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Interpolatory subdivision schemes with infinite masks originated from splines

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A generic technique for the construction of diversity of interpolatory subdivision schemes on the base of polynomial and discrete splines is presented in the paper. The devised schemes have rational symbols and infinite masks but they are competitive (regularity, speed of convergence, computational complexity) with the schemes that have finite masks. We prove exponential decay of basic limit functions of the schemes with rational symbols and establish conditions, which guaranty the convergence of such schemes on initial data of power growth.

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1. Introduction

Subdivision started as a tool for efficient computation of spline functions. Now it is an independent subject with many applications. It is being used for developing new methods for curve and surface design, approximation, generating wavelets and multiresolution analysis and also for solving some classes of functional equations. Interpolatory subdivision schemes (ISS) are refinement rules, which iteratively refine the data by inserting values corresponding to intermediate points, using linear combinations of values in initial points, while the data in these initial points are retained. Non-interpolatory schemes also update the initial data, in addition to the insertion values into intermediate points. Stationary schemes use the same insertion rule at each refinement step. And a scheme is called uniform if its insertion rule does not depend on the location in the data. To be more specific, a univariate stationary uniform subdivision scheme with binary refinement S_a consists in the following: A function f^j that is defined on the grid $\mathbf{G}^j = \{k/2^j\}_{k\in\mathbb{Z}}$: $f^j(k/2^j) = f_k^j$, is extended onto the grid $\mathbf{G}^{j+1}$ by filtering the array $\{f_k^j\}_{k\in\mathbb{Z}}$:

$$f_k^{j+1} = \sum_{l\in\mathbb{Z}} a_{k-2l} f_l^j. \quad (1)$$

This is one refinement step. Next refinement step employs f^{j+1} as an initial data. The filter $a = \{a_k\}_{k\in\mathbb{Z}}$ is called the refinement mask of S_a. We define the z-transform of a sequence $f := \{f_k\}_{k\in\mathbb{Z}}$ belonging to the space l_1 of summable sequences as $f(z) := \sum_{k\in\mathbb{Z}} z^k a_k$. The z-transform of the mask $a(z) = \sum_{k\in\mathbb{Z}} z^k a_k$ is called the symbol of S_a. Throughout the paper we assume that $z = \mathrm{e}^{-\mathrm{i}\omega}$. If f^j and f^{j+1} belong to l_1 then equation (1) is equivalent to the following relation in the z-domain:

$$f^{j+1}(z) = a(z) f^j\big(z^2\big). \tag{2}$$

If the subdivision scheme is interpolatory, then $a_0 = 1$, $a_{2k} = 0\ \forall k \neq 0$. In this case, the symbol is represented by the sum

$$a(z) = 1 + zU\big(z^2\big), \quad \text{where } U(z) := \sum_{k\in\mathbb{Z}} z^k u_k,\ u_k = a_{2k+1}, \tag{3}$$

and the insertion rule (1) is split into two rules:

$$f_{2k}^{j+1} = f_k^j,\ f_{2k+1}^{j+1} = \sum_{l\in\mathbb{Z}} u_{k-l} f_l^j \Leftrightarrow f_e^{j+1}(z) = f^j(z),\ f_o^{j+1}(z) = U(z) f^j(z). \tag{4}$$

Here $f^j(z)$, $f_e^j(z)$ and $f_o^j(z)$ are the z-transforms of the arrays $\{f_k^j\}_{k\in\mathbb{Z}}$, $\{f_{2k}^j\}_{k\in\mathbb{Z}}$, $\{f_{2k+1}^j\}_{k\in\mathbb{Z}}$, respectively.

The well-known interpolatory uniform subdivision scheme by Dubuc and Deslauriers [7] can be formulated in the following way:

Polynomial Insertion Rule. The polynomial spline $Q_j^{2r}(x)$ of an even order $2r$ (degree $2r-1$) of deficiency $2r-1$ is constructed, which interpolates the function f^j on the grid $\mathbf{G}^j$: $Q_j^{2r}(k/2^j) = f_k^j$. Then, the samples f_k^{j+1} are defined as the values of the spline: $f_k^{j+1} = Q_j^{2r}(k/2^{j+1})$.

We recall that a spline of order $2r$ of deficiency $2r-1$ is a continuous function consisting of central arcs of interpolatory polynomials of degree $2r-1$. Even the first derivative may have breaks at grid points. For the spline $Q_j^{2r}(x)$ the mask $a := \{a_k\}$ comprises $2r$ non-zero terms and the symbol $a(z)$ is a Laurent polynomial.

