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WAVELETS BASED ON PERIODIC SPLINES

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Wavelet bases of function spaces possess a variety of properties attractive for numerical analysis. For example: 1) The basis functions are orthogonal. 2) All basis functions can be obtained by means of translations and dilations of a single generating function. At the same time, a principal feature of wavelet bases is their spatial and frequency locality. Therefore the property 3) Basis functions have compact supports, is very desirable. Daubechies [1] succeeded in joining together in one wavelet basis all three of these properties. It should be pointed out, however, that, for problems where the symmetry and the smoothness of basis functions are required, these wavelet-bases appear not so suitable as ones constructed via spline functions. The drawback to spline-wavelets is that it is impossible to join together the properties 1)–3).

However, periodic splines of defect 1 allow us to implement the wavelet decomposition and reconstruction of functions in two bases simultaneously. One of these bases consists of compactly supported spline-wavelets subject to 2); the other one is an orthogonal basis. To change from one basis to the other, one must carry out the fast Fourier transform (FFT). Such a duality leads to extremely simple, flexible, and fast algorithms in both the one-dimensional and multidimensional cases. The coordinates of a function in these bases can be computed by the Monte Carlo method.

We define the concept of multiresolution analysis (MRA) of the space $\mathbf{L}^2(T)$ of 1-periodic functions (see [2] and [3]).

Definition 1. An *MRA of the space* $\mathbf{L}^2(T)$ is a sequence of closed spaces $\mathbf{V}^j$ $(j \geq 0)$ with the following four properties:

(1) $\mathbf{V}^0 \subset \mathbf{V}^1 \subset \cdots \subset \mathbf{V}^j \subset \cdots \subset \mathbf{L}^2(T)$.

(2) $\bigcup_{j\geq 0} \mathbf{V}^j$ is dense in $\mathbf{L}^2(T)$.

(3) $\mathbf{V}^0$ is {constant functions}, $f(x) \in \mathbf{V}^j \Rightarrow f(2x) \in \mathbf{V}^{j+1}$, and $f(x) \in \mathbf{V}^{j+1} \Rightarrow f(x/2) + f(x/2 + 1/2) \in \mathbf{V}^j$.

(4) $\dim \mathbf{V}^j = 2^j$, and for any value of j there exists a function ϕ_0^j such that its shifts $\phi_k^j(x) = \phi_0^j(x - k/2^j)$, $k = 0, 1, \ldots, 2^j - 1$, form a basis of $\mathbf{V}^j$. □

Because of property (1), the space $\mathbf{V}^j$ can be represented as $\mathbf{V}^j = \mathbf{V}^{j-1} \oplus \mathbf{W}^{j-1}$, where $\mathbf{W}^{j-1}$ is the orthogonal complement of $\mathbf{V}^{j-1}$ in $\mathbf{V}^j$. Property (4) implies that $\dim \mathbf{W}^{j-1} = 2^{j-1}$, and it can be established [4] that for any j there exists a function ψ_0^{j-1} whose shifts $\psi_k^{j-1}(x) = \psi_0^{j-1}(x - k/2^{j-1})$, $k = 0, 1, \ldots, 2^{j-1} - 1$, form a basis of $\mathbf{W}^{j-1}$. We call the functions $\psi_k^{j-1}(x)$ *wavelets*. Properties (1) and (2) entail the representation $\mathbf{L}^2(T) = \mathbf{V}^0 \oplus \bigcup_{j\geq 0} \mathbf{W}^j$ as well as the fact that the functions in the set $\{1, \psi_k^j(x), j = 0, 1, \ldots, k = 0, 1, \ldots, 2^j - 1\}$ form a basis of the space $\mathbf{L}^2(T)$.

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We will construct MRA on the grounds of periodic splines.

