{j-1} \rangle &= |a_r^j|^2 u_r^j + |a_{r-N/2}|^2 u_{r-N/2}^j \\ &= u_r^j\, u_{r-N/2}^j (|b_r^j|^2\, {}_{2p}u_r^j + |b_{r-N/2}^j|^2\, {}_{2p}u_{r-N/2}^j) = v_r^{j-1}. \end{aligned}$$

■

We will call the splines w_r^{j-1} the *ortwavelets* (OW). Note that the OW just as the OS are eigenvectors of the operator of shift.

Proposition 14. *There holds the relation*

$$w_r^{j-1}(x+2k/N) = \omega^{2kr} w_r^j(x). \tag{4.3.3}$$

Proof: Since $\omega^{2k(r-N/2)} = \omega^{2kr} e^{-2\pi ik} = \omega^{2kr}$, we have

$$\begin{aligned} w_r^{j-1}(x+2k/N) &= a_r^j\, m_r^j(x+2k/N) + a_{r-N/2}^j\, m_{r-N/2}^j(x+2k/N) \\ &= \omega^{2kr}(a_r^j\, m_r^j(x) + a_{r-N/2}^j\, m_{r-N/2}(x)) = \omega^{2kr} w_r^{j-1}(x). \end{aligned}$$

■

Theorem 4. *Let the spline*

$$W^{j-1}(x) = \sum_r^{j-1} \eta_r^{j-1}\, w_r^{j-1}(x)$$

be the orthogonal projection of a spline

$$S^j(x) = \sum_l^j \xi_l^j\, m_l^j(x) \in {}_p\mathcal{V}^j$$

onto the space ${}_p\mathcal{W}^{j-1}$*. Then the coordinates are*

$$\begin{aligned} \eta_r^{j-1} &= \langle S^j, w_r^{j-1}\rangle / v_r^{j-1} && (4.3.4) \\ &= \frac{1}{v_r^{j-1}}(\xi_r^j\, u_r^j\, \overline{a}_r^j + \xi_{r-N/2}^j\, u_{r-N/2}^j\, \overline{a}_{r-N/2}) && (4.3.5) \\ &= \frac{\omega^{-r}}{u_r^{j-1}}(\xi_r^j\, b_{r-N/2}^j - \xi_{r-N/2}^j\, b_r^j). && (4.3.6) \end{aligned}$$

Now we have carried out the first step of the decomposition

$$S^j(x) = S^{j-1}(x) \oplus W^{j-1}(x).$$

In the spectral domain we have split the frequency band $[-N/2, N/2]$ into the sub-band $[-N/4, N/4]$ and two strips $[-N/2, -N/4]$, $[N/4, N/2]$. Subjecting the spline S^{j-1} to the procedures suggested we carry out the second step and so long until we get (4.1.2). The frequency band will be split in accordance with the logarithmic scale.

4.4 High- and low-frequency wavelet subspaces

If a refined frequency resolution in the strips $[-N/2, -N/4]$, $[N/4, N/2]$ is wanted, we suggest to project a spline $W^{j-1}(x) \in {}_p\mathcal{W}^{j-1}$ onto two subspaces ${}^l_p\mathcal{W}^{j-2}$, ${}^h_p\mathcal{W}^{j-2}$ one of which is an orthogonal complement of the other in ${}_p\mathcal{W}^{j-1}$ and such that ${}^l_p\mathcal{W}^{j-2}$ is, loosely speaking, concentrated at the frequency strips $[-\frac{3N}{8}, -\frac{N}{4}]$, $[\frac{N}{4}, \frac{3N}{8}]$ and ${}^h_p\mathcal{W}^{j-2}$ in the strips $[-\frac{N}{2}, -\frac{3N}{8}]$, $[\frac{3N}{8}, \frac{N}{2}]$.

To construct these subspaces we start with bases.

Let us call the splines

$$ {}^h w_r^{j-2}(x) = b_r^{j-1}\, w_r^{j-1}(x) + b_{r-N/4}^{j-1}\, w_{r-N/4}^{j-1}(x) \in {}_p\mathcal{W}^{j-1}, $$

$r = 0, 1, ..., 2^{j-2} - 1$, the *high-frequency* OW (HOW) and the splines

$$ \begin{aligned} {}^l w_r^{j-2}(x) &= {}^1 a_r^{j-1}\, w_r^{j-1}(x) + {}^1 a_{r-N/4}^{j-1}\, w_{r-N/4}^{j-1}(x) \in {}_p\mathcal{W}^{j-1}, \\ {}^1 a_r^{j-1} &= \omega^{2r} \bar{b}_{r-N/4}^{j-1}\, v_{r-N/4}^{j-1} = 2^{-p} \omega^{2r} (1-\omega^{2r})^p v_{r-N/4}^{j-1}, \end{aligned} $$

$r = 0, 1, ..., 2^{j-2} - 1$, the *low-frequency* OW (LOW).