Our construction is based on a simple idea: To replace the Polynomial Insertion Rule by the following rule:

Spline Insertion Rule. We construct the polynomial spline of order p (degree $p-1$) $V_j^p(x) \in C^{p-2}$ of deficiency 1, which interpolates the function f^j on the grid $\mathbf{G}^j$: $V_j^p(k/2^j) = f_k^j$. Then, the samples f_k^{j+1} are defined as the values of the spline: $f_k^{j+1} = V_j^p(k/2^{j+1})$.

If a spline of even order V_j^{2r} is used in this insertion rule then the limit function of the subdivision scheme is the same spline V_0^{2r}, which interpolates the initial data. But splines of odd order possess the property of super-convergence in the midpoints between the interpolation points [17]. Due to this property, the limit function for a spline of odd order is more regular than the spline itself. Moreover, employment of these splines allows to achieve a certain approximation order and smoothness of a limit function with lower computational complexity than by using splines of even orders. Therefore, splines of odd order are more suitable for this scheme.

Together with the polynomial splines we explore the so-called interpolatory discrete splines as a source for devising refinement masks [3,11]. The derived masks are related to the Butterworth filters, which are commonly used in signal processing [12].

A seeming drawback in using interpolatory splines is that it requires a convolution of the data with the infinite mask. But, due to rational structure of the symbols, this obstacle could be circumvented by employing recursive filtering [3,13]. As a result, the computational complexity implementing these schemes is even lower than the complexity of implementation of schemes with finite masks, which have comparable properties.

We analyze convergence and regularity of the designed subdivision schemes. Our analysis is based on the technique developed in [8,9] for schemes with finite masks. The extension of the technique to schemes with infinite masks requires some modifications. We prove that the basic limit functions of subdivision schemes with rational masks decay exponentially as their arguments tend to infinity. Obviously, this result is not surprising. There are hints on that in [4,10]. But the author never saw a proof of this result. In some sense a reciprocal fact was established in [5]. Under certain assumptions exponential decay of a refined function implies exponential decay of the refinement mask.

The rest of the paper is organized as follows. In section 2 we discuss properties of polynomial and discrete splines, which are necessary for our construction, and devise refinement masks for ISS's using interpolatory splines. Section 3 is devoted to the investigation of convergence and regularity of subdivision schemes with rational masks. In addition, the exponential decay of basic limit functions is proved. In section 4 we apply the above theory to the construction and analysis of three ISS's with rational masks. We compare their properties with the properties of two ISS's by Dubuc and Deslauriers and argue that the newly designed schemes are just competitive for applications with schemes that have finite masks.

2. Refinement masks derived from polynomial and discrete splines

2.1. Auxiliary results

In this preparatory section we recall known properties of B-splines and establish a few relations, which are necessary for the design of refinement masks and for the proof of the super-convergence property of splines of odd order.

2.1.1. Some properties of B-splines

The centered B-spline of order p is the convolution $M^p(x) = M^{p-1}(x) * M^1(x)$, $p \geqslant 2$, where $M^1(x)$ is the characteristic function of the interval $[-1/2, 1/2]$. Note that the B-spline of order p is supported on the interval $(-p/2, p/2)$. It is positive within its support and symmetric about zero. Nodes of B-splines of even orders are located at points $\{k\}_{k\in\mathbb{Z}}$ and of odd orders at points $\{k+1/2\}_{k\in\mathbb{Z}}$.

The Fourier transform of the B-spline of order p is

$$\widehat{M}^p(\omega) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} M^p(x)\,\mathrm{d}x = \left(\frac{\sin \omega/2}{\omega/2}\right)^p. \tag{5}$$

We introduce two sequences, which are important for further construction:

$$v^p := \{M^p(k)\}_{k\in\mathbb{Z}}, \qquad w^p := \left\{M^p\left(k+\frac{1}{2}\right)\right\}_{k\in\mathbb{Z}}.$$

Due to the compact support of B-splines, these sequences are finite. In table 2 in appendix B we present the sequences v^p and w^p for some values of p. The discrete-time Fourier transforms of these sequences are

$$\hat{v}^p(\omega) := \sum_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega k} M^p(k) = P^p\left(\cos\frac{\omega}{2}\right),$$
$$\widehat{w}^p(\omega) := \sum_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega k} M^p\left(k+\frac{1}{2}\right) = \mathrm{e}^{\mathrm{i}\omega/2} Q^p\left(\cos\frac{\omega}{2}\right). \tag{6}$$

Here the functions P^p and Q^p are real-valued polynomials. If $p = 2r-1$ then P^p is a polynomial of degree $2r-2$ and Q^p is a polynomial of degree $2r-3$. If $p = 2r$ then P^p is a polynomial of degree $2r-2$ and Q^p is a polynomial of degree $2r-1$.

