Integral representation of slowly growing equidistant splines

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Abstract

In this paper we consider equidistant polynomial splines $S(x)$ which may grow as $\mathrm{O}|x|^s$. We present an integral representation of such splines with a distribution kernel. This representation is related to the Fourier integral of slowly growing functions. The part of the Fourier exponentials herewith play the so called exponential splines by Schoenberg. The integral representation provides a flexible tool for dealing with the growing equidistant splines. First, it allows us to construct a rich library of splines possessing the property that translations of any such spline form a basis of corresponding spline space. It is shown that any such spline is associated with a dual spline whose translations form a biorthogonal basis. As examples we present solutions of the problems of projection of a growing function onto spline spaces and of spline interpolation of a growing function. We construct formulas for approximate evaluation of splines projecting a function onto the spline space and establish therewith exact estimations of the approximation errors.

1 Introduction

In this paper we consider equidistant polynomial splines $S(x)$ which may grow as $\mathrm{O}|x|^s$. We present an integral representation of such splines with a distribution kernel. This representation is related, to some extent, to the Fourier integral of slowly growing functions. The part of the Fourier exponentials herewith play the so called exponential splines introduced by Schoenberg [8]. These exponential splines possess a number of nice properties which relate them to the Fourier exponentials. In particular, they are eigenvectors of the operator of shifts and generalized eigenvectors of the operators of differentiation and of convolution with a spline. Being supplied with some multipliers, the exponential splines interpolate the Fourier exponentials and serve as projections of the Fourier exponentials onto spline spaces.

It is worth to note of periodized exponential splines. It was discovered in [4] that these splines form orthogonal bases of spaces of periodic splines. In the papers [13], [14], [7] by the author this idea was used to come up with a concept of Spline Harmonic Analysis which is a version of the harmonic analysis in spaces of periodic splines. This concept enabled us, in particular, to develop a flexible computational scheme of the spline wavelet analysis. Later a related approach was applied to an extended class of periodic functions [5].

The integral representation together with a Parseval type identity provides a flexible tool for dealing with the growing equidistant splines. First, it allows us to construct a rich library of splines possessing the property that translations of any such spline form a basis of corresponding spline space. It is shown that any such spline is associated with a dual spline whose translations form a biorthogonal basis. Once such bases are available, one can match a basis to a problem under consideration.

As examples we present solutions of the problems of projection of a growing function onto spline spaces and of spline interpolation of a growing function. It should be pointed out that the latter problem had been solved completely by Schoenberg [10]. We look at this problem because using our approach it is easy to solve – almost trivial.

We examine thoroughly the splines dual to the fundamental ones and by means of these dual splines we construct formulas for approximate evaluation of splines projecting a function onto the spline space. We establish therewith exact estimations of the approximation errors.

The techniques of integral representation is found to be highly relevant to construction of the scheme of wavelet analysis in the spaces of growing splines. It is done in the work [17] by the author.

The paper is organized as follows. It consists of five sections.

In the introductory Section 2 we list necessary properties of splines with equidistant nodes and, especially, of the $\mathcal{B}$-splines.

Section 3 is basic for the whole work. At the beginning of this one we discuss some properties of periodic distributions. Then we introduce exponential splines. In the concluding subsection the integral representation of growing splines is given and a Parseval type identity is derived.

In Section 4 we establish conditions to be imposed on a spline to ensure that its translations form a basis of the corresponding spline space. We call such splines the $\mathcal{TB}$-splines. Dual splines are constructed as well and examples of $\mathcal{TB}$-splines are given. Further we discuss the problems of projection of a growing function onto spline spaces and of spline interpolation of growing functions.

Section 5 is concerned with approximate evaluation of splines projecting a function onto the spline space. Some properties of moments of the $\mathcal{B}$-splines (([15],[16]) provide us with means to derive an efficient quadrature formula for integrals of smooth functions convolved with the $\mathcal{B}$-splines and to establish therewith exact estimations of the approximation errors. From this quadrature formula we derive the desired evaluation of the projection splines.

2 Some properties of splines with equidistant nodes

This section is an introductory one. We outline here for later use a number of known properties of polynomial splines with equidistant nodes ([3], [8]).
A function ${}_pS_h(x)$ defined on the whole real line will be referred to as a spline of order p if

1. ${}_pS_h(x)$ is $p-2$ times continuously differentiable,
2. ${}_pS_h(x) = P_k(x) \quad \text{as} \quad x \in (x_k, x_{k+1}), \quad P_k(x) \in \Pi_{p-1}$

where Π_{p-1} is the space of polynomials whose degree doesn't exceed $p-1$.
In what follows we deal exclusively with splines whose nodes $\{x_k\}$ are equidistant $x_k = hk$, $k = -\infty, ..., \infty$. These splines are called the *cardinal* ones. The remarkable feature of cardinal splines is that the space of these splines is the *shift invariant* space [2]. It means that the space of cardinal splines of an order p - can be looked upon as the span of translates of a single spline, the so called $\mathcal{B}$-spline.
The $\mathcal{B}$-spline ${}_1B_h(x)$ of the order 1 is the probability density of a random variable uniformly distributed over the interval $[0, h]$

$$
{}_1B_h(x) = \begin{cases} \frac{1}{h} & \text{as } x \in (0, h) \\ 0 & \text{else.} \end{cases}
$$

In turn, the $\mathcal{B}$-spline ${}_pB_h(x)$ of an order p

$$
{}_pB_h(x) =: {}_1B_h(x)^{[p]}
$$

is the probability density of the sum of p uniformly distributed over the interval $[0, h]$ random variables. Here $f^{[p]}$ means the p-th convolution power of a function f. Throughout $\sum_r$ will stand for $\sum_{r=-\infty}^{\infty}$.
Properties of the $\mathcal{B}$-splines:

1. $\operatorname{supp}_p B_h(x) = (0; hp)$.
2. ${}_pB_h(x) > 0$ as $x \in (0, ph)$.
3. The $\mathcal{B}$-spline ${}_pB_h(x)$ is symmetric about $x = hp/2$ where it attains its unique maximum.
4. $h \sum_r {}_pB_h(x - rh) \equiv 1$
5. The product
$$
h\, {}_pB_h(xh) = {}_pB_1(x) \tag{1}
$$
i.e. does not depend on h.
6. The convolution is
$$
{}_pB_h * {}_qB_h(x) = \int_{-\infty}^{\infty} {}_pB_h(x-y)\, {}_qB_h(y)\, dy = {}_{p+q}B_h(x).
$$

7. The derivatives are

$$ {}_pB_h^{(q)}(x) = h^{-q}\nabla_h^q({}_{p-q}B_h(x)) = h^{-s}\sum_{l=0}^{q}(-1)^l\binom{q}{l}\,{}_{p-q}B_h(x-hl). \quad (2)$$

The $\mathcal{B}$-splines of any order can be computed immediately. In what follows we shall use repeatedly the 1-periodic function

$$ {}_pu(v) = h\sum_k e^{2\pi ivk}\,{}_pB_h((p/2-k)h) = \sum_k e^{2\pi ivk}\,{}_pB_1(p/2-k). \quad (3)$$

These functions have been studied extensively [11], [8]. They are related to the Euler-Frobenius polynomials. It is important for us the following property.

Proposition 2.1 *For any real v the functions ${}_pu(v)$ are strictly positive, moreover*

$$0 < K_p = {}_pu(\pi) \le {}_pu(v) \le {}_pu(0) = 1, \quad \lim_{p\to\infty} K_p = 0.$$

The Fourier Transform of the $\mathcal{B}$-spline is:

$$ {}_p\widehat{B}_h(v) = \Big(\frac{1-e^{-ivh}}{ivh}\Big)^p = e^{-\frac{ipvh}{2}}\Big(\frac{\sin vh/2}{vh/2}\Big)^p. \quad (4)$$

Proposition 2.2 *[9]. Any cardinal spline of order p ${}_pS_h(x)$ with its nodes at the points $\{hk\}_{-\infty}^{\infty}$ can be represented as follows*

$$ {}_pS_h(x) = h\sum_k q_k\,{}_pB_h(x-hk). \quad (5)$$

Remark If x is any fixed value, $lh \le x \le (l+1)h$ then the series (5) contains only p nonzero addends, $l-p+1 \le k \le l$. So, given a set of coefficients $\{q_k\}$, values of the spline ${}_pS_h(x)$ can be computed immediately. Moreover, Property 4 of the $\mathcal{B}$-splines implies that in this case the inequality holds:

$$\Big|{}_pS_h(x)\Big| \le \max\{|q_l|\}, \quad l-p+1 \le k \le l. \quad (6)$$

3 Integral representation of splines

In this section we restrict the class of splines under consideration and introduce a transform in spline spaces which results in an integral representation of splines which is related to the Fourier integral.