Theorem 5. *There hold the relations*

$$ \langle {}^l w_r^{j-2}, {}^l w_s^{j-2} \rangle = \delta_r^s\, {}^l v_r^{j-2}, \qquad \langle {}^h w_r^{j-2}, {}^h w_s^{j-2} \rangle = \delta_r^s\, {}^h v_r^{j-2} $$

where

$$ {}^h v_r^{j-2} = |b_r^{j-1}|^2 v_r^{j-1} + |b_{r-N/4}^{j-1}|^2 v_{r-N/4}^{j-1} \tag{4.4.1} $$

$$ = (\cos(2\pi r/N))^{2p} v_r^{j-1} + (\sin(2\pi r/N))^{2p} v_{r-N/4}^{j-1}, $$

$$ {}^l v_r^{j-2} = |a_r^{j-1}|^2 v_r^{j-1} + |a_{r-N/4}^{j-1}|^2 v_{r-N/4}^{j-1} \tag{4.4.2} $$

$$ = (\sin(2\pi r/N))^{2p} (v_{r-N/4}^{j-1})^2 v_r^{j-1} + (\cos(2\pi r/N))^{2p} (v_r^{j-1})^2 v_{r-N/4}^{j-1} (v_r^{j-1})^2. $$

Moreover, $\langle {}^l w_r^{j-2}, {}^h w_s^{j-2} \rangle = 0\ \forall r, s.$

Corollary 4. *The splines* $\{ {}^l w_r^{j-2}(x), {}^h w_r^{j-2}(x) \}_r^{j-2}$ *form an orthogonal basis of the space* ${}_p\mathcal{W}^{j-1}$.

Proof: The relations $\langle {}^i w_r^{j-2}, {}^k w_s^{j-2}\rangle = 0$ if $r \neq s$, $i = l,h$, $k = l,h$ are readily apparent due to the orthogonality of the OS $\{w_r^{j-1}\}$ each to another. The formulas (4.4.1), (4.4.2) are evident also. Let us examine the inner product

$$\langle {}^h w_r^{j-2}, {}^l w_r^{j-2}\rangle = \omega^{-2r}(b_r^{j-1}\, b_{r-N/4}^{j-1}\, v_r^{j-1}\, v_{r-N/r}^{j-1} - b_{r-N/4}^{j-1}\, b_r^{j-1}\, v_{r-N/4}^{j-1}\, v_r^{j-1}) = 0.$$

We have exploited the fact that $v_{r-N/4-N/4}^{j-1} = v_r^{j-1}$, just as b_r^{j-1}. ■

These results enable us to decompose the wavelet space ${}_p\mathcal{W}^{j-1}$ into an orthogonal sum of spaces. To be specific, define the space ${}^l\mathcal{W}^{j-2} \subset {}_p\mathcal{W}^{j-1}$ as ${}^l\mathcal{W}^{j-2} =:$ span$\{{}^l w_r^{j-2}(x)\}_r^{j-2}$ and the space ${}^h\mathcal{W}^{j-2} \subset {}_p\mathcal{W}^{j-1}$ as ${}^h\mathcal{W}^{j-2} =:$ span$\{{}^h w_r^{j-2}(x)\}_r^{j-2}$. It can be verified immediately that

$${}_p\mathcal{W}^{j-1} = {}^l\mathcal{W}^{j-2} \oplus {}^h\mathcal{W}^{j-2}.$$

It is reasonable that the space ${}^l\mathcal{W}^{j-2}$ to be referred as to the low-frequency wavelet subspace and the space ${}^h\mathcal{W}^{j-2}$ as to the high-frequency wavelet subspace. The space ${}^l\mathcal{W}^{j-2}$ is "concentrated" at the bands $[-\frac{3}{8N}, -\frac{N}{4}]$, $[\frac{N}{4}, \frac{3}{8N}]$, whereas ${}^h\mathcal{W}^{j-2}$ at $[-\frac{1}{2N}, -\frac{3}{8N}]$, $[\frac{3}{8N}, \frac{1}{2N}]$.

If need be we can decompose in a similar manner one (or both) of the subspaces ${}^l\mathcal{W}^{j-2}$, ${}^h\mathcal{W}^{j-2}$ into orthogonal sums of subspaces ${}^{ll}\mathcal{W}^{j-3} \oplus {}^{hl}\mathcal{W}^{j-3}$ and ${}^{lh}\mathcal{W}^{j-3} \oplus {}^{hh}\mathcal{W}^{j-3}$ respectively and to iterate this process.

Proposition 14 entails the following fact.

Proposition 15. *There hold the relations*

$${}^i w_r^{j-2}(x + 4k/N) = \omega^{4kr} \cdot {}^i w_r^{j-2}(x), \quad i = l,h.$$

Similar formulas hold for ${}^{ik}w_r^{j-3}$, $i = l,h$; $k = l,h$.

To project a spline

$$W^{j-1}(x) = \sum_r^{j-1} \eta_r^{j-1}\, w_r^{j-1}(x) \in {}_p\mathcal{W}^{j-1}$$

onto the spaces ${}^l\mathcal{W}^{j-2}$ and ${}^h\mathcal{W}^{j-2}$ one should act in a way similar to that used for establishing (4.2.3) and (4.3.4). So, we have

$$\begin{aligned}
W^{j-1}(x) &= {}^l W^{j-2}(x) \oplus {}^h W^{j-2}(x), \\
{}^h W^{j-2}(x) &= \sum_r^{j-2} {}^h\eta_r^{j-2}\ {}^h w_r^{j-2}(x), \\
{}^h\eta_r^{j-2} &= \frac{1}{{}^h v_r^{j-2}}\big(\eta_r^{j-1}\, v_r^{j-1}\, \bar{b}_r^{j-1} + \eta_{r-N/4}^{j-1}\, v_{r-N/4}^{j-1}\, \bar{b}_{r-N/4}^{j-1}\big) \\
{}^l W^{j-2}(x) &= \sum_r^{j-2} {}^l\eta_r^{j-2}\ {}^l w_r^{j-2}(x), \\
{}^l\eta_r^{j-2} &= \frac{1}{{}^l v_r^{j-2}}\big(\eta_r^{j-1}\, v_r^{j-1}\ {}^1\bar{a}_r^{j-2} + \eta_{r-N/4}^{j-1}\, v_{r-N/4}^{j-1}\ {}^1\bar{a}_{r-N/4}^{j-2}\big).
\end{aligned}$$

4.5 Reconstruction of a spline

Let a spline ${}_pS^j(x)$ be given in the decomposed form. It is required to reconstruct it in the conventional form suited for computation. To be specific, suppose we have two splines

$$S^{j-1}(x) = \sum_r^{j-1} m_r^{j-1}(x)\xi_r^{j-1} \in {}_p\mathcal{V}^{j-1},$$

$$\mathcal{W}^{j-1}(x) = \sum_r^{j-1} w_r^{j-1}(x)\eta_r^{j-1} \in {}_p\mathcal{W}^{j-1}.$$

Let $S^j(x) = S^{j-1}(x) \oplus \mathcal{W}^{j-1}(x)$. We are able to come up with the following assertion.