The z-transform s of the sequences v^p and w^p

$$v^p(z) = \sum_{-\infty}^{\infty} z^k M^p(k), \qquad w^p(z) = \sum_{-\infty}^{\infty} z^k M^p\left(k+\frac{1}{2}\right)$$

are the so-called Euler–Frobenius polynomials [14]. These polynomials were extensively studied in [14,15]. In particular, the recurrence relations for their computation were established as well as the following fact.

Proposition 2.1 ([14]). On the unit circle $z = \mathrm{e}^{-\mathrm{i}\omega}$ the following inequalities hold:

$$0 < v^p(z) \leqslant 1. \tag{7}$$

The roots of the Laurent polynomials $v^p(z)$ are all simple and negative. Each root ζ can be paired with a dual root θ such that $\zeta\theta = 1$. Thus, if $p = 2r-1$, $p = 2r$ then $v^p(z)$ can be represented as:

$$v^p(z) = \prod_{n=1}^{r} \frac{1}{\gamma_n}(1+\gamma_n z)\left(1+\gamma_n z^{-1}\right),$$
$$0 < |\gamma_1| < |\gamma_2| < \cdots |\gamma_r| = \mathrm{e}^{-g} < 1, \; g > 0. \tag{8}$$

2.1.2. Euler–Frobenius polynomials and their ratios

The following facts are needed to establish the approximation properties of the forthcoming subdivision schemes that are based on the Spline Insertion Rule. Using (5) we can write

$$\begin{aligned} M^p(x-k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(x-k)}\left(\frac{\sin\omega/2}{\omega/2}\right)^p \mathrm{d}\omega \\ &= \sum_{l=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}lx}\int_0^1 \mathrm{e}^{2\pi\mathrm{i}\xi(x-k)}\frac{(\sin\pi\xi)^p(-1)^{lp}}{\pi(l+\xi))^p}\,\mathrm{d}\xi \\ &= \int_0^1 \mathrm{e}^{-2\pi\mathrm{i}\xi k} m_x^p(\xi)\,\mathrm{d}\xi, \quad \text{where} \end{aligned}$$

$$m_x^p(\xi) := \mathrm{e}^{2\pi\mathrm{i}\xi x}(\sin\pi\xi)^p \sum_{l=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}lx}\frac{(-1)^{lp}}{(\pi(l+\xi))^p}. \tag{9}$$

Relation (9) means that $M^p(x-k)$ is a Fourier coefficient of the 1-periodic function $m_x^p(\xi)$ and this function can be represented as the sum

$$m_x^p(\xi) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}k\xi}\, M^p(x+k). \tag{10}$$

Equations (6) and (10) imply the following representations:

$$P^p\left(\cos\frac{\omega}{2}\right) = \hat{v}^p(\omega) = m_0^p\left(\frac{\omega}{2\pi}\right) = \left(\sin\frac{\omega}{2}\right)^p \sum_{l=-\infty}^{\infty} \frac{(-1)^{lp}}{(\pi l+\omega/2)^p}, \tag{11}$$

$$Q^p\left(\cos\frac{\omega}{2}\right) = \mathrm{e}^{-\mathrm{i}\omega/2}\widehat{w}^p(\omega) = \mathrm{e}^{-\mathrm{i}\omega/2} m_{1/2}^p\left(\frac{\omega}{2\pi}\right) = \left(\sin\frac{\omega}{2}\right)^p \sum_{l=-\infty}^{\infty} \frac{(-1)^{l(p+1)}}{(\pi l+\omega/2)^p}.$$

It is seen from (11) that

$$P^p(1) = Q^p(1) = 1. \tag{12}$$

We also introduce two rational functions:

$$R^p(y) := \frac{Q^p(y)}{P^p(y)}, \qquad U_I^p(z) := \frac{w^p(z)}{v^p(z)}. \tag{13}$$

In section 2.2 we show that the function $1+zU_I^p(z^2)$ is the symbol for the ISS based on Spline Insertion Rule. It is readily verified that

$$\mathrm{e}^{-\mathrm{i}\omega/2}U_I^p\big(\mathrm{e}^{-\mathrm{i}\omega}\big) = R^p\left(\cos\frac{\omega}{2}\right), \qquad 1-zU_I^p\big(z^2\big) = 1-R^p(\cos\omega), \quad z=\mathrm{e}^{-\mathrm{i}\omega}. \tag{14}$$

Example.