Definition 3.1 *The space of sequences $\vec{a} = \{a_k\}_{-\infty}^{\infty}$ each of which meets the requirement $|a_k| \le M|k|^s\,\forall k$ with the integer s and a positive constant M we will denote by $\mathbf{G}^s$. The space $\mathbf{G} =: \bigcup_{-\infty}^{\infty}\mathbf{G}^s$ we will call the space of sequences of slow growth. The space of locally integrable functions $f(x)$ each of which meets the requirement $|f(x)| \le M|x|^s\,\forall x$ we will denote by $\mathbf{F}^s$. The space $\mathbf{F} =: \bigcup_{-\infty}^{\infty}\mathbf{F}^s$ we will call the space of functions of slow growth.*

Definition 3.2 *We denote by ${}_p\mathbf{V}_h^s = \{{}_pS_h(x)\}$ the space of splines such that the sequence $\vec{q} = \{q_k\}_{-\infty}^{\infty}$ in the representation (5) of ${}_pS_h(x)$ belongs to $\mathbf{G}^s$ and the space ${}_p\mathbf{V}_h =: \bigcup_{-\infty}^{\infty}{}_p\mathbf{V}_h^s$.*

Remark. We stress that for any spline ${}_pS_h(x) \in {}_p\mathbf{V}_h^s$ the inequality holds:

$$\Big|{}_pS_h(x)\Big| \le L|x|^s \quad (7)$$

with some positive constant L. It follows straightforward from (6). Therefore we may maintain that ${}_p\mathbf{V}_h^s \subset \mathbf{F}^s$

3.1 Some remarks on periodic distributions

Let $\vec{a} = \{a_k\}_{-\infty}^{\infty} \in \mathbf{G}$. Denote

$$\mathcal{F}(\vec{a}, v) = \sum_k e^{-2\pi ikv} a_k. \tag{8}$$

This series is an 1-periodic distribution [12], p.331.

Definition 3.3 *We denote by $\mathbf{D}^s$ the space of 1-periodic distributions given by (8) with $\vec{a} \in \mathbf{G}^s$, and $\mathbf{D} =: \bigcup_{-\infty}^{\infty} \mathbf{D}^s$. The space of 1-periodic complex-valued functions which have s continuous derivatives we will denote as $\mathbf{C}^s$.*

Emphasize that $\mathbf{D}^{-s-2} \subset \mathbf{C}^s$.
Given a sequence $\vec{a} \in \mathbf{G}^s$, we define the function

$$\Phi(\vec{a}, v) =: \Big(\sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty} \Big) \frac{a_k}{(-2\pi ik)^{s+2}} e^{-2\pi ikv} \in \mathbf{D}^{-2} \subset \mathbf{C}^0.$$

Then the distribution $\mathcal{F}_h(\vec{a}, v)$ can be represented as follows:

$$\mathcal{F}(\vec{a}, v) = a_0 + \Phi^{(s+2)}(\vec{a}, v), \tag{9}$$

where the derivative is used in the sense of the distribution theory [12].
The distribution $\mathcal{F}(\vec{a}, v)$ defines the functional on the space $\mathbf{C}^{s+2}$ which will be denoted as the integral with the central dot. To be specific, $\forall\, g(v) \in \mathbf{C}^{s+2}$

$$\int_{\alpha}^{\alpha+1} \mathcal{F}(\vec{a}, v) \ \cdot \overline{g(v)}\, dv =: a_0 \int_{\alpha}^{\alpha+1} \overline{g(v)}\, dv \tag{10}$$
$$+ \int_{\alpha}^{\alpha+1} \Phi(\vec{a}, v)\, \overline{g^{(s+2)}}\, dv = \sum_k a_k\, \overline{g_k}.$$

Here $\vec{g} = \{g_k\}$ is the sequence of the Fourier coefficients of the function g. The integrals in the right hand side should be understood in a conventional sense.
Remark. Of course, if $\vec{a} \in l_1$ then $\mathcal{F}_h(\vec{a}, v) \in \mathbf{C}^0$ and the integral (10) with any integrable function g is to be understood in a conventional sense.
The series in (7) is the Fourier series of the distribution $\mathcal{F}_h(\vec{a}, v)$. Hence

$$a_k = \int_0^1 \mathcal{F}(\vec{a}, v) \cdot e^{2\pi ivk}\, dv$$

are the Fourier coefficients of the distribution.
Let us discuss multiplication of the distribution $\mathcal{F}(\vec{a}, v) \in \mathbf{D}^s$ with a continuous function $g(v) = \mathcal{F}(\vec{g}, v)$ such that the sequence $\vec{g}$ of its Fourier coefficients belongs to $\mathbf{G}^{-s-2}$. The discrete convolution of the sequences $\vec{a}$ and $\vec{g}$ is:

$$\vec{b} =: \{b_k\} = \vec{a} * \vec{g} = \{ \sum_l a_{k-l}\, g_l \}$$

The following assertion is readily verified.

Proposition 3.1 *Provided a sequence $\vec{g}$ belongs to $\mathbf{G}^{-s-2}$ and a sequence $\vec{a} \in \mathbf{G}^s$, its discrete convolution $\vec{b}$ belongs to the space $\mathbf{G}^s$ as well as $\vec{a}$. Provided a function $f(x)$ belongs to $\mathbf{F}^{-s-2}$ and function $h(x) \in \mathbf{F}^s$, its continuous convolution $f * h(x)$ belongs to the space $\mathbf{F}^s$ as well as $h(x)$.*

The proposition implies that the series

$$b(v) =: \sum_k e^{-2\pi ikv} b_k = \mathcal{F}(\vec{b}, v)$$

is the distribution of the space $\mathbf{D}^s$ as well as $\mathcal{F}(\vec{a}, v)$. Now let $\varphi(v)$ be any testing function of $\mathbf{C}^{s+2}$ and $\{\varphi_k\}$ are its Fourier coefficients. Let us consider the integral

$$\int_0^1 b(v) \cdot \overline{\varphi(v)}\, dv = \sum_k \overline{\varphi_k} \sum_l g_{k-l} a_l$$
$$= \sum_l a_l \sum_k \overline{\varphi_k\, g_{l-k}} = \int_0^1 \mathcal{F}_h(\vec{a}, v) \cdot \overline{g(v)\varphi(v)}\, dv.$$

This relation justifies the following

Definition 3.4 *The product of a distribution of* $\mathbf{D}^s$ *with a function of* $\mathbf{D}^{-s-2}$ *will be understood as follows:*

$$\mathcal{F}(\vec{a}, v)\, g(v) =: \mathcal{F}(\vec{a} * \vec{g}, v) \in \mathbf{D}^s. \tag{11}$$

It corresponds with the conventional definition of multiplication of a distribution with a function.

3.2 Exponential splines

Let us return to the $\mathcal{B}-$spline. Eq. (4) implies

$$\begin{aligned} {}_pB_h(x-kh) &= \int_{-\infty}^{\infty} e^{2\pi i\omega(x-kh)} \Big(\frac{1-e^{-2\pi i\omega h}}{2\pi i\omega h}\Big)^p d\omega \\ &= \frac{1}{h}\sum_l \int_0^1 e^{2\pi i(v-l)(x/h-k)} \frac{(1-e^{-2\pi iv})^p}{(2\pi i(v-l))^p}\, dv. \end{aligned}$$

Integration and summation here can be transposed and we appear at the following representation

$$ {}_pB_h(x-kh) = \frac{1}{h}\int_0^1 {}_pm_h(v,x) e^{-2\pi ikv}\, dv, \quad \text{where} \tag{12}$$

$$\begin{aligned} {}_pm_h(v,x) &=: \sum_l e^{2\pi i(v-l)x/h} \Big(\frac{1-e^{-2\pi iv}}{2\pi i(v-l)}\Big)^p \\ &= e^{2\pi iv(x/h-p/2)}(\sin\pi v)^p \sum_l e^{-2\pi ilx/h}(\pi(v-l))^{-p}. \end{aligned} \tag{13}$$

Here the product $h\, {}_pB_h(x-kh)$ appears as the k-th Fourier coefficient of the 1-periodic with respect to the variable v function ${}_pm_h(v,x)$. Hence

$$ {}_pm_h(v,x) = h\sum_k e^{2\pi ikv} {}_pB_h(x-kh) \in {}_p\mathbf{V}_h^0. \tag{14}$$

It is apparent from this relation that with any x, ${}_pm_h(v,x) \in \mathbf{C}^\infty$. As for the variable x, with any v, ${}_pm_h(v,x)$ is a spline of ${}_p\mathbf{V}_h^0$.
The spline ${}_pm_h(v,x)$ is the exponential spline in a sense by Schoenberg, [8], p.17, ${}_pm_h(v,x) = \Phi_n(x;t)$ with $t = e^{2\pi iv}, n = p-1$.
We shall not discuss here numerous noteworthy properties of the splines m but point out only ones to be used in sequel.
Point out first that, due to (1),

$$ {}_pm_h(v,x) = {}_pm_1(v, \frac{x}{h}). \tag{15}$$

Proposition 3.2 *The splines ${}_pm_h(v,x)$ are eigenvectors of the shift operator. Namely*

$$ {}_pm_h(v,x+lh) = e^{2\pi ilv}\,{}_pm_h(v,x). \tag{16}$$

Now put $x = ph/2$. Then we have

$$\begin{aligned} {}_pm_h(v,ph/2) &= h\sum_k e^{2\pi ikv}\,{}_pB_h((p/2-k)h) \qquad (17)\\ = {}_pu(v) &= \left(\frac{\sin \pi v}{\pi}\right)^p \sum_n \frac{(-1)^{pn}}{(v-n)^p}. \end{aligned}$$

Recall that ${}_pu(v)$ was primarily defined in (3).