Theorem 6. *There hold the relations*

$$S^j(x) = \frac{1}{N}\sum_k^j q_k^j\, M^j(x - k/N) = \sum_r^j \xi_r^j\ {}_pm_r^j(x), \tag{4.5.1}$$

$$\xi_r^j = b_r^j\, \xi_r^{j-1} + a_r^j\, \eta_r^{j-1}, \qquad q_k^j = \sum_r^j \omega^{kr} \xi_r^j. \tag{4.5.2}$$

Proof: Due to (4.2.1) we can write

$$S^{j-1}(x) = \sum_r^{j-1} \xi_r^{j-1}\big(b_r^j\, m_r^j(x) + b_{r-N/2}\, m_{r-N/2}^j\big) = \sum_r^j \xi_r^{j-1}\, b_r^j\, m_r^j(x).$$

Similarly, (4.3.1) entails

$$W^{j-1}(x) = \sum_r^j \eta_r^{j-1}\, a_r^j\, m_r^j(x).$$

These two relations imply (4.5.1), (4.5.2). ■

By this means, given the representation of a spline in the form (4.1.2), it is possible to reconstruct it into the conventional form (4.5.1) in line with the diagram (4.1.3).

The algorithm suggested allows a fast implementation.

Remark 6. To compute values and display graphically the spline

$$\mathcal{W}^{j-1}(x) = \sum_r^{j-1} w_r^{j-1}(x)\eta_r^{j-1}$$

one may carry out the suggested reconstruction procedure assuming $\xi_r^{j-1} = 0$.

§5 Wavelets and multichannel processing a signal

In Section 4 we have carried out decomposition of a spline belonging to ${}_p\mathcal{V}^j$ into into the set of its projections onto the subspaces ${}_p\mathcal{V}^{j-m}$, ${}_p\mathcal{W}^{j-m}$. Now we are going to process the spline in these subspaces. To do it we need relevant bases of the subspaces ${}_p\mathcal{V}^{j-m}$, ${}_p\mathcal{W}^{j-m}$. We start with the space ${}_p\mathcal{V}^j$.

5.1 Father wavelets

Definition 1. *A spline* ${}^s\varphi^j(x) \in {}_p\mathcal{V}^j$ *will be referred to as the father wavelet (FW) if its shifts* ${}^s\varphi^j(x - k/2^j)$, $k = 0, 1, ..., 2^j - 1$ *form a basis of the space* ${}_p\mathcal{V}^j$. *Two FW are said to be the dual ones if* $\langle {}^s\varphi^j(\cdot - k/2^j), {}^\sigma\varphi^j(\cdot - l/2^j)\rangle = \delta_k^l$.

We establish conditions to a spline to be the FW and to two FW to be the dual ones.

Theorem 7. *A spline*

$$
{}^s\varphi^j(x) = 2^{-j/2} \sum_r^j {}^s\rho_r^j \, m_r^j(x) \tag{5.1.1}
$$

is the FW if and only if ${}^s\rho_r^j \neq 0 \; \forall r$. *Two FWs are dual each to the other if and only if*

$$
{}^s\rho_r^j \, {}^\sigma\overline{\rho}_r^j \; {}_{2p}u_r^j = 1. \tag{5.1.2}
$$

Proof: Let a spline ${}^s\varphi^j(x)$ be written as in (5.1.1) Due to (3.2.8) we have

$$
{}^s\varphi^j(x - k/2^j) = 2^{-j/2} \sum_r^j {}^s\rho_r^j \, \omega^{-kr} m_r^j(x). \tag{5.1.3}
$$

Hence it follows

$$
{}^s\rho_r^j \, m_r^j(x) = 2^{-j/2} \sum_k^j \omega^{kr} \; {}^s\varphi^j(x - k/2^j).
$$

These two relations imply the first assertion. Indeed, if some of $\{\rho_r^j\}$ are zero, then the dimension of the *span*$\{ {}^s\varphi^j(x - k/2^j)\}$ is less than 2^j; if all of $\{\rho_r^j\}$ are nonzero then all of m_r^j belong to the span. To establish the second assertion write the inner product keeping in mind (5.1.3):

$$
\langle {}^s\varphi^j(\cdot - k/2^j), {}^\sigma\varphi^j(\cdot - l/2^j)\rangle = 2^{-j} \sum_r^j {}^s\rho_r^j \; {}^\sigma\overline{\rho}_r^j \; {}_{2p}u_r^j \, \omega^{(l-k)r}.
$$

The latter sum is equal to δ_k^l if and only if (5.1.2) holds. ■

The following assertion relates the coordinates of a spline with respect to a FW basis with these in the OS one.