Quadratic spline:

$$U_I^3(z) = 4\frac{1+z^{-1}}{z+6+z^{-1}}, \qquad 1+zU_I^3(z^2) = \frac{(1+z)^4}{z^4+6z^2+1},$$
$$1-zU_I^3(z^2) = \frac{(z^{-1}-2+z)^2}{z^2+6+z^{-2}}.$$

Cubic spline:

$$U_I^4(z) = \frac{(z^{-1}+1)(z^{-1}+22+z)}{8(z+4+z^{-1})}, \qquad 1+zU_I^4(z^2) = \frac{(1+z)^4(z+4+z^{-1})}{8(z^4+4z^2+1)}.$$

Spline of fourth degree:

$$U_I^5(z) = \frac{16(z+10+z^{-1})(1+z^{-1})}{z^2+76z+230+76z^{-1}+z^{-2}},$$
$$1+zU_I^5(z^2) = \frac{(1+z)^6\,(z^2+10z+1)}{z^8+76z^6+230z^4+76z^2+1}.$$

We observe that the symbols, originated from the splines of second and fourth degrees (of orders $2r-1$, $r=2,3$, respectively), comprise the factors $(z+1)^{2r}$. We show that this factorization is common to splines of even degrees and results in the so-called super-convergence property, which is valuable for subdivision.

Lemma 2.1. If $p = 2r-1$ then in the neighborhood of $\omega = 0$

$$1 - R^{2r-1}\left(\cos\frac{\omega}{2}\right) = \frac{\sin^{2r}\omega/2}{P^{2r-1}(\cos\omega/2)}\left(A_r + \mathrm{O}\left(\sin^2\frac{\omega}{2}\right)\right),$$
$$A_r := \frac{(4^r-1)}{r(2r-2)!}|b_{2(r)}|, \tag{15}$$

where b_s is the Bernoulli number of order s.

The proof is given in appendix A. Recall that the degree of a Laurent polynomial $\sum_{k=-\mu}^{\nu}\alpha_k z^k$ is defined as $\mu+\nu$.

Corollary 2.1. If $p = 2r-1$ then the following factorization formula holds

$$1 - zU_I^{2r-1}(z^2) = \frac{(-z+2-z^{-1})^r}{v^{2r-1}(z^2)}\left(4^{-r}A_r + \left(z-2+z^{-1}\right)q^r(z)\right), \tag{16}$$

where $q^2(x) \equiv 0$ and $q^r(z)$ is a symmetric Laurent polynomial of degree $2(r-3)$ for $r \geqslant 3$.

Proof. The case $r = 2$ is explicitly presented in the above example. We have

$$\begin{aligned}
1 - zU_I^{2r-1}\big(z^2\big) &= \frac{v^{2r-1}(z^2) - zw^{2r-1}(z^2)}{v^{2r-1}(z^2)} \\
&= 1 - R^p(\cos\omega) = \frac{\sin^{2r}\omega}{P^{2r-1}(\cos\omega)}\big(A_r + \mathrm{O}\big(\sin^2\omega\big)\big) \\
&= \frac{(-z+2-z^{-1})^r}{v^{2r-1}(z^2)}\big(4^{-r}A_r + \mathrm{O}\big(z-2+z^{-1}\big)\big).
\end{aligned}$$

The numerator of the rational function $1-zU_I^{2r-1}(z^2)$ is a symmetric Laurent polynomial of degree $4(r-1)$. Thus, $\mathrm{O}(z-2+z^{-1}) = (z-2+z^{-1})q^r(z)$. □

We conclude the section by a fact about splines of even order.