Proposition 3.3 *The exponential spline ${}_pm_h(v,x)$ is a generalized eigenvector of the operator of differentiation in a sense that:*

$$ {}_pm_h(v,x)^{(s)}_x = \left(\frac{1-e^{-2\pi iv}}{h}\right)^s {}_{p-s}m_h(v,x). \tag{18}$$

Proof: Eq. (13) implies

$$\begin{aligned} {}_pm_h(v,x)^{(s)}_x &= h^{-s}\sum_l e^{2\pi i(v-l)x/h}\frac{(1-e^{-2\pi iv})^p}{(2\pi i(v-l))^{p-s}} \\ &= \left(\frac{1-e^{-2\pi iv}}{h}\right)^s \sum_l e^{2\pi i(v-l)x/h}\left(\frac{1-e^{-2\pi iv}}{2\pi i(v-l)}\right)^{p-s} \\ &= \left(\frac{1-e^{-2\pi iv}}{h}\right)^s {}_{p-s}m_h(v,x). \end{aligned}$$

■

Proposition 3.4 *The derivative ${}_pm_h(v,x)^{(s)}_v$ is a spline of the space ${}_p\mathbf{V}^s_h$.*

Proof: The statement stems immediately from (14). ■

Let us discuss the convolution of the exponential spline with a spline of ${}_p\mathbf{V}^s$. We start with the convolution with the $\mathcal{B}$-spline. From Property 6 of the $\mathcal{B}$-splines it is seen that

$$\begin{aligned} {}_pm_h(v,\cdot) * {}_rB_h(x) &= h\sum_k e^{2\pi ikv}\,{}_pB_h(x-kh) * {}_rB_h(x) \\ &= h\sum_k e^{2\pi ikv}\,{}_{p+r}B_h(x-kh) = {}_{p+r}m_h(v,x). \qquad (19) \end{aligned}$$

Hence it follows

$$\begin{aligned} {}_pm_h(v,\cdot) * h\sum_{k=-n}^{n} q_k\,{}_rB_h(x-hk) &= h\sum_{k=-n}^{n} q_k\,{}_{p+r}m_h(v,x-hk) \\ &= {}_{p+r}m_h(v,x)h\sum_{k=-n}^{n} e^{-2\pi ikv}\,q_k. \qquad (20) \end{aligned}$$

Due to the symmetry of the $\mathcal{B}$-splines we have ${}_pB_h(x) = {}_pB_h(ph - x)$. Hence there holds

$$\begin{aligned}\int_{-\infty}^{\infty} {}_pm_h(v,x)\,{}_pB_h(x-kh)\,dx &= {}_pm_h(v,\cdot) * {}_pB_h((k+p)h) \\ = {}_{2p}m_h(v,(k+p)h) &= e^{2\pi ikv}\,{}_{2p}m_h(v,ph) = e^{2\pi ikv}\,{}_{2p}u(v). \end{aligned} \tag{21}$$

Respectively, we have

$$\int_{-\infty}^{\infty} {}_pm_h(v,x)\,h\sum_{k=-n}^{n} q_k\,{}_pB_h(x-kh)\,dx = {}_{2p}u(v)h\sum_{k=-n}^{n} e^{-2\pi ikv}\,q_k. \tag{22}$$

Now, assume that a a spline

$${}_rS_h(x) = h\sum_k q_k\,{}_rB_h(x-hk), \quad \text{and } \vec{q} = \{q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s.$$

We denote

$${}_rS_h^n(x) = h\sum_{k=-n}^{n} q_k\,{}_rB_h(x-hk) \in {}_r\mathbf{V}^{-\infty}$$

Let $\varphi(v)$ be any testing function of $\mathbf{C}^{s+2}$. Then (19) implies that

$$I_n(\varphi) =: \int_0^1 \varphi(v)\Big[{}_pm_h(v,\cdot) *_p S_h^n(x)\Big]\,dv = \int_0^1 dv\,\varphi(v)\,{}_{p+r}m_h(v,x)h\sum_{k=-n}^{n} e^{-2\pi ikv}\,q_k.$$

It is readily verified that

$$\lim_{n\to\infty} I_n = \int_0^1 \varphi(v)\psi(v)\,dv,$$

where the distribution

$$\psi(v) =: h\mathcal{F}(\vec{q},v)\,{}_{p+r}m_h(v,x) \in \mathbf{D}^s.$$

It is pertinent now to define the distributional convolution as follows

$${}_pm_h(v,\cdot)\tilde{*}_p S_h(x) =: \psi(v) = h\mathcal{F}(\vec{q},v)\,{}_{p+r}m_h(v,x). \tag{23}$$

Similarly we define the integral

$$\int_{-\infty}^{\infty} {}_pS_h(x)h\,\tilde{}\,\overline{{}_pm_h(v,x)}\,dx =: h\mathcal{F}(\vec{q},v)\,{}_{2p}u(v). \tag{24}$$

Point out that in the event when the sequence $\vec{q} \in l^1$, we have in (23), (24) the conventional convolution and the integral. We may maintain that exponential splines ${}_pm_h(v,x)$ are generalized eigenvectors of the operator of convolution with a spline in a sense of Eq. (23)

3.3 Integral representation

We proceed now to establishing central results of this section.

Theorem 3.1 *Let a distribution $\xi(v) \in \mathbf{D}^s$ with an integer s. Then the function*

$$\sigma(x) =: \int_0^1 \xi(v)\cdot{}_pm_h(v,x)\,dv \tag{25}$$

is a spline of ${}_p\mathbf{V}^s \subset \mathbf{F}^s$*:*

$$\sigma(x) = {}_pS_h(x) = h\sum_k q_k\, {}_pB_h(x-hk), \quad \vec{q} = \{q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s \tag{26}$$

therewith the coefficients in (26) are:

$$q_k = \int_0^1 \xi(v)\cdot e^{2\pi ivk}\, dv.$$

Conversely, any spline belonging to ${}_p\mathbf{V}^s$ *may be represented as the integral (25) with* $\xi(v) \in \mathbf{D}^s$*.*

Proof: Let

$$q_k = c_k\Big(\xi(\cdot)\Big) = \int_0^1 \xi(v)\cdot e^{2\pi ivkh}\, dv.$$

be the Fourier coefficients of a distribution $\xi(v) \in \mathbf{D}^s$. The sequence $\vec{q} = \{q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s$. Then (10) entails

$$\begin{aligned} \sigma(x) &= \sum_k q_k\, c_k\Big({}_pm(\cdot, x)\Big) \\ &= h\sum_k q_k\, {}_pB_h(x-hk) = {}_pS_h(x) \in {}_p\mathbf{V}^s. \end{aligned}$$

Conversely, assume that ${}_pS_h(x)$ is a spline belonging to ${}_p\mathbf{V}^s$ and is given as in (26). Then

$$\mathcal{F}(\vec{q}, v) = \sum_k e^{-2\pi ikv} q_k \in \mathbf{D}^s \quad \text{and}$$

$$\sigma(x) =: \int_0^1 \mathcal{F}(\vec{q}, v)\cdot {}_pm_h(v, x)\, dv$$

is a spline of ${}_p\mathbf{V}^s$. Its $\mathcal{B}-$spline coefficients are

$$Q_k = \int_0^1 \xi(v)\cdot e^{2\pi ivk}\, dv = q_k.$$

Therefore,

$${}_pS_h(x) \equiv \sigma(x) = \int_0^1 \mathcal{F}_h(\vec{q}, v)\cdot {}_pm_h(v, x)\, dv \tag{27}$$

■

Remark. Point out the distribution $\xi(v)$ may be written in the integral form in accordance with (24):

$$\xi(v) = \mathcal{F}(\vec{q}, v) = \Big(h\; {}_{2p}u(v)\Big)^{-1} \int_{-\infty}^{\infty} {}_pS_h(x) h \tilde{\ } \overline{{}_pm_h(v, x)}\, dx.$$

Example. For the spline ${}_pm_h(u, x)$ the $\mathcal{B}-$spline coefficients are $q_k = e^{2\pi iuk}$ and, therefore,

$$\mathcal{F}(\vec{q}, v) = h\sum_k e^{-2\pi ik(v-u)} = \sum_l \delta(u - v - l). \tag{28}$$

Here $\delta(u)$ is the Dirac delta. We will derive now an identity related to the Parseval one. It is basic for dealing with integral-represented splines.