We introduce some notions. Throughout, $N = 2^j$ and $\sum_k^j$ stands for $\sum_{k=0}^{2^j-1}$. We write $\omega = \exp(2\pi i/N)$. The discrete Fourier transform (DFT) of a vector $\mathbf{a} = \{a_k\}_0^{N-1}$ is

$$(1) \qquad T_n^j(\mathbf{a}) = \frac{1}{N}\sum_k^j \omega^{-nk} a_k, \qquad a_k = \sum_n^j \omega^{nk} T_n^j(\mathbf{a}),$$

The function ${}_pB^j(x) = N^p \nabla_j^p (x_+^{p-1}/(p-1)!)$, where $x_+ = (x+|x|)/2$, is the B-spline of degree $p-1$ with knots at the points $\{k/N\}$. The symbol ∇_j denotes the descending difference with step $1/N$. We point out that the support $\operatorname{supp}\, {}_pB^j(x)$ of the B-spline is $(0, p/N)$. The symbol ${}_pM^j(x)$ will denote the 1-periodic B-spline of degree $p-1$:

$$(2) \qquad {}_pM^j(x) = \sum_{l=-\infty}^{\infty} {}_pB^j(x+l) = \sum_{n=-\infty}^{\infty} e^{-\pi inp/N} \left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^p e^{2\pi inx}.$$

Throughout, ${}_p\mathfrak{B}^j$ will denote the spaces of 1-periodic splines of degree $p-1$ and of defect 1 with their knots at the points $\{k/2^j\}$, $j = 0, 1, \dots$, $k = 0, \dots, 2^j - 1$. As is easily seen, the spaces ${}_p\mathfrak{B}^j$ generate an MRA of the space $\mathbf{L}^2(T)$. The shifts of the B-spline ${}_pM^j(x)$ form a basis of ${}_p\mathfrak{B}^j$. Any spline ${}_pS^j \in {}_p\mathfrak{B}^j$ can be represented as follows:

$$(3) \qquad {}_pS^j(x) = \frac{1}{N}\sum_k^j q_k \, {}_pM^j(x - k/N).$$

Writing $\mathbf{q} = \{q_k^j\}_0^{N-1}$ and exploiting the relations (1), we write the spline as

$$(4) \qquad {}_pS^j(x) = \frac{1}{N}\sum_k^j {}_pM^j(x-k/N) \sum_r^j \omega^{rk} T_r^j(\mathbf{q}) = \sum_r^j \xi_r^j \, {}_pm_r^j(x),$$

where

$$(5) \qquad {}_pm_r^j(x) = \frac{1}{N}\sum_k^j {}_pM^j(x-k/N)\omega^{rk} = \frac{1}{N}\sum_k^j {}_pM^j(x+k/N)\omega^{-rk},$$

and $\xi_r^j = T_r^j(\mathbf{q})$. Hence we see that the splines ${}_pm_r^j(x)$, $r = 0, 1, \dots, 2^j - 1$, form a basis of ${}_p\mathfrak{B}^j$, and $\{\xi_r^j\}$ are the coordinates of the spline ${}_pS^j(x)$ with respect to this basis.

By reference to relations established in [5], one many readily check the following property of the splines m_r^j:

$$(6) \qquad ({}_pm_r^j, {}_pm_k^j) = \int_0^1 {}_pm_r^j(y) \, {}_p\overline{m}_k^j(y)\, dy = \delta_n^r \, {}_{2p}u_r^j,$$

where δ_k^r is the Kroneker delta, and

$${}_pu_r^j = \sin(\pi r/N)^p \sum_{l=-\infty}^{\infty} (-1)^{pl} (\pi(r+lN)/N)^{-p} = \frac{1}{N}\sum_k^j \omega^{-rk} {}_pM^j(k/N + p/2N) > 0.$$

The functions ${}_pu_r^j$ were studied in [6] and [7].

The relation (6) implies, in particular, that the splines ${}_pm_r^j(x)$ form an orthogonal basis of ${}_p\mathfrak{B}^j$. Therefore we call these splines *ortsplines.* We point out the relations

$$\langle {}_pm_r^j, {}_pm_r^j\rangle = {}_{2p}u_r^j, \qquad {}_pM^j(x+k/N) = \sum_r^j {}_pm_r^j(x)\omega^{rk}.$$

Our account will follow the conventional scheme: first we expand the splines ${}_pm_k^{j-1}(x)$ with respect to the basis $\{{}_pm_r^j(x)\}$, as well as the splines ${}_pM^{j-1}(x-2k/N)$ with respect to the basis $\{{}_pM^j(x-l/N)\}$; then we construct wavelets, and establish formulae for decomposition and reconstruction of splines as well as formulae for the projection of a function onto the space ${}_p\mathfrak{B}^j$.