Theorem 8. *Let*

$$ {}^s\varphi^j(x) = 2^{-j/2} \sum_r^j {}^s\rho_r^j \, m_r^j(x) $$

be a FW and a spline $S^j(x)$ is expanded with respect to the two bases

$$ S^j(x) = \sum_k^j {}^s q_k^j \, {}^s\varphi^j(x - k/N) = \sum_r^j \xi_r^j \, m_r^j(x). $$

Then

$$ {}^s q_k^j = 2^{-j/2} \sum_r^j \omega^{rk} \, \xi_r^j / {}^s\rho_r^j, \qquad \xi_r^j = 2^{-j/2} \, {}^s\rho_r^j \sum_k^j {}^s q_k^j \, \omega^{-rk}. \tag{5.1.4} $$

Proof: Let us employ (5.1.3) once more

$$ \begin{aligned} S^j(x) &= \sum_k^j {}^s q_k^j \, {}^s\varphi^j(x - k/N) \\ &= \sum_k^j {}^s q_k^j \, 2^{-j/2} \sum_r^j {}^s\rho_r^j \, \omega^{-rk} \, m_r^j(x) \\ &= \sum_r^j m_r^j \, {}^s\rho_r^j \, 2^{-j/2} \sum_k^j \omega^{-rk} \, {}^s q_k^j. \end{aligned} $$

Hence

$$ \xi_r^j = 2^{-j/2} \, {}^s\rho_r^j \sum_k^j {}^s q_k^j \, \omega^{-rk}. $$

The second relation of (5.1.4) can be obtained immediately by means of DFT. ∎

Proposition 16. *If FW ${}^\sigma\varphi^j$ is dual to FW ${}^s\varphi^j$ then*

$$ {}^s q_k^j = \langle S^j, {}^\sigma\varphi^j(\cdot - k/2^j) \rangle. $$

Remark 7. Eq. (5.1.4) implies that to make the change from a FW basis to the OS one or the reciprocal change, one have to carry out DFT. Of course, it should be employed a FFT algorithm.

We present some examples of FWs.

Examples

1. *B-spline.* Suppose ${}^1\rho_r^j = 1$. Then we can derive immediately from Eq.(3.2.5) that ${}^1\varphi^j(x) = 2^{-j/2} M^j(x)$.

2. *FW dual to* ${}^1\varphi^j(x)$. Suppose ${}^2\rho_r^j = 1/{}_{2p}u_r^j$. Then, in accordance with Eq. (5.1.2) the FW ${}^2\varphi^j(x)$ is dual to ${}^1\varphi^j(x)$.

   Emphasize that if $S^j(x) = \sum_k^j {}^2q_k^j\, {}^2\varphi^j(x - k/N)$ then

$$ {}^2q_k^j = 2^{-j/2} \int_0^{p/N} S^j(x - k/N) M^j(x)\, dx. $$

   Provided $S^j(x) = S^j(f, x)$ is an orthogonal projection of a function f onto the spline space ${}_p\mathcal{V}^j$, we have

$$ {}^2q_k^j = 2^{-j/2} \int_0^{p/N} f(x - k/N) M^j(x)\, dx. $$

3. Setting ${}^3\rho_r^j = ({}_{2p}u_r^j)^{-1/2}$ we obtain the *self-dual FW* ${}^3\varphi^j(x)$ those shifts form an orthonormal basis of ${}_p\mathcal{V}^j$ [1], [10].

4. *Interpolating FW.* If we set ${}^4\rho_r^j = 1/{}_p u_r^j$ then FW ${}^1\varphi^j(x) = 2^{-j/2}\, {}_pL^j(x)$, ${}_pL^j(x)$ is so called fundamental spline, namely

$$ {}_pL^j((k + p/2)/N) = \begin{cases} 1 & \text{if} k = 0 \\ 0 & \text{if} k = 1, ..., N - 1 \end{cases} $$

   Therefore the spline

$$ S^j(x) = \sum_k^j z_k\, {}_pL^j(x - k/N) $$

   interpolates the vector $\{z_k\}_k^j$. To be specific, $S^j(k/N + p/2N) = z_k \forall k$.

5.2 Mother wavelets

We present here a family of bases of the space ${}_p\mathcal{W}^{j-1}$. The contents of this section is related to that of the Subsection 5.1 where we have introduced FWs.

Definition 2. *A spline* ${}^s\psi^{j-1}(x) \in {}_p\mathcal{W}^{j-1}$ *will be referred to as the mother wavelet (MW) if its shifts* ${}^s\psi^{j-1}(x - k/2^{j-1})$, $k = 0, 1, ..., 2^{j-1} - 1$ *form a basis of the space* ${}_p\mathcal{W}^{j-1}$. *Two MW are said to be the dual ones if*

$$ \left\langle {}^s\psi^{j-1}(\cdot - k/2^{j-1}),\, {}^\sigma\psi^{j-1}(\cdot - l/2^{j-1}) \right\rangle = \delta_k^l. $$

We will establish conditions to a spline to be MW and to two MW to be dual ones.