Theorem 3.2 *Let a spline*

$$ {}_pS_h(x) = \int_0^1 \xi(v) \cdot {}_pm_h(v,x)\,dv = h\sum_k q_k\,{}_pB_h(x-hk) $$

belong to ${}_p\mathbf{V}^s$, $s \geq 0$ *and a spline*

$$ {}_pT_h(x) = \int_0^1 \eta(v)\,{}_pm_h(v,x)\,dv = h\sum_k t_k\,{}_pB_h(x-hk) $$

belong to ${}_p\mathbf{V}^{-s-4}$. *Then the following identity holds:*

$$ \int_{-\infty}^{\infty} {}_pS_h(x)\,\overline{{}_pT_h(x)}\,dx = h\int_0^1 \xi(v)\cdot\overline{\eta(v)}\,{}_{2p}u(v)\,dv. \tag{29} $$

Proof: We recall first of all Eq.(7) to maintain that the integral in the left hand side of (29) converges absolutely. As for the right hand side integral, it exists in the sense of (10) since $\xi(v) \in \mathbf{D}^s, \quad \eta(v) \in \mathbf{C}^{s+2}$. Let us consider the integral

$$ \begin{aligned} I_n &=: \int_{-nh}^{nh} {}_pS_h(x)\,\overline{{}_pT_h(x)}\,dx \\ &= \int_{-nh}^{nh} h\sum_{k=-n-p+1}^{n-1} q_k\,{}_pB_h(x-hk)\,\overline{{}_pT_h(x)}\,dx \\ &= I_n^1 + I_n^2 + I_n^3. \end{aligned} $$

We examine first the integral

$$ I_n^1 =: \int_{-nh}^{nh} h\sum_{k=-n-p+1}^{-n-1} q_k\,{}_pB_h(x-hk)\,\overline{{}_pT_h(x)}\,dx. $$

Since $\operatorname{supp}{}_pB_h(x) = (0;hp)$, the integral I_n^1 is being computed, in fact, over the interval $[(-n-p+1)h, (-n+p-1)h]$. This fact results in the inequality

$$ |I_n^1| \leq \sum_{k=-n-p+1}^{-n-1} |q_k| \int_{(-n-p+1)h}^{(-n+p-1)h} |{}_pT_h(x)|\,dx. $$

Appealing again to (7), we appear at

$$ |I_n^1| \leq Mp(n+p-1)^s \cdot Lp(n+p-1)^{-s-4} \to 0 \quad \text{as} \quad n \to \infty. $$

By the same reasons we affirm that the integral

$$ I_n^3 =: \int_{-nh}^{nh} h\sum_{k=n-p+1}^{n-1} q_k\,{}_pB_h(x-hk)\overline{{}_pT_h(x)}\,dx. $$

tends to zero as $\quad n \to \infty$. Let us turn now to the integral

$$ \begin{aligned} I_n^2 &=: \int_{-nh}^{nh} dx\,h\sum_{k=-n}^{n-p} q_k\,{}_pB_h(x-hk)\overline{{}_pT_h(x)}\,dx \\ &= \int_{-nh}^{nh} dx\,h\sum_{k=-n}^{n-p} q_k\,{}_pB_h(x-hk)\int_0^1 \overline{\eta(v)\,{}_pm_h(v,x)}\,dv \\ &= \int_0^1 \overline{\eta(v)}\,dv\,h\sum_{k=-n}^{n-p} q_k \int_{-\infty}^{\infty} {}_pB_h(x-hk)\overline{{}_pm_h(v,x)}\,dx. \end{aligned} $$

Eq.(21) enables us to write

$$
\begin{aligned}
I_n^2 &= \int_0^1 \overline{\eta(v)}\, {}_{2p}u(v)\, dv \left[hq_0 + h\Big(\sum_{k=-n}^{-1} + \sum_{k=1}^{n-p}\Big) q_k\, e^{-2\pi ikv}\right] \\
&= hq_0 \int_0^1 \overline{\eta(v)}\, {}_{2p}u(v)\, dv \\
&+ \int_0^1 \left(\overline{\eta(v)}\, {}_{2p}u(v)\right)^{(s+2)} dv\, h\Big(\sum_{k=-n}^{-1} + \sum_{k=1}^{n-p}\Big) \frac{q_k}{(-2\pi ik)^{s+2}}\, e^{-2\pi ikv}.
\end{aligned}
$$

The series

$$\Phi(\vec{q}, v) =: h\Big(\sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty}\Big) \frac{q_k}{(-2\pi ik)^{s+2}} e^{-2\pi ikv}$$

converges absolutely and uniformly with respect to the variable v. Consequently, we have:

$$
\begin{aligned}
\lim_{n\to\infty} I_n &= \int_{-\infty}^{\infty} {}_pS_h(x)\, \overline{{}_pT_h(x)}\, dx \\
&= \lim_{n\to\infty} I_n^2 = hq_0 \int_0^1 \overline{\eta(v)}\, {}_{2p}u(v)\, dv \\
&+ h \int_{-1/2}^{1/2} \left(\overline{\eta(v)}\, {}_{2p}u(v)\right)^{(s+2)} \Phi(\vec{q}, v)\, dv \\
&= h \int_0^1 \xi(v) \cdot \overline{\eta(v)}\, {}_{2p}u(v)\, dv.
\end{aligned}
$$

■

4 Splines of type $\mathcal{B}$

4.1 $\mathcal{TB}$-splines and their duals

Definition 4.1 *We will refer to a set of splines $\{{}_ps_k(x)\}_{-\infty}^{\infty}$ as to a basis of the space ${}_p\mathbf{V}_h$ if any spline ${}_pS_h(x) \in {}_p\mathbf{V}_h$ can be represented uniquely as the series*

$${}_pS_h(x) = \sum_{-\infty}^{\infty} a_k\, {}_ps_k(x)$$

which converges uniformly at any compact set of the real line.

The translations of the $\mathcal{B}$-spline form a basis of the space ${}_p\mathbf{V}_h$. We describe now a class of splines which offer the similar property.

Definition 4.2 *We call a spline ${}_p\varphi_h(x) \in {}_p\mathbf{V}_h^{-\infty}$ the spline of type $\mathcal{B}$ ($\mathcal{TB}$-spline) if its shifts $\{{}_p\varphi_h(x - kh)\}_{-\infty}^{\infty}$ form a basis of the space ${}_p\mathbf{V}_h$.*

We stress that any spline ${}_p\varphi_h(x) \in {}_p\mathbf{V}_h^{-\infty}$ can be represented as the integral

$${}_p\varphi_h(x) = \int_0^1 \rho(v)\, {}_pm_h(v, x)\, dv. \tag{30}$$

with a function $\rho(v) \in D^{-\infty} = \mathbf{C}^{\infty}$. Moreover, the inequality (7) enables us to affirm that for all x belonging to any compact set of the real line the estimate is true:

$$|{}_p\varphi_h(x-kh)| \le C\nu_k, \tag{31}$$

with a sequence $\{\nu_k\} \subset \mathbf{G}^{-\infty}$.