Theorem 1. *For* $r = 0, 1, \dots, 2^{j-1}-1$,

$$\begin{aligned} {}_pm_r^{j-1}(x) &= {}_pb_r^j\, {}_pm_r^j(x) + {}_pb_{r+N/2}^j\, {}_pm_{r+N/2}^j(x), \\ {}_pb_r^j &= e^{-\pi irp/N}(\cos \pi r/N)^p = 2^{-p}(1+\omega^{-r})^p, \\ {}_pM^{j-1}(x) &= 2^{-p}\sum_{l=0}^{p}\binom{p}{l}\, {}_pM^j(x-l/N). \end{aligned}$$

We point out a useful identity which follows immediately from the latter relations:

$${}_{2p}u_r^{j-1} = {}_{2p}m_r^{j-1}(2p/N) = 4^{-p}\omega^{rp}[(1+\omega^{-r})^{2p}\, {}_{2p}u_r^j + (-1)^p(1-\omega^{-r})^{2p}\, {}_{2p}u_{r+N/2}^j].$$

Define the space of wavelets ${}_p\mathfrak{W}^{j-1}$ as the orthogonal complement of ${}_p\mathfrak{B}^{j-1}$ in ${}_p\mathfrak{B}^j$. We start with splines which will be called *ortwavelets.* Specifically, these are the splines $\{{}_pw_r^{j-1}(x)\}_0^{N/2-1}$ of ${}_p\mathfrak{W}^{j-1} \subset {}_p\mathfrak{B}^j$ with

$${}_pw_r^{j-1}(x) = {}_pa_r^j\, {}_pm_r^j(x) + {}_pa_{r+N/2}^j\, {}_pm_{r+N/2}^j(x).$$

If $s \neq r$, then ${}_pw_r^{j-1} \perp {}_pw_s^{j-1}$ and ${}_pw_r^{j-1} \perp {}_pm_s^{j-1}$ because of orthogonality of the ortsplines ${}_pm_r^j$. Find the coefficients ${}_pa_r^j$ so as to ensure that ${}_pw_r^{j-1} \perp {}_pm_r^{j-1}$. Denote ${}_sv_r^j = {}_su_{r+N/2}^j$. Then

$$\langle {}_pw_r^{j-1}, {}_pm_r^{j-1}(x)\rangle = {}_pa_r^j\, {}_p\overline{b}_r^j\, {}_{sp}u_r^j + {}_pa_{r+N/2}^j\, {}_p\overline{b}_{r+N/2}^j\, {}_{2p}v_r^j = 0.$$

We can write a variety of solutions of the latter equation:

$$(7) \qquad {}_p^\nu a_r^j = e^{2\pi ir/N}\, {}_p^\nu a_r^j\, {}_p\overline{b}_{r+N/2}^j = 2^{-p}\omega^r(1-\omega^r)^p\, {}_p^\nu a_r^j, \qquad {}_p^\nu a_r^j = {}_p^\nu\tau_r^{j-1}({}_{2p}u_r^j)^{-1},$$

where $\{{}_p^\nu\tau_r^{j-1}\}$ is any 2^{j-1}-periodic sequence with ${}_p^\nu\tau_r^{j-1} \neq 0$ for all r. The assertion follows.