Proposition 2.2. If $p = 2r$, then

$$1 + R^{2r}\left(\cos\frac{\omega}{2}\right) = \frac{2(\cos\omega/4)^{2r}P^{2r}(\cos\omega/4)^{2r}}{P^{2r}(\cos\omega/2)}, \tag{17}$$

$$1 + zU_I^{2r}\big(z^2\big) = \frac{(1+z)^{2r}v^{2r}(z)}{2^{2r-1}z^r v^{2r}\big(z^2\big)}, \qquad 1 - zU_I^{2r}\big(z^2\big) = \frac{(z-2+z^{-1})^r v^{2r}(z)}{2^{2r-1}v^{2r}(z^2)}. \tag{18}$$

Proof. From (11) and (12) we have

$$\begin{aligned}
1 + R^{2r}\left(\cos\frac{\omega}{2}\right) &= 71 + \frac{Q^{2r}(\cos\omega/2)}{P^{2r}(\cos\omega/2)} = \frac{2(\sin\omega/2)^{2r}\sum_{l=-\infty}^{\infty}(\pi 2l + \omega/2)^{-2r}}{P^{2r}(\cos\omega/2)} \\
&= \frac{2(\cos\omega/4)^{2r}(\sin\omega/4)^{2r}\sum_{l=-\infty}^{\infty}(\pi l + \omega/4)^{-2r}}{P^{2r}(\cos\omega/2)} \\
&= \frac{2(\cos\omega/4)^{2r}P^{2r}(\cos\omega/4)}{P^{2r}(\cos\omega/2)}.
\end{aligned}$$

Equations (18) follow immediately from definition (13) and equation (17). □

2.2. *Refinement masks derived from interpolatory polynomial splines*

In this section we devise the refinement mask according to Spline Insertion Rule. We also evaluate the approximation error at midpoints between points of interpolation and prove the super-convergence property.

Shifts of B-splines form a basis in the space $\mathbf{V}_j^p$ of splines of order p on the grid $\mathbf{G}^j = \{k/2^j\}_{k\in\mathbb{Z}}$. Namely, any spline $V_j^p \in \mathbf{V}_j^p$ can be represented as

$$V_j^p(x) = \sum_l c_l\, M^p\big(2^j x - l\big). \tag{19}$$

Denote $c = \{c_l\}_{l\in\mathbb{Z}}$ and let $c(z)$ be the z-transform of c. We introduce the sequences $\epsilon^p := \{\epsilon_k^p = V_j^p(k/2^j)\}_{k\in\mathbb{Z}}$, $o^p := \{o_k^p = V_j^p((2k+1)/2^{j+1})\}_{k\in\mathbb{Z}}$ and $s^p = \{s_k^p =$

$V_j^p(k/2^{j+1})\}_{k\in\mathbb{Z}}$ of spline values at the sparse-grid points, at the midpoints and on the refined grid $\{k/2^{j+1}\}_{k\in\mathbb{Z}}$, respectively. The z-transform of the sequence s^p is

$$s^p(z) = \epsilon^p(z^2) + zo^p(z^2). \tag{20}$$

We have

$$\epsilon_k^p = \sum_l c_l\, M^p(k-l), \qquad o_k^p = \sum_l c_l\, M^p\left(k-l+\frac{1}{2}\right).$$

Thus $\epsilon^p(z) = c(z)v^p(z)$, and $o^p(z) = c(z)w^p(z)$.

From these equations we can derive expressions for the coefficients of the spline V_j^p, which interpolates a given sequence $e = \{e_k\} \in l_1$ on the sparse grid $\{k/2^j\}_{k\in\mathbb{Z}}$:

$$\epsilon_k^p = e_k,\ \forall k \in \mathbb{Z}, \Leftrightarrow c(z)v^p(z) = e(z) \Leftrightarrow c(z) = \frac{e(z)}{v^p(z)} \Leftrightarrow c_l = \sum_{n=-\infty}^{\infty} \lambda_{l-n}^p e_n. \tag{21}$$

Here, $\lambda^p = \{\lambda_k^p\}_{k\in\mathbb{Z}}$ is the sequence, which is defined via its z-transform:

$$\lambda^p(z) = \sum_{k=-\infty}^{\infty} z^k \lambda_k^p = \frac{1}{v^p(z)}.$$

It follows from (8) that the coefficients $\{\lambda_k^p\}_{k\in\mathbb{Z}}$ decay exponentially as $|k| \to \infty$. We will prove a general statement about this fact in proposition 3.1. Substitution of (21) into (19) results in an alternative representation of the interpolatory spline:

$$V_j^p(x) = \sum_{l=-\infty}^{\infty} e_l\, L^p(2^j x - l), \qquad \text{where } L^p(x) := \sum_l \lambda_l^p\, M^p(x-l). \tag{22}$$