Theorem 4.1 *Let a spline ${}_p\varphi_h(x) \in {}_p\mathbf{V}_h^{-\infty}$ be represented as in(30). Then it is a $\mathcal{TB}$-spline if and only if the function $|\rho(v)|$ is strictly positive for all real v. Herewith, if a spline ${}_pS_h(x) \in {}_p\mathbf{V}_h^s$ is represented in two forms*

$$ {}_pS_h(x) = \sum_k Q_k\, {}_p\varphi_h(x-kh) = \int_0^1 \xi(v)\cdot {}_pm_h(v,x)\,dv.$$

$$\begin{aligned} \textit{Then} \qquad \xi(v) &= \rho(v)\sum_k e^{-2\pi ikv}Q_k, \\ Q_k &= \int_0^1 \frac{\xi(v)}{\rho(v)}\cdot e^{2\pi ivk}\,dv. \end{aligned} \tag{32}$$

Proof: Point out first that $\rho(v) \in \mathbf{C}^{\infty}$. Since it is 1-periodic, we may write, keeping in mind (16),

$$ {}_p\varphi_h(x-kh) = \int_0^1 e^{-2\pi ikv}\rho(v)\,{}_pm_h(v,x)\,dv. \tag{33}$$

These are the Fourier coefficients of the function $\rho(v)\,{}_pm_h(v,x) \in \mathbf{C}^{\infty}$ of the variable v. Hence

$$\rho(v)\,{}_pm_h(v,x) = \sum_k e^{2\pi ikv}\,{}_p\varphi_h(x-kh). \tag{34}$$

1.Let $|\rho(v)|$ be strictly positive. Suppose a spline ${}_pS_h(x) \in {}_p\mathbf{V}_h^s$ be represented as in (25):

$$ {}_pS_h(x) = \int_0^1 \xi(v)\cdot {}_pm_h(v,x)\,dv.$$

Then in accordance with (10) we may write

$$\begin{aligned} {}_pS_h(x) &= \int_0^1 \frac{\xi(v)}{\rho(v)}\cdot\rho(v)\,{}_pm_h(v,x)\,dv = \sum_k Q_k\varphi_h(x-kh), \\ Q_k &= \int_0^1 \frac{\xi(v)}{\rho(v)}\cdot e^{2\pi ikv}\,dv. \end{aligned}$$

Since the values Q_k are the Fourier coefficients of the distribution $\frac{\xi(v)}{\rho(v)} \in \mathbf{D}^s$, the sequence $\vec{Q} = \{Q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s$. Then from (31) we see that the series in the right hand side converges uniformly at any compact set of the real line. This implies that ${}_p\varphi_h(x)$ is a $\mathcal{TB}$-spline.
2.Conversely, suppose that ${}_p\varphi_h(x)$ given by (30) is a $\mathcal{TB}$-spline. Its translations form a basis of the space ${}_p\mathbf{V}_h$. Therefore

$$ {}_pm_h(v,x) = \sum_k \mu_k(v)\,{}_p\varphi_h(x-kh)$$

and, substituting it to (34) we appear at

$$\rho(v) \sum_k \mu_k(v)_p\varphi_h(x-kh) = \sum_k e^{2\pi ikv}{}_p\varphi_h(x-kh) \Longrightarrow \rho(v)\,\mu_k(v) = e^{2\pi ikv}.$$

Hence it follows that $\rho(v) \neq 0$ for all real v. But $\rho(v)$ is a continuous 1-periodic function. Therefore $|\rho(v)|$ is strictly positive. ■

Generally, bases formed from translations of $\mathcal{TB}$- splines are non-orthogonal in the L_2 sense. However, biorthogonal bases exist.

Definition 4.3 *Let two splines* ${}^1_p\varphi_h(x)$, ${}^2_p\varphi_h(x) \in {}_p\mathbf{V}_h$ *be* $\mathcal{TB}$- *splines. We refer to these as to the dual ones if there holds the relation*

$$\int_{-\infty}^{\infty} {}^1_p\varphi_h(x-kh)\,\overline{{}^2_p\varphi_h(x-lh)}\,dx = \delta_l^k,$$

where δ_l^k *means the Kroneker delta.*

We will show that any $\mathcal{TB}$ -spline has a dual one.

Theorem 4.2 *Let a* $\mathcal{TB}$ *-spline be represented as follows:*

$${}^1_p\varphi_h(x) = \int_0^1 {}^1\rho(v)\,{}_pm_h(v,x)\,dv.$$

Then there exists an unique $\mathcal{TB}$ *-spline dual to* ${}^1_p\varphi_h(x)$*:*

$$\begin{aligned} {}^2_p\varphi_h(x) &= \int_0^1 {}^2\rho(v)\,{}_pm_h(v,x)\,dv, \\ {}^2\rho(v) &= (\overline{{}^1\rho(v)}\,{}_{2p}u(v)\,h)^{-1}. \end{aligned} \tag{35}$$

Proof: Let a spline ${}^2_p\varphi_h(x)$ be given as in (35). Then, in accordance with the identity (29) and Eq. (33) we have

$$\int_{-\infty}^{\infty} {}^1_p\varphi_h(x-kh)\,\overline{{}^2_p\varphi_h(x-lh)}\,dx = h\int_0^1 e^{-2\pi i(k-l)v}\,{}^1\rho(v)\,\overline{{}^2\rho(v)}\,{}_{2p}u(v)\,dv.$$

If (35) holds then

$$\int_{-\infty}^{\infty} {}^1_p\varphi_h(x-kh)\,\overline{{}^2_p\varphi_h(x-lh)}\,dx = \int_0^1 e^{-2\pi i(k-l)v}\,dv = \delta_l^k.$$

We stress that in this case ${}^2_p\varphi_h(x) \in {}_p\mathbf{V}_h^{-\infty}$ just as ${}^1_p\varphi_h(x)$.
Conversely, the relation

$$h\int_0^1 e^{-2\pi i(k-l)v}\,{}^1\rho(v)\,\overline{{}^2\rho(v)}\,{}_{2p}u(v)\,dv = \delta_l^k$$

means that all of the Fourier coefficients c_k of the continuous function ${}^1\rho(v)\,\overline{{}^2\rho(v)}\,{}_{2p}u(v)$, with $k \neq 0$ are zero, whereas $c_0 = \frac{1}{h}$. Therefore (35) is true. It means that the dual spline is unique.■

Theorem 4.3 *Let two $\mathcal{TB}$ -splines ${}^i_p\varphi_h(x), \quad i = 1, 2,$ be represented as follows:*

$$ {}^i_p\varphi_h(x) = \int_0^1 {}^i\rho(v)\, {}_pm_h(v, x)\, dv. $$

and a spline ${}_pS_h(x) \in {}_p\mathbf{V}_h^s$ is expanded with respect to the two bases

$$ {}_pS_h(x) = \sum_k {}^1Q_k\, {}^1_p\varphi_h(x - kh) = \sum_k {}^2Q_k\, {}^2_p\varphi_h(x - kh). \tag{36} $$

Then the coordinates are related as follows:

$$ {}^2Q_k = \sum_l b^{1,2}_{k-l}\, {}^1Q_l, \qquad where \quad b^{1,2}_r = \int_0^1 \frac{{}^1\rho(v)}{{}^2\rho(v)} e^{2\pi ivr}\, dv \tag{37} $$

are the Fourier coefficients of the function $\frac{{}^1\rho(v)}{{}^2\rho(v)}$. In particular,

$$ {}^1_p\varphi_h(x - lh) = \sum_k b^{1,2}_{k-l}\, {}^2_p\varphi_h(x - kh). \tag{38} $$

Proof: The spline ${}_pS_h(x)$ may be written as in (32). Then

$$ \xi(v) = {}^1\rho(v)\mathcal{F}_h({}^1\vec{Q}, v) = {}^2\rho(v)\mathcal{F}_h({}^2\vec{Q}, v). $$

Hence we come to the equation

$$ \mathcal{F}_h({}^2\vec{Q}, v) = \frac{{}^1\rho(v)}{{}^2\rho(v)}\mathcal{F}_h({}^1\vec{Q}, v). $$

Since $\frac{{}^1\rho(v)}{{}^2\rho(v)} \in \mathbf{D}^{-\infty}$, we appear at (36), keeping in mind (11). To derive (38) we should put in (36) ${}^1Q_k = \delta^l_k$. ■

Let us set aside the special case of (38) when ${}^1_p\varphi_h(x) = h\, {}_pB_h(x)$ is the $\mathcal{B}$-spline. At this case due to (12) we have ${}^1\rho(v) \equiv 1$ and, therefore,

$$ {}^2_p\varphi_h(x - lh) = h\sum_k \beta^{2,1}_{k-l}\, {}_pB_h(x - kh), \qquad h\, {}_pB_h(x - lh) = \sum_k \beta^{1,2}_{k-l}\, {}^2_p\varphi_h(x - kh). \tag{39} $$

$$ \beta^{1,2}_r = \int_0^1 \frac{1}{{}^2\rho(v)} e^{2\pi ivr}\, dv, \qquad \beta^{2,1}_r = \int_0^1 {}^2\rho(v) e^{2\pi ivr}\, dv. \tag{40} $$

Emphasize that the sequences $\{\beta^{2,1}_r\} \in \mathbf{G}^{-\infty}$.