Theorem 2. *There exists a family of ortwavelets*

$${}_p^\nu w_r^{j-1}(x) = {}_p^\nu a_r^j\, {}_pm_r^j(x) + {}_p^\nu a_{r+N/2}^j\, {}_pm_{r+N/2}^j(x), \qquad r = 0, 1, \dots, 2^{j-1}-1,$$

whose coefficients are determined in (7). *These ortwavelets form an orthogonal basis of the space* ${}_p\mathfrak{W}^{j-1}$, *and*

$$\langle {}_p^\nu w_r^{j-1}, {}_p^\nu w_r^{j-1}\rangle = |{}_p^\nu\tau_r^{j-1}|^2({}_{2p}v_r^j\, {}_{2p}u_r^j)^{-1}\, {}_{2p}u_r^{j-1}.$$

We distinguish three special cases:

$$\begin{aligned} {}_p^1a_r^j &= 2^{-p}\omega^r(1-\omega^r)^p\, {}_{2p}v_r^j, \\ {}_p^0a_r^j &= 2^{-p}\omega^r(1-\omega^r)^p/{}_{2p}u_r^j, \\ {}_p^2a_r^j &= 2^{-p}\omega^r(1-\omega^r)^p({}_{2p}v_r^j/{}_{2p}u_r^j)^{1/2}. \end{aligned}$$

We emphasize that $\langle {}^2_p w_r^{j-1}, {}^2_p w_r^{j-1} \rangle = {}_{2p}u_r^{j-1}$.

We now denote ${}^\nu_p A_k^{j-1} = \sum_r^j \omega^{rk} {}^\nu_p a_r^j$ and define the splines

$$ {}^\nu_p \psi^{j-1}(x) = \frac{1}{N} \sum_k^j {}^\nu_p A_k^{j-1} {}_p M^j(x - k/N) = \sum_r^j {}^\nu_p a_r^j {}_p m_r^j(x). $$

As is readily seen, the dual relations hold:

$$ {}^\nu_p \psi^{j-1}(x + 2l/N) = \sum_r^{j-1} \omega^{2rl}\, {}^\nu_p w_r^{j-1}(x), \tag{8} $$

$$ {}^\nu_p w_r^{j-1}(x) = \frac{2}{N} \sum_l^{j-1} \omega^{-2rl}\, {}^\nu_p \psi^{j-1}(x + 2l/N). \tag{9} $$

These relations imply, in particular, that ${}^\nu_p \psi^{j-1}(x + 2l/N) \in {}_p\mathfrak{W}^{j-1}$.

Theorem 3. *The splines* $\{{}^\nu_p \psi^{j-1}(x - 2l/N)\}_{l=0}^{N/2-1}$ *form a basis of the space* ${}_p\mathfrak{W}^{j-1}$; *any spline* ${}_pW^{j-1}(x) \in {}_p\mathfrak{W}^{j-1}$ *can be written as*

$$ {}_pW^{j-1}(x) = \sum_r^{j-1} {}^\nu_p \eta_r^{j-1}\, {}^\nu_p w_r^{j-1}(x) = \frac{2}{N} \sum_l^{j-1} {}^\nu_p t_l^{j-1}\, {}^\nu_p \psi^{j-1}(x - 2l/N). $$

Moreover,

$$ {}^\nu_p t_l^{j-1} = \sum_r^{j-1} \omega^{2rl}\, {}^\nu_p \eta_r^{j-1} \Leftrightarrow {}^\nu_p \eta_r^{j-1} = \frac{2}{N} \sum_l^{j-1} \omega^{-2rl} {}^\nu_p t_l^{j-1}. $$

The theorem enables us to affirm that the splines ${}^\nu_p \psi^{j-1}(x)$ appear as wavelets in terms of the definition given at the beginning of the paper.

Now let us examine the wavelets ${}^\nu_p \psi^{j-1}(x)$ with $\nu = 0, 1, 2$.

1. Let ${}_{2p}L^j(x) \in {}_{2p}\mathfrak{B}^j$ be a spline such that ${}_{2p}L^j(p/N) = N$ and ${}_{2p}L^j(k/N) = 0$ when $k \neq p$. These splines are said to be the *fundamental* ones. It can be shown that

$$ (2N)^{-p}\, {}_{2p}L^j(x - 1/N)^{(p)} = (-1)^p\, {}^0_p \psi^{j-1}(x). $$

Hence we see that the wavelet ${}^0_p \psi^{j-1}(x)$ appears (up to a constant multiplier) as a "periodization" of the wavelet suggested by Chui and Wang in [8].