Theorem 9. *A spline*

$$ {}^{s}\psi^{j-1}(x) = 2^{(1-j)/2} \sum_{r}^{j-1} {}^{s}\tau_r^{j-1}\, w_r^{j-1}(x) \tag{5.2.1} $$

is a MW if and only if ${}^{s}\tau_r^{j-1} \neq 0\ \forall r$. *Two MW are dual each to the other if and only if*

$$ {}^{s}\tau_r^{j-1}\, \overline{{}^{\sigma}\tau_r^{j-1}}\, v_r^{j-1} = 1\, \forall r. \tag{5.2.2} $$

Proof: Let a spline ${}^{s}\psi^{j-1}(x)$ be written as in (5.2.1). Due to (4.3.3) we have

$$ {}^{s}\psi^{j-1}(x - k/2^{j-1}) = 2^{(1-j)/2} \sum_{r}^{j-1} {}^{s}\tau_r^{j-1} \omega^{-2kr} w_r^{j-1}(x). $$

Hence it follows

$$ {}^{s}\tau_r^{j-1} w_r^{j-1} = 2^{(1-j)/2} \sum_{k}^{j-1} \omega^{2kr}\ {}^{s}\psi(x - k/2^{j-1}). $$

These two relations imply the first assertion of the theorem. To establish the second assertion, we write the inner product

$$ \left\langle {}^{s}\psi^{j-1}(\cdot - k/2^{j-1}),\ {}^{\sigma}\psi^{j-1}(\cdot - l/2^{j-1}) \right\rangle = 2^{1-j} \sum_{r}^{j-1} {}^{s}\tau_r^{j-1}\, \overline{{}^{\sigma}\tau_r^{j-1}}\, v_r^{j-1} \omega^{2(l-k)r} = \delta_l^k $$

provided (5.2.2) holds. ■

The following assertion relates the coordinates of a spline with respect to a MW basis with these in the OW one.

Theorem 10. *Let*

$$ {}^{s}\psi^{j-1}(x) = 2^{(1-j)/2} \sum_{r}^{j-1} {}^{s}\tau_r^{j-1} w_r^{j-1}(x) \tag{5.2.3} $$

be a MW and a spline $W^{j-1}(x) \in {}_{p}\mathcal{W}^{j-1}$ *is expanded with respect to the two bases*

$$ W^{j-1}(x) = \sum_{k}^{j-1} {}^{s}p_k^{j-1}\ {}^{s}\psi^{j-1}(x - 2k/N) = \sum_{r}^{j-1} \eta_r^{j-1} w_r^{j-1}(x). \tag{5.2.4} $$

Then

$$ {}^{s}p_k^{j-1} = 2^{(1-j)/2} \sum_{r}^{j-1} \omega^{2rk} \eta_r^{j-1} /\, {}^{2}\tau_r^{j-1}, \tag{5.2.5} $$

$$ \eta_r^{j-1} = {}^{s}\tau_r^{j-1}\, 2^{(1-j)/2} \sum_{k}^{j-1} {}^{s}p_k^{j-1} \omega^{-2rk}. \tag{5.2.6} $$

Proof: Substituting (5.2.3) into (5.2.4) we obtain in view of (4.3.3)

$$\begin{aligned} W^{j-1}(x) &= \sum_k^{j-1} {}^s p_k^{j-1} \, 2^{(1-j)/2} \sum_r^{j-1} \omega^{-2kr} \ {}^s\tau_r^{j-1} w_r^{j-1}(x) \\ &= \sum_r^{j-1} w_r^{j-1}(x) \cdot \ {}^s\tau_r^{j-1} \, 2^{(1-j)/2} \sum_k^{j-1} {}^s p_k^{j-1} \omega^{-2kr}. \end{aligned}$$

This implies (5.2.6). Carrying out DFT we derive hence (5.2.5). ■

Remark 8. If MW ${}^\sigma\psi^{j-1}$ is dual to MW ${}^s\psi^{j-1}$ then for any spline $W^{j-1}(x)$ given as in (5.2.4)

$${}^s p_k^{j-1} = \left\langle W^{j-1}, \ {}^\sigma\psi^{j-1}(\cdot - k/2^{j-1}) \right\rangle.$$

Provided the spline $W^{j-1}(x)$ is an orthogonal projection of a spline $S^j(x)$ onto ${}_p\mathcal{W}^{j-1}$,

$${}^s p_k^{j-1} = \left\langle S^j, \ {}^\sigma\psi^{j-1}(\cdot - k/2^{j-1}) \right\rangle.$$

Remark 9. Theorem 10 implies that to make the change from a MW basis to the OW one or the reciprocal change, one has to carry out DFT.

Present some examples of MW.

Examples

1. *B-wavelet*. Suppose ${}^1\tau_r^{j-1} = 1 \ \forall r$. The determining feature of wavelet ${}^1\psi^{j-1}(x)$ is the compactness (up to periodization) of its support. To be precise, $\mathrm{supp}\,{}^1\psi^{j-1}(x) \subseteq ((-2p)/N, (2p-2)/N)(\mathrm{mod} 1)$. The wavelet $\psi^{j-1}(x) = 2^{(-1+j)/2} \cdot {}^1\psi^{j-1}(x)$ is a periodization of the B-wavelet invented by Chui and Wang [3].

2. *MW dual to* ${}^1\psi^{j-1}(x)$. Suppose ${}^2\tau_r^{j-1} = 1/v_r^{j-1}$. Then, in accordance with Eq. (5.2.2), the MW ${}^2\psi^{j-1}(x)$ is dual to ${}^1\psi^{j-1}(x)$.

   Emphasize that if $S^j(x) = S^{j-1}(x) \oplus W^{j-1}(x)$ and

$$W^{j-1}(x) = \sum_k^{j-1} {}^2p_k^{j-1} \ {}^2\psi^{j-1}(x - 2k/N) \qquad (5.2.7)$$

   then

$${}^2p_k^{j-1} = \int_{-2p/N}^{(2p-2)/N} S^j(x - 2k/N) \ {}^1\psi^{j-1}(x)\, dx.$$

   Provided $S^j(x) = S^j(f, x)$ is an orthogonal projection of a function f onto the spline space ${}_p\mathcal{V}^j$ we have

$${}^2p_k^{j-1} = \int_{-2p/N}^{(2p-2)/N} f(x - 2k/N) \ {}^1\psi^{j-1}(x)\, dx. \qquad (5.2.8)$$

3. Setting ${}^3\tau_r^j = (v_r^{j-1})^{-1/2}$ we obtain the *self-dual MW* ${}^3\psi^{j-1}(x)$ those shifts form an orthonormal basis of ${}_p\mathcal{W}^{j-1}$. This MW as a periodization of the Battle–Lemarié wavelet ([9], [24].