The spline $L^p(x)$, defined in (22), is called the fundamental spline. It interpolates the Kronecker delta $\delta(k)$, that is $L^p(x)$ vanishes at all integer points except $x = 0$ where $L^p(0) = 1$. Due to the decay of the coefficients $\{\lambda_k^p\}_{k\in\mathbb{Z}}$, the spline $L^p(x)$ decays exponentially as $|x| \to \infty$. Therefore, the representation (22) of the interpolatory spline remains valid for the sequences $\{e_k\}_{k\in\mathbb{Z}}$, which may grow not faster than a power of k [16] (the sequences of power growth). The values of the fundamental spline at midpoints are

$$\widetilde{L}_k^p := L^p\left(k+\frac{1}{2}\right) = \sum_l \lambda_l^p M^p\left(k-l+\frac{1}{2}\right), \qquad \widetilde{L}^p(z) = \frac{w^p(z)}{v^p(z)} = U_I^p(z), \tag{23}$$

where $\widetilde{L}^p(z)$ denotes the z-transform of the sequence $\{\widetilde{L}_k^p\}$. Hence, the values of the interpolatory spline at midpoints are

$$o_k^p = \sum_n \widetilde{L}_{k-n}^p e_n \Leftrightarrow o^p(z) = U_I^p(z)e(z). \tag{24}$$

The spline V_j^p is interpolatory. Therefore, substituting (24) into (20) we obtain

$$s^p(z) = a_I^p(z)e(z^2), \quad \text{where } a_I^p(z) := 1 + zU_I^p(z^2).$$

If a subdivision scheme S_a is defined in accordance with the Spline Insertion Rule, which was formulated in section 1, then the rational function $a_I^p(z)$ is its symbol. The mask $a_I^p = \{a_k^p\}_I$ is infinite but decays exponentially as $|k| \to \infty$.

Super-convergence property. The interpolatory splines of odd orders possess the so-called super-convergence property, which is valuable for subdivision. Recall that in general the spline of order p (degree $p-1$), which interpolates the values of a polynomial of degree $p-1$, coincides with this polynomial. However, we show that the spline of odd order $2r-1$ (degree $2r-2$), which interpolates the values of a polynomial of degree $2r-1$ on the equispaced grid, restores the values of this polynomial at mid-points between the points of interpolation. We claim that the mid-points are points of super-convergence of the spline V_j^{2r-1}.

Denote by $\mathbf{D}^2$ the operator of centered second difference: $\mathbf{D}^2 f_k = f_{k-1} - 2f_k + f_{k+1}$. Application of this operator in z-domain reduces to multiplication with the Laurent polynomial $D^2(z) = z - 2 + z^{-1}$. The Laurent polynomial $D^{2r}(z) = (z - 2 + z^{-1})^r$ corresponds to the $2r$-order difference, which we denote as $\mathbf{D}^{2r}$.

Theorem 2.1 (Superconvergence property). Let a spline V_j^{2r-1} of order $2r-1$ interpolate $f(x)$ on the grid $\{k/2^j\}_{k\in\mathbb{Z}}$. If the function f is a polynomial of degree $2r-1$ then

$$V_j^{2r-1}\left(\frac{2k+1}{2^{j+1}}\right) = f\left(\frac{2k+1}{2^{j+1}}\right) \quad \forall k \in \mathbb{Z}.$$

If f is a polynomial of degree $2r+1$ then

$$V_j^{2r-1}\left(\frac{2k+1}{2^{j+1}}\right) = f\left(\frac{2k+1}{2^{j+1}}\right) - A_r F,$$

where the constant $F := \mathbf{D}^{2r} f(x)$.