2. The determining feature of wavelet ${}^1_p \psi^{j-1}(x)$ is the compactness (up to periodization) of its support. The following representation holds:

$$ {}^1_p \psi^{j-1}(x) = \frac{1}{N} \sum_{k=-2p}^{p-2} {}^1_p A_k^{j-1} {}_p M^j(x - k/N), $$

$$ {}^1_p A_k^{j-1} = 2^{-p}(-1)^{k+1} \sum_{l=0}^{p} \binom{p}{l} {}_{2p}M^j((k + l + 1 + p)/N). $$

The support $\operatorname{supp} {}^1_p \psi^{j-1}(x) \subseteq ((-2p)/N,\ (2p-2)/N) \pmod N$. The wavelet ${}^1_p \psi^{j-1}(x)$ appears as a periodization of the B-wavelet suggested by Chui and Wang in [9].

3. Now we turn to the case $\nu = 2$. Suppose

$$ {}_p \psi^{j-1}(x) = (2/N)^{1/2} \sum_r^{j-1} {}^2_p w_r^{j-1}(x) / ({}_{2p}u_r^{j-1})^{1/2}. $$

The wavelet ${}_p\psi^{j-1}(x)$ appears as a periodization of the Battle-Lemarié wavelet (see [10], [11], and [3]). The shifts $\{{}_p\psi^{j-1}(x-2l/N)\}_{l=0}^{N/2-1}$ form an orthonormal basis of the space ${}_p\mathfrak{W}^{j-1}$. Moreover, a more general assertion holds.

Theorem 4. *The relation*

$$\langle {}^{\nu}_{p}\psi^{j-1}(x-2l/N),\ {}^{\mu}_{p}\psi^{j-1}(x-2s/N)\rangle = \delta_s^l$$

is true if and only if

$${}^{\nu}_{p}\tau_r^{j-1}\,\overline{{}^{\mu}_{p}\tau_r^{j-1}}\,({}_{2p}v_r^j\,{}_{2p}u_r^j)^{-1}\,{}_{2p}u_r^{j-1} = 1/2N.$$

The theorem lets us construct together with a wavelet basis $\{{}^{\nu}_{p}\psi^{j-1}(x-2l/N)\}$ its dual basis $\{{}^{\mu}_{p}\psi^{j-1}(x-2s/N)\}$ in the sense of Chui and Wang [9], which belongs to the same family of wavelet bases as the original basis.

We now establish decomposition formulae.

Theorem 5. *For* $r=0,1,\dots,2^j-1$ *the representation*

$$\begin{aligned} {}_pm_r^j(x) &= {}_ph_r^j\,{}_pm_r^{j-1}(x) + {}^{\nu}_{p}g_r^j\,{}^{\nu}_{p}w_r^{j-1}(x),\\ {}_ph_r^j &= \omega^r(1+\omega^r)^p\,{}_{2p}u_r^j/(2^p\,{}_{2p}u_r^{j-1}),\\ {}^{\nu}_{p}g_r^j &= (1-\omega^{-r})^p\omega^{-r}\,{}_{2p}v_r^j\,{}_{2p}u_r^j/(2^p\,{}_{2p}u_r^{j-1}\,{}^{\nu}_{p}\tau_r^{j-1}) \end{aligned} \tag{10}$$

holds. Any spline

$${}_pS^j(x) = \frac{1}{N}\sum_k^j q_{k}^j\,{}_pM^j(x-k/N) = \sum_r^j \xi_r^j\,{}_pm_r^j(x) \in {}_p\mathfrak{B}^j,$$

$$\mathbf{q} = \{q_k\}_0^{N-1}, \qquad \xi_r^j = T_r^j(\mathbf{q}),$$

can be represented as the following orthogonal sum:

$${}_pS^j(x) = {}_pS^{j-1}(x) + {}^{\nu}_{p}W^{j-1}(x),$$

where

$${}_pS^{j-1}(x) = \sum_r^{j-1} {}_pm_r^{j-1}(x)\xi_r^{j-1} = \frac{2}{N}\sum_k^{j-1} q_k^{j-1}\,{}_pM^{j-1}(x-2k/N) \in {}_p\mathfrak{B}^{j-1},$$