4. *Cardinal MW.* If we set ${}^4\tau_r^{j-1} = 1/(u_r^j\, u_{r-N/2}^j)$ then we obtain MW ${}^4\psi^{j-1}(x) = 2^{(1-j)/2}\, {}_{2p}L^j(x+1/N)^{(p)}$, where ${}_{2p}L^j(x)$ is the fundamental spline of the degree $2p-1$ introduced in Subsection 7. MW $2^{(1-j)/2}\ {}^4\psi^{j-1}(x)$ is a periodization of the cardinal wavelet suggested by Chui and Wang in [2].

5.3 Wavelet packets

Now we discuss briefly bases in the low- and high-frequency wavelet spaces ${}^l\mathcal{W}^{j-2}$ and ${}^h\mathcal{W}^{j-2}$.

Just as in previous sections we can find splines whose shifts form bases of the subspaces ${}^l\mathcal{W}^{j-2}$ and ${}^h\mathcal{W}^{j-2}$. For example, a spline ${}^l_s\psi^{j-2}(x) \in {}^l\mathcal{W}^{j-2}$ will be referred to as the low-frequency MW (LMW) if its shifts ${}^l_s\psi^{j-2}(x-k/2^{j-2})$, $k = 0, 1, ..., 2^{j-2}-1$ form a basis of the space. Two LMW are said to be the dual ones if

$$\left\langle {}^l_s\psi^{j-2}(\cdot - k/2^{j-2}), {}^l_\sigma\psi^{j-2}(\cdot - l/2^{j-2})\right\rangle = \delta_k^l.$$

Theorem 11. *A spline*

$${}^l_s\psi^{j-2} = 2^{(2-j)/2} \sum_r^{j-2} {}^l_s\tau_r^{j-2}\ {}^l w_r^{j-2}(x)$$

is the LMW if and only if ${}^l_s\tau_r^{j-2} \neq 0\ \forall r$. *Two MW are dual each to the other if and only if*

$${}^l_s\tau_r^{j-2}\, \overline{{}^l_\sigma\tau_r^{j-2}} \cdot {}^l v_r^{j-2} = 1 \forall r.$$

There holds an assertion related to Theorem 8 and Theorem 10.

Point out that, setting ${}^l_1\tau_r^{j-2} = 1$, we obtain the LMW of minimal support, so to say, B-LMW.

Similar considerations can be conducted in the space ${}^h\mathcal{W}^{j-2}$. We are now able to construct a diversity of bases of the space ${}_p\mathcal{W}^{j-1}$ for refined frequency resolution of a certain signal f under processing. For example, one of such bases may be structured as follows:

$$\{{}^l_s\psi^{j-2}(x-k/2^{j-2})\}_k^{j-2}, \quad \{{}^{lh}_\sigma\psi^{j-3}(x-l/2^{j-3})\}_l^{j-3}, \quad \{{}^{hh}_\gamma\psi^{j-3}(x-\nu/2^{j-3})\}_\nu^{j-3}.$$

The MWs of type $\{{}^l_s\psi^{j-2}, {}^{lh}_\sigma\psi^{j-3}, {}^{hh}_\gamma\psi^{j-3}\}$ are called the *wavelet packets* (compare with [7].

5.4 Digital processing a periodic signal by means of spline wavelets

We discuss here a scheme of processing a periodic signal $f(x)$ belonging to $\mathcal{C}^p$. The commonly encountered situation is when the array of samples is available: $\vec{f^j} = \{f_k^j = f(p/2N + k/N)\}_k^j$. The goal of the processing is to transform the original data array into a more informative array. We will process the signal by means of spline wavelets of order p.

First we establish a quadrature formula. Denote

$$F_k^\nu = \int_0^{p/N} f(x - k/N) M^\nu(x)\, dx. \tag{5.4.1}$$

Theorem 12. *If $f \in \mathcal{C}^p$ and $p < N/2$ then*

$$F_k^j = 2^{-j} \sum_l^j f((l + p/2 - k)/N) M^j((l + p/2)/N) + {}_pG^j,$$

where ${}_pG^j = O(N^{-p})$ as p is an even and ${}_pG^j = o(N^{-p})$ as p is an odd number.

Proof: Without loss of generality assume that $k = 0$. Provided $p < N/2$, inside the interval $[-1/2, 1/2]$ the periodic $\mathcal{B}$-spline ${}_pM^j(x)$ coincides with $\mathcal{B}$-spline ${}_pB^j(x)$. Therefore, the Proposition 1 is valid for ${}_pM^j$ as well as for the cardinal $\mathcal{B}$-splines ${}_pB_j$. Namely $\forall t \in [0, 1]$

$$\frac{1}{N} \sum_l^j [(t + l - p/2)/N]^s M^j((t + l)/N) = \mu_s(t),$$

$$\begin{aligned} \mu_s(t) &= M_s = \int_0^{p/N} (x - p/2N)^s M^j(x)\, dx \qquad \text{if} s < p, \\ \mu_p(t) &= N^{-p}(-1)^{p-1}\beta_p(t) + M_p, \end{aligned}$$

$\beta_p(t)$ is the Bernoulli polynomial. If $f \in \mathcal{C}^p$ we may write

$$\begin{aligned} F_0^j &= \int_0^{p/N} f(x) M^j(x)\, dx \\ &= \sum_{s=0}^p \frac{f^{(s)}(p/2N)}{s!} \int_0^{p/N} (x - p/2N)^s M^j(x)\, dx + o(N^{-p}) \\ &= \sum_{s=0}^p \frac{f^{(s)}(p/2N)}{s!} M_s + o(N^{-p}). \end{aligned}$$

Distinguish now two cases.