4.2 Selfdual $\mathcal{TB}$-splines

Let us consider the $\mathcal{TB}$-spline

$$ {}^o_p\varphi_h(x) = \int_0^1 \rho(v)\, {}_pm_h(v, x)\, dv $$

with

$$ \rho(v) = \frac{1}{\sqrt{{}_{2p}u(v)\, h}}. \tag{41} $$

It is readily seen from (35) that this $\mathcal{TB}$-spline coincides with its dual one. So, it is pertinent to call it the selfdual $\mathcal{TB}$-spline. The shifts $\{{}^o_p\varphi_h(x - kh)\}_{-\infty}^{\infty}$ form an orthonormal basis in a sense by L_2 of the space ${}_p\mathbf{V}_h$. These $\mathcal{TB}$-splines had been discovered by Battle and Lemarié [1], [6].

4.3 Galerkin projections

Since the spaces of functions we operate with are not the Hilbert ones, we should introduce, instead of the notion of orthogonal projection, its weak substitution.

Definition 4.4 *Let $f(x)$ be a function of slow growth. We refer to a spline $S(f,x) \in {}_p\mathbf{V}_h$ as to the Galerkin projection (GP) of the function f onto the spline space ${}_p\mathbf{V}_h$ if for all integers k the following relations are true:*

$$\int_{-\infty}^{\infty} S(f,x)\,{}_pB_h(x-kh)\,dx = \int_{-\infty}^{\infty} f(x)\,{}_pB_h(x-kh)\,dx =: \Phi_k. \tag{42}$$

Remark. In the case when the function f is square summable on the real line, its GP is just the same as the conventional orthogonal projection.

Proposition 4.1 *Let ${}_p\varphi_h(x)$ be a $\mathcal{TB}$ -spline and a function $f \in \mathbf{F}^s$. Then a spline ${}_pS_h(x) \in {}_p\mathbf{V}_h^s$ is the GP of the function onto the spline space ${}_p\mathbf{V}_h$ iff*

$$\int_{-\infty}^{\infty} S(f,x)\,{}_p\varphi_h(x-kh)\,dx = \int_{-\infty}^{\infty} f(x)\,{}_p\varphi_h(x-kh)\,dx.$$

Proof: It is an immediate consequence of Eqs.(39). ■

Let us write explicitly the $\mathcal{TB}$*-spline dual to the* $\mathcal{B}$*-spline.* Since for the $\mathcal{B}$-spline ${}_pB_h(x)$, $\quad \rho(v) \equiv \frac{1}{h}$, the condition (35) implies that the spline

$${}_p^d\varphi_h(x) = \int_0^1 \frac{{}_pm_h(v,x)}{{}_{2p}u(v)}\,dv$$

is dual to the $\mathcal{B}$-spline. We stress that ${}_{2p}u(v)^{-1} \in \mathbf{C}^\infty$. Therefore, ${}_p^d\varphi_h(x) \in {}_p\mathbf{S}_h^{-\infty}$, and it belongs to $\mathbf{F}^{-\infty}$. In fact, the coefficients of the $\mathcal{B}$-spline representation of the spline ${}_p^d\varphi_h(x)$ are of exponential decay (see [8] e.g.) and the same may be said about the very spline ${}_p^d\varphi_h(x)$.

Theorem 4.4 *Let $f(x)$ be a function of slow growth. Then there exists an unique GP of the function onto ${}_p\mathbf{V}_h$. Moreover, if $f \in \mathbf{F}^s$ then the corresponding spline $S(f,x) \in {}_p\mathbf{V}_h^s \subset \mathbf{F}^s$.*

Proof: It is readily seen that the sequence $\vec{\Phi} = \left\{\Phi_k\right\}$ defined by (42) belongs to $\mathbf{G}^s$ and, consequently, $\mathcal{F}_h(\vec{\Phi}, v) \in \mathbf{D}^s$. Let a spline we are looking for is written in the basis formed from the $\mathcal{TB}$ -splines dual to the $\mathcal{B}$ -spline:

$${}_pS_h(f,x) = \sum_k Q_k\,{}_p^d\varphi_h(x-kh), \tag{43}$$

Then the integrals are

$$\int_{-\infty}^{\infty} {}_pS_h(f,x)\,{}_pB_h(x-kh)\,dx = Q_k = \Phi_k.$$

Hence

$${}_pS_h(f,x) = \sum_k \Phi_k\,{}_p^d\varphi_h(x-kh), \tag{44}$$

In the integral form

$${}_pS_h(f,x) = \int_0^1 \frac{\mathcal{F}(\vec{\Phi}, v)}{{}_{2p}u(v)} \cdot {}_pm_h(v,x)\,dv. \tag{45}$$

Theorem 3.1 guarantees that this spline belongs to ${}_p\mathbf{V}_h^s$. ■

Corollary 4.1 *The GP of a polynomial $P(x) \in \Pi_{p-1}$ onto the spline space ${}_p\mathbf{V}_h$ is the very polynomial $P(x)$.*

Example: *GP of the exponential function.* Eq. (4) implies

$$\int_{-\infty}^{\infty} e^{2\pi ivx}\, {}_pB_h(x-kh)\,dx = (-1)^p\, e^{2\pi ivkh}\Big(\frac{1-e^{2\pi ivh}}{2\pi ivh}\Big)^p. \qquad (46)$$

At the same time, we see from (21) that

$$\int_{-\infty}^{\infty} {}_pm_h(vh,x)\,{}_pB_h(x-kh)\,dx = e^{2\pi ikhv}\,{}_{2p}u(hv).$$

Comparing the latter relation with (46), we conclude that the spline

$${}^e m_h(v,x) =: \Big(\frac{1-e^{2\pi ivh}}{2\pi ivh}\Big)^p \frac{{}_pm_h(vh,x)}{{}_{2p}u(hv)}$$

is the GP of the exponential function $e^{2\pi ivx}$ onto the spline space ${}_p\mathbf{V}_h$.

4.4 Cardinal spline interpolation

Following I. Schoenberg [8], we formulate the problem of cardinal spline interpolation (CSI) in such a manner:
Problem:*Let a sequence $\vec{y} = \{y_k\}_{-\infty}^{\infty}$ belongs to $\mathbf{G}$. Find a spline ${}_pS_h(x) \in {}_p\mathbf{S}_h$ subject to the conditions:*

$${}_pS_h((k+\frac{p}{2})h) = y_k, \quad \forall k \in \mathbf{Z} \qquad (47)$$

The spline ${}_pS_h(x)$ is the interpolatory spline for the sequence $\vec{y}$.
The problem had been solved by I. Schoenberg [10], nonetheless we take some time to discuss it. The reason is that within our considerations, the solution of the problem turns out to be almost trivial.

Theorem 4.5 *Problem CSI has an unique solution. Moreover, if $\vec{y}$ belongs to $\mathbf{G}^s$ then the interpolatory spline ${}_pS_h(x) \in {}_p\mathbf{V}_h^s \subset \mathbf{F}^s$. The solution can be written as follows:*

$${}_pS_h(x) = \sum_k y_k\, {}_pL_h(x-kh), \qquad (48)$$

where the so called fundamental spline ${}_pL_h(x)$ is the spline interpolating the sequence $\vec{\delta} = \{\delta_k^0\}_{-\infty}^{\infty}$.

Proof: We start with construction of the fundamental spline. We will look for such spline within the space ${}_p\mathbf{V}_h$. This spline, if exists, is being represented by the integral

$${}_pL_h(x) = \int_0^1 \rho(v)\cdot {}_pm_h(v,x)\,dv$$

with a distribution $\rho(v)$. Then

$$\begin{aligned}
{}_pL_h((k+\frac{p}{2})h) &= \int_0^1 \rho(v)\cdot {}_pm_h(v,(k+\frac{p}{2})h)\,dv \\
&= \int_0^1 e^{2\pi ivk}\,\rho(v)\cdot {}_pm_h(v,\frac{hp}{2})\,dv \\
&= \int_0^1 e^{2\pi ivk}\,\rho(v)\cdot {}_pu(v)\,dv = \delta_k^0.
\end{aligned}$$

In other words, the Fourier coefficients of the distribution $\rho(v)\,{}_pu(v), \quad c_k = \delta_k^0$. Hence

$$\rho(v) = \frac{1}{{}_pu(v)} \quad \text{and} \quad {}_pL_h(x) = \int_0^1 \frac{{}_pm_h(v,x)}{{}_pu(v)}\,dv.$$

We see from this equation that ${}_pL_h(x)$ is a $\mathcal{TB}$ -spline and, therefore, any spline of ${}_p\mathbf{V}_h$ may be represented as follows:

$${}_pS_h(x) = \sum_l \eta_l\,{}_pL_h(x - lh).$$

Then

$${}_pS_h((k + \frac{p}{2})h) = \sum_l \eta_l\,{}_pL_h((k - l + \frac{p}{2})h) = \eta_k.$$