$${}^{\nu}_{p}W^{j-1}(x) = \sum_r^{j-1} {}^{\nu}_{p}w_r^{j-1}(x)\,{}^{\nu}\eta_r^{j-1} = \frac{2}{N}\sum_k^{j-1} {}^{\nu}t_k^{j-1}\,{}^{\nu}_{p}\psi^{j-1}(x-2k/N) \in {}_p\mathfrak{W}^{j-1},$$

$$\xi_r^{j-1} = \xi_r^j\,{}_ph_r^j + \xi_{r+N/2}^j\,{}_ph_{r+N/2}^j, \qquad {}^{\nu}\eta_r^{j-1} = \xi_r^j\,{}^{\nu}_{p}g_r^j + \xi_{r+N/2}^j\,{}^{\nu}_{p}g_{r+N/2}^j. \tag{11}$$

$$q_k^{j-1} + \sum_{r=0}^{N/2-1}\omega^{2kr}\xi_r^{j-1}, \qquad {}^{\nu}t_k^{j-1} = \sum_{r=0}^{N/2-1}\omega^{2kr}\,{}^{\nu}\eta_r^{j-1}. \tag{12}$$

Further, following the conventional pyramidal diagram of decomposition

$$\begin{array}{ccccccccc} {}_pS^j(x) & \longrightarrow & {}_pS^{j-1}(x) & \longrightarrow & {}_pS^{j-2}(x) & \longrightarrow & \cdots & \longrightarrow & {}_pS^{j-m}(x) \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ & & {}^{\nu}_{p}W^{j-1}(x) & & {}^{\nu}_{p}W^{j-2}(x) & & \cdots & & {}^{\nu}_{p}W^{j-m}(x), \end{array}$$

$m \le j$, we obtain the representation of the spline

$${}_pS^j(x) = {}_pS^{j-m}(x) + {}^{\nu}_{p}W^{j-1}(x) + {}^{\nu}_{p}W^{j-2}(x) + \cdots + {}^{\nu}_{p}W^{j-m}(x) \tag{13}$$

as the sum of the *smeared* version ${}_pS^{j-m}(x)$, and the *details* $\{{}^{\nu}_{p}W^{j-k}(x)\}_1^m$.

After establishing, in accordance with (11), the expansion (13) via ortsplines and ortwavelets, it is natural to change, by means of the FFT, from these orthogonal bases to the bases of shifts of B-splines and of the wavelets ${}^{\nu}_{p}\psi^{j-k}(x)$ which provide analysis of a function which is local in the frequency and spatial domains simultaneously. Nevertheless, it should be pointed out that spectral properties of splines ${}_pS^j(x)$ can be investigated most efficiently with the ortspline-ortwavelet bases. We emphasize that the relations $\xi_r^j = T_r^j(\mathbf{q})$ imply an opportunity for fast decomposition algorithms related to the FFT algorithms.

We now establish formulae for reconstructing a spline from its wavelet representation.

Theorem 6. *Suppose that a spline* ${}_pS^j(x) \in {}_p\mathfrak{B}^j$ *is represented as the sum* ${}_pS^j(x) = {}_pS^{j-1}(x) + {}^{\nu}_{p}W^{j-1}(x)$, *where*

$$ {}_pS^{j-1}(x) = \frac{2}{N}\sum_k^{j-1} q_k^{j-1}\,{}_pM^{j-1}(x-2k/N) = \sum_r^{j-1} {}_pm_r^{j-1}(x)\xi_r^{j-1} \in {}_p\mathfrak{B}^{j-1}, $$

$$ {}^{\nu}_{p}W^{j-1}(x) = \frac{2}{N}\sum_k^{j-1} {}^{\nu}t_k^{j-1}\,{}^{\nu}_{p}\psi^{j-1}(x-2k/N) = \sum_r^{j-1} {}^{\nu}_{p}w_r^{j-1}(x)\,{}^{\nu}\eta_r^{j-1} \in {}_p\mathfrak{W}^{j-1}, $$