1.The number p is even. Then

$$\frac{1}{N}\sum_{l}^{j} f((l+p/2)/N)M^j((l+p/2)/N) = \frac{1}{N}\sum_{l}^{j} f(l/N)M^j(l/N)$$

$$= \frac{1}{N}\sum_{l}^{j} M^j(l/N)\sum_{s=0}^{p}\frac{f^{(s)}(p/2N)}{s!}((l-p/2)/N)^s + o(N^{-p})$$

$$= \sum_{s=0}^{p}\frac{f^{(s)}(p/2N)}{s!}\mu_s(0) + o(N^{-p})$$

$$= \sum_{s=0}^{p}\frac{f^{(s)}(p/2N)}{s!}M_s + \frac{N^{-p}\beta_p(0)}{p!}f^{(p)}(p/N) + o(N^{-p})$$

$$= F_0^j + \frac{N^{-p}\beta_p(0)}{p!}f^{(p)}(p/N) + o(N^{-p}).$$

2.The number p is odd. Then

$$\frac{1}{N}\sum_{l}^{j} f((l+p/2)/N)M^j((l+p/2)/N)$$

$$= \frac{1}{N}\sum_{l}^{j} f((l+1/2)/N)M^j((l+1/2)/N)$$

$$= \frac{1}{N}\sum_{l}^{j} M^j((l+1/2)/N)\sum_{s=0}^{p}\frac{f^{(s)}(p/2N)}{s!}((l+1/2-p/2)/N)^s + o(N^{-p})$$

$$= \sum_{s=0}^{p}\frac{f^{(s)}(p/2N)}{s!}\mu_s(1/2) + o(N^{-p})$$

$$= \sum_{s=0}^{p}\frac{f^{(s)}(p/2N)}{s!}M_s + \frac{N^{-p}\beta_p(1/2)}{p!}f^{(p)}(p/N) + o(N^{-p})$$

$$= F_0^j + \frac{N^{-p}\beta_p(1/2)}{p!}f^{(p)}(p/N) + o(N^{-p}).$$

If p is an odd number then $\beta_p(1/2) = 0$. Hence it follows that for the odd p

$$F_k^j = 2^{-j}\sum_{k}^{j} f((l+p/2-k)/N)M^j((l+p/2)/N) + o(N^{-p}).$$

■

Corollary 5. *If $f \in \mathcal{C}^p$ then*

$$\mathcal{T}_r^j(\vec{F}^j) = \mathcal{T}_r^j(\vec{f}^j)\, {}_p u_r^j + {}_p g^j,$$

where $\vec{F}^j = \{F_k^j\}_k^j$ and ${}_p g^j = O(N^{-p})$ as p is an even and ${}_p g^j = o(N^{-p})$ as p is an odd number.

The assertion becomes apparent if we note that the expression $2^{-j}\sum_k^j f((l+p/2-k)/N)M^j((l+p/2)/N)$ is a discrete convolution.

Theorem 13. *Suppose $f(x)$ is an 1-periodic, integrable signal and $\psi^\nu(x) \in \mathcal{W}^\nu$ is the $\mathcal{B}$-wavelet. Let F_k^ν be defined as in (5.4.1) and*

$$\Phi_k^\nu =: \int_{-p/2^\nu}^{(p-1)/2^\nu} f(x - k/2^\nu)\psi^\nu(x)\, dx.$$

Then the following relations hold

$$\mathcal{T}_r^{j-1}(\vec{F}^{j-1}) = \mathcal{T}_r^j(\vec{F}^j)\overline{b_r^j} + \mathcal{T}_{r+N/2}^j\overline{b_{r+N/2}^j}, \tag{5.4.2}$$

$$F_k^{j-1} = \sum_r^{j-1} \omega^{2kr}\mathcal{T}_r^{j-1}(F^{j-1}) = \sum_r^j \omega^{2kr}\overline{b_r^j}\mathcal{T}_r^j(\vec{F}_j),$$

$$\mathcal{T}_r^{j-1}(\vec{\Phi}^{j-1}) = \mathcal{T}_r^j(\vec{F}^j)\,\overline{a}_r^j + \mathcal{T}_{r+N/2}^j(\vec{F}^j)\,a_{r+N/2}^j, \tag{5.4.3}$$

$$\Phi_k^{j-1} = \sum_r^{j-1} \omega^{2kr}\mathcal{T}_r^{j-1}(\vec{F}^{j-1}) = \sum_r^j \omega^{2kr}\,\overline{a}_r^j\,\mathcal{T}_r^j(\vec{F}^j).$$

Proof: Let $R^\nu(x)$ be the FW dual to the $\mathcal{B}$-spline $M^\nu(x)$ and $\Psi^\nu(x)$ be the MW dual to the $\mathcal{B}$-wavelet $\psi^\nu(x)$ If the spline $S^j(f,x)$ is the orthogonal projection of a signal f onto the spline space ${}_p\mathcal{V}^j$ then

$$\begin{aligned} S^j(f,x) &= \sum_k^j F_k^j\, R(x-k/N) = \sum_r^j \xi_r^j m_r^j(x), \\ \xi_r^j &= = \frac{1}{u_r^j}\mathcal{T}_r^j(\vec{F}^j), \qquad \mathcal{T}_r^j(\vec{F}^j) = \xi_r^j\, u_r^j. \end{aligned}$$