Proof. From equations (22)–(24) we obtain

$$\begin{aligned} f\left(\frac{2k+1}{2^{j+1}}\right) - V_j^{2r-1}\left(\frac{2k+1}{2^{j+1}}\right) &= f\left(\frac{2k+1}{2^{j+1}}\right) - \sum_n \widetilde{L}_{k-n}^{2r-1} f\left(\frac{n}{2^j}\right) \\ &= \sum_n g_{2k+1-n}^{2r-1} f\left(\frac{n}{2^{j+1}}\right), \end{aligned} \tag{25}$$

where

$$g_k^p = \begin{cases} \delta(l), & \text{if } k = 2l, \\ -\widetilde{L}_l^p, & \text{if } k = 2l+1. \end{cases}$$

Equation (25) means that to obtain the difference $f((2k+1)/2^{j+1}) - o_k^{2r-1}$, we must apply the mask $g^{2r-1} := \{g_k^{2r-1}\}_{k\in\mathbb{Z}}$ to the data $\{f(k/2^{j+1})\}_{k\in\mathbb{Z}}$ and take odd samples from the produced array. The series in (25) converges absolutely due to the exponential decay of the coefficients of the mask g^{2r-1}. The symbol of the mask g^{2r-1} is $g^{2r-1}(z) = 1 - zU_I^{2r-1}(z^2)$. Due to (16),

$$g^{2r-1}(z) = \frac{1}{v^{2r-1}(z^2)} G^{2r-1}(z),$$
$$G^{2r-1}(z) := 4^{-r} A_r \big(z - 2 + z^{-1}\big)^r + q^r(z)\big(z - 2 + z^{-1}\big)^{r+1}.$$

Application of the filter $g^{2r-1}(z)$ to the array $\{f(k/2^{j+1})\}_{k\in\mathbb{Z}}$ reduces to the subsequent application of the filters $G^{2r-1}(z)$ and $1/v^{2r-1}(z^2)$. The result of application of the filter $G^{2r-1}(z)$ is the array

$$\gamma_k := 4^{-r} A_r \mathbf{D}^{2r} f\left(\frac{k}{2^{j+1}}\right) + \sum_{l=-r+3}^{r-3} q_l^r \mathbf{D}^{2r+2} f\left(\frac{k-l}{2^{j+1}}\right),$$

where $\{q_l^r\}_{l=-r+3}^{r-3}$ are the coefficients of the Laurent polynomial $\{q^r(z)\}$. If f is a polynomial of degree $2r-1$ then $\gamma_k = 0\ \forall k$. If f is a polynomial of degree $2r+1$ then $\gamma_k = 4^{-r} A_r F\ \forall k$. The result of application of the filter $1/v^{2r-1}(z^2)$ to the constant $4^{-r} A_r F$ is the constant $4^{-r} A_r F / v^{2r-1}(1) = 4^{-r} A_r F$ due to (12). □

Originally, this property was established by other means in [17].

Remark. The interpolatory splines of even order $2r$ do not have this super-convergence property. They are exact on polynomials of degree $2r-1$ but also on splines of order $2r$ in the following sense.

Theorem 2.2. Let $f(x) = V_1^{2r}(x)$ be a spline of order $2r$ with nodes on the grid $\{k\}_{k\in\mathbb{Z}}$, $k \in \mathbb{Z}$, and the initial data $f_k^0 = f(k)$. Then, all the subsequent steps of subdivision reproduce the values of this spline: $f_k^j = f(k/2^j),\ k \in \mathbb{Z},\ j \in \mathbb{N}$.

Proof. Without loss of generality, assume that $f(x) = L^{2r}(x)$ is the fundamental spline of order $2r$ with nodes on the grid $\{k\}_{k\in\mathbb{Z}}$. Due to the well known property of minimal norm [2], the integral

$$\mu := \int_{-\infty}^{\infty} \left|f^{(r)}(x)\right|^2 \mathrm{d}x \leqslant \int_{-\infty}^{\infty} \left|g^{(r)}(x)\right|^2 \mathrm{d}x,$$

where $g(x)$ is any function such that $g^{(r)}(x)$ is square integrable and $g(k) = \delta(k)$. Let $F(x)$ be a spline of order $2r$, which interpolates the values $\{\varphi_k = f(k/2)\}_{k\in\mathbb{Z}}$. Then,

$$\nu := \int_{-\infty}^{\infty} \left|F^{(r)}(x)\right|^2 \mathrm{d}x \leqslant \int_{-\infty}^{\infty} \left|G^{(r)}(x)\right|^2 \mathrm{d}x,$$

where $G(x)$ is any function such that $G^{(r)}(x)$ is square integrable and $G(k/2) = \varphi_k$. Hence, $\nu \leqslant \mu$. On the other hand, $F(k) = \delta(k)$ and, therefore, $\mu \leqslant \nu$. Thus,

$$\int_{-\infty}^{\infty} \left| f^{(r)}(x) \right|^2 \mathrm{d}x = \int_{-\infty}^{\infty} \left| F^{(r)}(x) \right|^2 \mathrm{d}x.$$

Hence, it follows that $F(x) \equiv f(x)$. Due to the representation (22), the assertion is extended to any spline of order $2r$, which interpolates the data of power growth. □

Remark. The above result remains true for splines, which interpolate initial data on a non-equispaced grid.