where $\xi_r^{j-1} = \frac{2}{N}\sum_k^{j-1}\omega^{-2kr}q_k^{j-1}$ *and* ${}^{\nu}\eta_r^{j-1} = \frac{2}{N}\sum_k^{j-1}\omega^{-2kr}\,{}^{\nu}t_k^{j-1}$. *Then*

$$ {}_pS^j(x) = \frac{1}{N}\sum_k^j q_k^j\,{}_pM^j(x-k/N) = \sum_r^j \xi_r^j\,{}_pm_r^j(x), $$

where $\xi_r^j = {}_pb_r^{j-1}\xi_r^{j-1} + {}^{\nu}_{p}a_r^{j-1}\,{}^{\nu}\eta_r^{j-1}$ *and* $q_k^j = \sum_r^j \omega^{kr}\xi_r^j$.

Hence we see that, given the representation of a spline in the form (13), it is possible to reconstruct it in the conventional form by means of the inverse diagram

$$ \begin{array}{ccccccc} {}_pS^{j-m}(x) & \longrightarrow \cdots & {}_pS^{j-2}(x) & \longrightarrow & {}_pS^{j-1}(x) & \longrightarrow & {}_pS^j(x) \\ & \nearrow & & \nearrow & & \nearrow & \\ {}^{\nu}_{p}W^{j-m}(x) & & \cdots {}^{\nu}_{p}W^{j-2}(x) & & {}^{\nu}_{p}W^{j-1}(x) & & \end{array} $$

Our algorithm allows a fast implementation.

In closing we discuss the procedure for orthogonal projection of any function $f \in \mathbf{L}^2(T)$ onto the space ${}_p\mathfrak{B}^j$, which is the original one for a decomposition of the latter. Denote by ${}_pS^j(f, x)$ such a projection.

Theorem 7. *The representation*

$$ {}_pS^j(f, x) = \frac{1}{N}\sum_k^j q_k^j\,{}_pM^j(x-k/N) = \sum_r^j \xi_r^j\,{}_pm_r^j(x) $$

holds, where $\xi_r^j = T_r^j(\mathbf{F})/{}_{2p}u_r^j$, $\mathbf{F} = \{F_k\}_0^{N-1}$, *and*

$$ F_k = \int_0^1 f(x+k/N)\,{}_pM^j(x)\,dx, \qquad q_k^j = \sum_r^j \xi_r^j\omega^{kr}. $$

It may be shown that the coefficients $\{q_k^j\}$ of the ${}_pS^j(f, x) \in {}_p\mathfrak{B}^j$ coincide with those of the spline ${}_{2p}S^j(x) \in {}_{2p}\mathfrak{B}^j$ for which ${}_{2p}S^j((k+p)/N) = F_k$, $k = 0, \dots, N-1$.

We say some words about computing the values of F_k. Exploiting properties of B-splines (see [12]) leads us, provided $f \in C^p(T)$, to the relation

$$F_k = \frac{1}{N}\sum_{l=0}^{p} {}_pM^j(l/N)f(l/N + k/N) + o((p/N)^p).$$

If $f \in C^2(T)$, then

$$F_k = f(p/2N - k/N) + pf''(p/2N - k/N)/24N^2 + o((p/N)^2).$$

For functions f of lesser smoothness one may exploit the fact that the B-spline ${}_pB^j(x)$ (recall that the spline ${}_pM^j(x)$ is a "periodization" of ${}_pB^j(x)$) is the probability density of the sum of p random variables uniformly distributed on $[0, 1/N]$. Therefore F_k can be looked upon as the mean value of the function $f(x + k/N)$ with respect to the distribution ${}_pB^j(x)$, and one can compute F_k by the Monte Carlo method.

We mention that the ortsplines ${}_pm_r^j(x)$, which are basic for our constructions, are generalized eigenvectors of operators of convolution and of differentiation. Therefore the techniques we have suggested appear to be an adequate tool for solving problems connected with these operators. This topic as well as the spectral properties of our wavelets will be the subject of subsequent papers by the author. Our algorithms can be extended readily to the multidimensional case.

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