The requirement (47) results in $\eta_k = y_k$. Appealing to (32), we may write

$${}_pS_h(x) = \int_0^1 \frac{\mathcal{F}(\vec{y},v)}{{}_pu(v)} \cdot {}_pm_h(v,x)\,dv.$$

Assume now that the sequence $\vec{y}$ belongs to $\mathbf{G}^s$. Then $\mathcal{F}_h(\vec{y},v) \in \mathbf{D}^s$. The same is true for the distribution $\mathcal{F}_h(\vec{y},v)({}_pu(v))^{-1}$. Theorem 3.1, implies that ${}_pS_h(x) \in {}_p\mathbf{V}_h^s \subset \mathbf{F}^s$. ■

Example: *The spline interpolating the exponential function.* Let us consider the spline

$${}_p^im_h(v,x) =: \frac{{}_pm_h(hv,x)}{{}_pu(hv)}.$$

Eq. (16) implies that

$${}_p^im_h(v, kh + \frac{ph}{2}) = e^{2\pi vkh}.$$

In other words, the spline ${}_p^im_h(v,x)$ interpolates the exponential function $e^{2\pi vx}$ on the grid $\{hk\}$. We point out that in the case when p is even, this spline is the exponential Euler spline [8], p.26, ${}_p^im_h(v,x) = S_n(x;t)$ with $t = e^{2\pi iv}, n = p - 1$.

Proposition 4.2 *The spline ${}_pS_h(x)$ satisfying the conditions (47) can be written as follows:*

$$\begin{aligned}
{}_pS_h(x) &= h\sum_k q_k\,{}_pB_h(x - kh), \qquad \textit{where} \\
q_k &= \sum_l \beta_{k-l}^{2,1}\,y_l, \qquad \beta_r^{2,1} = \int_0^1 \frac{e^{2\pi ivr}}{{}_pu(v)}\,dv.
\end{aligned}$$

Proof: It is an immediate consequence of Eqs.(37). ■

5 $\mathcal{TB}$-splines dual to the fundamental ones

The $\mathcal{TB}$-spline

$${}_p\lambda_h(x) = \frac{1}{h}\int_0^1 \frac{{}_pu(v)}{{}_{2p}u(v)}\,{}_pm_h(v,x)\,dv$$

is dual to the fundamental one ${}_pL_h(x)$. Point out first of all that, to some extent, the splines ${}_p\lambda_h(x)$ play the role of the Dirac delta in spline spaces.

Proposition 5.1 *Let a spline ${}_pS_h(x)$ belongs to ${}_p\mathbf{V}_h$. Then the integral is*

$$\int_{-\infty}^{\infty} {}_pS_h(x)\,{}_p\lambda_h(x - lh)\,dx = {}_pS_h((l + \frac{p}{2})h).$$

Proof: In accordance with (48) we can write

$$ {}_pS_h(x) = \sum_k {}_pS_h((k+\frac{p}{2})h)\, {}_pL_h(x-kh) $$

Hence the assertion follows. ■

Usually, when a signal f is being processed, only the array of samples

$$ \vec{f} = \{f_k^p =: f((k+p/2)h\} $$

is available as the initial data. However in the case when the GP of the signal onto the spline space ${}_p\mathbf{V}_h$ is required, the problem arises to compute the integrals

$$ \Phi_k = \int_{-\infty}^{\infty} f(x)\, {}_pB_h(x-kh)\, dx. $$

We will use the splines ${}_p\lambda_h(x)$ to evaluate the GP. Suppose that a function f possesses the continuous derivative of an order θ. Let us define two nonnegative sequences:

$$ C_k^\theta =: \max|f^{(\theta)}(x)| \quad \text{as}\, x \in [kh,(k+1)h], $$
$$ D_k^\theta =: \max|f^{(\theta)}(x) - f^{(\theta)}(y)| \quad \text{as}\, x,y \in [kh,(k+1)h]. $$

We stress that once $f^{(\theta)} \in \mathbf{F}^s$, the sequences defined belong to $\mathbf{G}^s$. Let $\{\nu_k^p\}$ be the Fourier coefficients of the function ${}_{2p}u(v)^{-1}$:

$$ \nu_k^p =: \int_0^1 \frac{e^{2\pi i v k}}{{}_{2p}u(v)}\, dv. \tag{49} $$

We point out that values $\{\nu_k^p\}$ serve simultaneously as the coordinates of the fundamental spline ${}_{2p}L_h(x)$ with respect to the $\mathcal{B}-$spline basis. This sequence is of exponential decay.
Let us define discrete and continuous moments of the $\mathcal{B}$-splines

$$ \begin{aligned} \mu_h^s(t) &= h\sum_r \Big(h\Big(t+r-\frac{p}{2}\Big)\Big)^s {}_pB_h(h(t+r)), t \in [0,1], \\ M_h^s &= \int_0^{ph} (x-hp/2)^s {}_pB_h(x)\, dx. \end{aligned} \tag{50} $$

Our constructions will be based on the following assertion.

Proposition 5.2 *([15],[16]). Provided $s \le p-1$, the discrete moments $\mu_h^s(t)$ does not depend on $t \in [0,1]$ and coincide with the corresponding continuous moments M_h^s. The moments*

$$ \begin{aligned} \mu_h^p(t) &= (-1)^{p+1}\beta_p(t)h^p + M_h^p \\ \mu_h^{p+1}(t) &= (-1)^{p+1}p\beta_{p+1}(t)h^{p+1} + M_h^{p+1} \end{aligned} $$

as $t \in [0,1]$ and are 1-periodic with respect to t.

Here $\beta_l(t)$ is the Bernoulli polynomial of degree l.
We put $\theta = p$, $q(p) = 0$ as p is even and $\theta = p+1$, $q(p) = \frac{1}{2}$ as p is odd.

Theorem 5.1 *Assume that a function f and its derivatives up to the order θ are of slow growth. Moreover, let $f^{(\theta)} \in \mathbf{F}^s$ and be continuous. Let the spline ${}_pS_h(f,x)$ be the GP of f onto the space ${}_p\mathbf{V}_h$ and the spline ${}_p\sigma_h(f,x)$ be defined as follows*

$$ {}_p\sigma_h(f,x) =: h\sum_k f_k^p\, {}_p\lambda_h(x-lh). $$

Then the following relation is true

$$ {}_pS_h(f,x) = {}_p\sigma_h(f,x) + {}_p\gamma_h(f,x). $$

where ${}_p\gamma_h(f,x)$ is the spline of ${}_p\mathbf{V}^s$:

$$ {}_p\gamma_h(f,x) = h\sum_k g_k^p\, {}_pB_h(x-hk) $$

those coefficients are subdued to the following exact inequalities:

$$ |g_k^p| \le Q_k^p =: \frac{h^\theta}{\theta!}\Big(M_1^\theta \sum_l D_{k-l}^\theta\, |\nu_l^p| + |\beta_\theta(q(p))| \sum_l C_{k-l}^\theta\, |\nu_l^p|\Big). \tag{51} $$

Respectively, the magnitude of ${}_p\gamma_h(f,x)$ does not exceed the value of the non-negative spline ${}_p\Gamma_h(f,x) \in {}_p\mathbf{V}_h^s$ with the $\mathcal{B}$-spline basis coordinates Q_k^p.

Corollary 5.1 *If p is even then for all $f \in \Pi_{p-1}$*

$$ {}_pS_h(f,x) \equiv {}_p\sigma_h(f,x). $$

If p is odd, this identity is true for all $f \in \Pi_p$.

Corollary 5.2 *For all $f \in \Pi_{p-1}$*

$$ {}_p\sigma_h(f,x) \equiv f(x). $$

We start with a quadrature formula. Denote $\vec{f^p} = \{f_k^p\}$, $\vec{B}^p = h\{{}_pB_h((k+p/2)h)\}$,

$$ \Psi_k =: h\sum_l f_l^p\, {}_pB_h((k-l+p/2)h) = \vec{f^p} * \vec{B}^p. $$

Point out that the sum in the right hand side contains only p nonzero terms with any k.