Similarly if projections of the signal f onto the spaces ${}_p\mathcal{V}^{j-1}$, ${}_p\mathcal{W}^{j-1}$ are:

$$\begin{aligned} S^{j-1}(f,x) &= \sum_r^{j-1} \xi_r^{j-1} m_r^{j-1}(x), \\ W^{j-1}(f,x) &= \sum_r^{j-1} \eta_r^{j-1} w_r^{j-1}(x) \end{aligned}$$

then

$$\mathcal{T}_r^{j-1}(\vec{F}^{j-1}) = \xi_r^{j-1}\, u_r^{j-1}, \qquad \mathcal{T}_r^{j-1}(\vec{\Phi}^{j-1}) = \eta_r^{j-1}\, v_r^{j-1}.$$

Now we see that (5.4.2) and (5.4.3) are immediate consequences of (4.2.3) and (4.3.4) respectively.

■

Remark 10. If $f \in \mathcal{C}^p$ then it is natural to employ Corollary 5.

As a result of the first step of decomposition we have derived the set $\{\Phi_k^{j-1}\}_k^{j-1}$ from the array $\{F_k^j\}_k^j$. We stress that the value Φ_k^{j-1} carries an information on the behavior of the signal f in the frequency strips $[-N/2, -N/4]$, $[N/4, N/2]$ and in the spatial interval $[\frac{2(k-p)}{N}, \frac{2(k+p-1)}{N}]$. By a similar means we acquire the values $\{\Phi_k^{j-\nu}\}_k^{j-\nu}$, $\nu = 2, ..., m$.

Remark 11. If $f \in \mathcal{C}^p$ then it is natural to employ Corollary 5.

By this means we have transformed the original array $\{f_k^j\}_k^j$ into the array

$$\mathcal{D}^j = \big\{\{\Phi_k^{j-\nu}\}_k^{j-\nu},\ \nu = 1, ..., m,\ \{F_k^{j-m}\}_k^{j-m}\big\}$$

those terms are localized in spatial and frequency domains.

Remark 12. We have described transformation of the original array $\{f_k^j\}$ into the array $\mathcal{D}^j$ associated with $\mathcal{B}$-splines and $\mathcal{B}$-wavelets. The elements of this array appear usually as most informative ones. However, for some special purposes, arrays associated with other FW–MW bases could be of use. The algorithms established in the paper allow to perform corresponding transformation straightforward as well as transformations to arrays allied with wavelet packets.

5.5 Reconstruction of a signal

We dwell now on the situation that is reciprocal to the situation considered in the previous section. We want to reconstruct a signal from the array $\mathcal{D}^j$. The case in point is an approximate reconstruction, of course.

Consider first a single step of the reconstruction.

Problem. *The arrays* $\{F_k^{j-1}\}$, $\{\Phi_k^{j-1}\}$ *are available, where*

$$\begin{aligned} F_k^{j-1} &= \int_0^{p/2^{j-1}} f(x - k/2^{j-1}) M^{j-1}(x)\, dx, \\ \Phi_k^{j-1} &= \int_{-p/2^{j-1}}^{p/2^{j-1}} f(x - k/2^{j-1}) \psi^{j-1}(x)\, dx, \end{aligned}$$

$f(x)$ *is any 1-periodic integrable signal. The coefficients* $\{q_k^j\}_k^j$ *are wanted of the spline*

$$S^j(f, x) = 2^{-j} \sum_k^j q_k^j M^j(x - k/2^j) \tag{5.5.1}$$

which is an orthogonal projection of the signal f onto the space ${}_p\mathcal{V}^j$.

Emphasize that written as in (5.5.1), a spline $S^j(f,x)$ can be computed and, if need be, displayed graphically immediately.

Solution to the Problem

Carrying out the fast Fourier transform we obtain the arrays $\{\mathcal{T}_r^{j-1}(\vec{F}^{j-1})\}$ and $\{\mathcal{T}_r^{j-1}(\vec{\Phi}^{j-1})\}$. Then, using the line of reasoning similar to that of Subsection 5.4, we can maintain that, if the splines

$$\begin{aligned} S^{j-1}(f,x) &= \sum_k^{j-1} \xi_r^{j-1}\, m_r^{j-1}, \\ W^{j-1}(f,x) &= \sum_r^{j-1} \eta_r^{j-1}\, w_r^{j-1}(x), \end{aligned}$$

are orthogonal projections of the signal f onto the spaces ${}_p\mathcal{V}^{j-1}$ ${}_p\mathcal{W}^{j-1}$ correspondingly, then

$$\begin{aligned} \xi_r^{j-1} &= \mathcal{T}_r^{j-1}(\vec{F}^{j-1})/u_r^{j-1}, \\ \eta_r^{j-1} &= \mathcal{T}_r^{j-1}(\vec{\Phi}^{j-1})/v_r^{j-1}. \end{aligned}$$

Now Theorem 6 enables us to write desired coefficients

$$q_k^j = \sum_r^j \omega^{kr}(b_r^j\, \xi_r^{j-1} + a_r^j\, \eta_r^{j-1}).$$

In this manner, given the arrays $\{\Phi_k^{j-\nu}\}$, $\nu = 1, ..., m$, $\{F_k^{j-m}\}$ we are able to reconstruct the spline $S^j(f,x)$ which is an orthogonal projection of the signal f onto ${}_p\mathcal{V}^j$. By the similar way the spline $S^j(f,x)$ can be reconstructed when arrays associated with wavelet packets are available.

In conclusion point out that algorithms suggested can be extended straightforward to the multidimensional case.

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