Lemma 5.1 *If $f \in \mathcal{C}^\theta$ then*

$$ \begin{aligned} \Phi_k &= \Psi_k + {}_pG_k, \\ {}_pG_k &=: \frac{M_h^\theta}{\theta!}(f^{(\theta)}(\xi_k) - f^{(\theta)}(\eta_k)) + \frac{h^\theta \beta_\theta(q(p))}{\theta!} f^{(\theta)}(\eta_k), \end{aligned} $$

where $\xi_k, \eta_k \in [kh,(k+p)h]$. Moreover, the following exact inequalities hold:

$$ |{}_pG_k| \le \frac{h^\theta}{\theta!}\Big(M_1^\theta\, D_k^\theta + |\beta_\theta(q(p))|\, C_k^\theta\Big) \tag{52} $$

Proof: The assertion of the lemma stems from Proposition 5.2. Without loss of generality assume that $k = 0$. Provided $f \in \mathcal{C}^r$ and r is even, we may write

$$\begin{aligned} \Phi_0 &= \int_0^{ph} f(x)\, {}_pB_h(x)\, dx \\ &= \sum_{s=0}^{r-1} \frac{f^{(s)}(ph/2)}{s!} \int_0^{ph} (x - ph/2)^s\, {}_pB_h(x)\, dx + R_r \\ &= \sum_{s=0}^{r-1} \frac{f^{(s)}(ph/2)}{s!} M_h^s + R_r, \\ R_r(x) &=: \int_0^{ph} \frac{f^{(r)}(\xi(x))}{r!} (x - ph/2)^r\, {}_pB_h(x)\, dx. \end{aligned} \tag{53}$$

Since ${}_pB_h(x) \geq 0$ as well as $(x - ph/2)^r$,

$$R_r = \frac{f^{(r)}(\xi)}{r!} \int_0^{ph} (x - ph/2)^r\, {}_pB_h(x)\, dx = \frac{f^{(r)}(\xi)}{r!} M_h^r \tag{54}$$

with some $\xi \in [0, ph]$. Let us turn to the sum Ψ_0. We distinguish two cases.
1.The order of the spline p is even. Then there holds

$$\begin{aligned} \Psi_0 &= h \sum_{l=0}^{p} f(lh)\, {}_pB_h(lh) \\ &= h \sum_{l=0}^{p} {}_pB_h(lh) \sum_{s=0}^{p-1} \frac{f^{(s)}(ph/2)}{s!} ((l - p/2)h)^s + \rho_p, \\ &= \sum_{s=0}^{p-1} \frac{f^{(s)}(ph/2)}{s!} \mu_h^s(0) + \rho_p = \sum_{s=0}^{p-1} \frac{f^{(s)}(ph/2)}{s!} M_h^s + \rho_p, \\ \rho_p &=: h \sum_{l=0}^{p} {}_pB_h(lh) \frac{f^{(p)}(\eta_l)}{p!} ((l - p/2)h)^p. \end{aligned}$$

Due to non-negativity of the $\mathcal{B}$-splines and of even powers, the following is true

$$\begin{aligned} \rho_p &= \frac{f^{(p)}(\eta)}{p!} h \sum_{l=0}^{p} {}_pB_h(lh)((l - p/2)h)^p \\ &= \frac{f^{(p)}(\eta)}{p!} \mu_h^p(0) = \frac{f^{(p)}(\eta)}{p!} (M_h^p - \frac{h^p \beta_p(0)}{p!}). \end{aligned}$$

with some $\eta \in [0, ph]$. Comparing this expression with (53) and (54) when $r = p$, we obtain for even values of p:

$${}_pG_0 = \frac{M_h^p}{p!} (f^{(p)}(\xi) - f^{(p)}(\eta)) + \frac{h^p \beta_p(0)}{p!} f^{(p)}(\eta), \quad \xi, \eta \in [0, ph]. \tag{55}$$

2.The order of the spline p is odd. In this case

$$\Psi_0 = h \sum_{l=0}^{p} f((l + 1/2)h)\, {}_pB_h((l + 1/2)h)$$

$$
\begin{aligned}
&= h\sum_{l=0}^{p} {}_pB_h((l+1/2)h)\sum_{s=0}^{p-1}\frac{f^{(s)}(ph/2)}{s!}((l+1/2-p/2)h)^s+\rho_p \\
&= \sum_{s=0}^{p-1}\frac{f^{(s)}(ph/2)}{s!}\mu_h^s(1/2)+\rho_p=\sum_{s=0}^{p-1}\frac{f^{(s)}(ph/2)}{s!}M_h^s+\rho_p, \\
\rho_p &=: \frac{f^{(p)}(ph/2)}{p!}h\sum_{l=0}^{p} {}_pB_h((l+1/2)h)((l-p/2)h)^p \\
&+ h\sum_{l=0}^{p+1} {}_pB_h((l+1/2)h)\frac{f^{(p+1)}(\eta_l)}{(p+1)!}((l-p/2)h)^{p+1} \\
&= \frac{f^{(p)}(ph/2)}{p!}(M_h^p-\frac{h^p\beta_p(1/2)}{p!})+\frac{f^{(p+1)}(\eta)}{(p+1)!}(M_h^{p+1}-\frac{ph^{p+1}\beta_{p+1}(1/2)}{(p+1)!}).
\end{aligned}
$$

If p is odd then $\beta_p(1/2)=0$. Hence it follows that for odd numbers p

$$
\rho_p=\frac{f^{(p)}(ph/2)}{p!}M_p^h+\frac{f^{(p+1)}(\eta)}{(p+1)!}(M_{p+1}^h-\frac{ph^{p+1}\beta_{p+1}(1/2)}{(p+1)!}).
$$

Comparing this expression with (53) and (54) when $r=p+1$ we obtain for odd values of p:

$$
{}_pG_0=\frac{M_{p+1}^h}{(p+1)!}(f^{(p+1)}(\xi)-f^{(p+1)}(\eta))+\frac{h^{p+1}\beta_{p+1}(1/2)}{(p+1)!}f^{(p+1)}(\eta),\quad \xi,\eta\in[0,ph]. \tag{56}
$$

Note that Eqs. (55), (56) hold for any ${}_pG_k$ with replacement ξ,η to $\xi_k,\eta_k\in[kh,(k+p)h]$. It can be readily verified that the moment

$$
M_h^\theta=h^\theta M_1^\theta.
$$

Hence we derive immediately the estimate (52). ■

Proof of Theorem 5.1: Denote $\vec{\Psi}=\{\Psi_k\}$, $\vec{\Phi}=\{\Phi_k\}$, $\vec{G}^p=\{G_k^p\}$. We stress that, once $f^{(\theta)}\in\mathbf{F}^s$, the sequence $\vec{G}^p$ belongs to $\mathbf{G}^s$. Since $\Psi_k=\vec{f}^p*\vec{B}^p$, we may write

$$
\mathcal{F}(\vec{\Psi},v)=\mathcal{F}(\vec{f}^p,v)\mathcal{F}_h(\vec{B}^p,v)=\mathcal{F}(\vec{f}^p,v)\,{}_pu(v).
$$

Hence, referring to Theorem 4.1, we have

$$
\begin{aligned}
{}_p\sigma_h(f,x) &= \int_0^1\mathcal{F}(\vec{f}^p,v)\frac{{}_pu(v)}{{}_{2p}u(v)}\,{}_pm_h(v,x)\,dv \\
&= \int_0^1\frac{\mathcal{F}(\vec{\Psi},v)}{{}_{2p}u(v)}\,{}_pm_h(v,x)\,dv.
\end{aligned}
$$

We recall that (45) implies that the GP spline ${}_pS_h(f,x)$ may be represented as follows

$$
{}_pS_h(f,x)=\int_0^1\frac{\mathcal{F}(\vec{\Phi},v)}{{}_{2p}u(v)}\cdot\,{}_pm_h(v,x)\,dv.
$$

Then Lemma 5.1 leads us to the relations:

$$
\begin{aligned}
{}_pS_h(f,x) &= {}_p\sigma_h(f,x)+{}_p\gamma_h(f,x), \\
{}_p\gamma_h(f,x) &=: \int_0^1\frac{\mathcal{F}(\vec{G}^p,v)}{{}_{2p}u(v)}\,{}_pm_h(v,x)\,dv \\
&= h\sum_k g_k^p\,{}_pB_h(x-hk).
\end{aligned}
$$

The coefficients are as follows:

$$g_k^p = \int_0^1 e^{2\pi ivk} \frac{\mathcal{F}(\vec{G}^p, v)}{{}_{2p}u(v)}\, dv = \sum_l G_{k-l}\, \nu_l^p.$$

The sequence ν_l^p was defined in (49). Note that the sequence $\vec{g}^p = \{g_k^p\}$ belongs to $\mathbf{G}^s$ as well as $\vec{G}^p$. Hence, keeping in mind ((52) we derive the estimate (51). ■

Concluding remark

We note that the approach being developed for one-dimensional polynomial splines can be applied to numerous classes of spline functions. We mention L-splines, box splines, discrete splines, Hermite splines.

The circle of problems solvable by means of the techniques established is rather wide. Point out that these techniques are especially relevant for solving problems concerned with the operators of convolution and of differentiation because an intimate relationship exponential splines to these operators.

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