2.3. *Refinement masks derived from discrete splines*

In this section we introduce refinement masks for ISS using the so-called discrete splines. The discrete splines are defined on grids $\{k\}_{k\in\mathbb{Z}}$ and present a counterpart to the continuous splines. For a detailed description of the subject, see [11].

The discrete B-spline $B^{1,n} = \{B_j^{1,n}\}_{j\in\mathbb{Z}}$ of first order is defined by the following sequence:

$$B_j^{1,n} = \begin{cases} 1 & \text{if } j = 0, \ldots, 2n-1, \ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The higher-order B-splines are defined as discrete convolutions by recurrence: $B^{p,n} = B^{1,n} * B^{p-1,n}$. Obviously, the z-transform of the B-spline of order p is

$$B^{p,n}(z) = \left(1 + z + z^2 + \cdots + z^{2n-1}\right)^p, \quad p = 1, 2, \ldots.$$

If $p = 2r$, $r \in \mathbb{N}$, then the B-spline $B^{p,n}$ is symmetric about the point $j = r$ where it attains its maximal value. The centered B-spline $M_j^{2r,n}$ of order $2r$ is defined as a shift of the B-spline:

$$M_j^{2r,n} := B_{j+r}^{2r,n}, \qquad M^{2r,n}(z) = z^{-r} B^{2r,n}(z).$$

The discrete spline $V^{2r,n} = \{V_k^{2r,n}\}_{k\in\mathbb{Z}}$ of order $2r$ is defined as a linear combination, with real-valued coefficients, of shifts of the centered B-spline of order $2r$:

$$V_k^{2r,n} := \sum_{l=-\infty}^{\infty} c_l M_{k-2nl}^{2r,n}.$$

Let $\{e_k\}_{k\in\mathbb{Z}}$ be a given sequence. The discrete spline $V^{2r,n}$ is called interpolatory if the following relations hold:

$$V_{2nk}^{2r,n} = e_k, \quad k \in \mathbb{Z}. \tag{26}$$

Proposition 2.3 ([11]). If the sequence $\{e_k\}_{k\in\mathbb{Z}}$ is of power growth then there exists a unique discrete spline of power growth $V^{2r,n}$ satisfying (26).

The points $\{2kn\}_{k\in\mathbb{Z}}$ are called the nodes of the spline.

In this paper we explore only the case $n = 1$ and denote $V^{2r} := V^{2r,1}$ and $M^{2r} := M^{2r,1}$. One refinement step of the corresponding interpolatory subdivision scheme consists in the following:

Discrete Spline Insertion Rule. Given the data $f^j := \{f_k^j\}_{k\in\mathbb{Z}}$ of power growth, we construct the discrete spline V^{2r} such that $V_{2k}^{2r} = f_k^j$, $k \in \mathbb{Z}$. Then, the entries of the refined array are defined as the values of the spline: $f_k^{j+1} := V_k^{2r}$, $k \in \mathbb{Z}$.

Analysis, which results in the symbol of the ISS, is similar to the analysis in the polynomial splines case but it is simpler. As before, we denote $\epsilon^{2r} = \{\epsilon_k^{2r} = V_{2k}^{2r}\}_{k\in\mathbb{Z}}$, $o^{2r} = \{o_k^{2r} = V_{2k+1}^{2r}\}_{k\in\mathbb{Z}}$ and $s^{2r} = \{s_k^{2r} = V_k^{2r}\}_{k\in\mathbb{Z}}$. If the spline V^{2r} interpolates the sequence $\{e_k\}_{k\in\mathbb{Z}}$ then

$$\epsilon_k^{2r} = \sum_{l=-\infty}^{\infty} c_l M_{2(k-l)}^{2r} = e_k \iff c(z) M_e^{2r}(z) = e(z),$$

where

$$M_e^{2r}(z^2) := \sum_{l=-\infty}^{\infty} z^{2l} M_{2l}^{2r} = \frac{1}{2}\big(M^{2r}(z) + M^{2r}(-z)\big)$$
$$= \frac{1}{2}\big(z^{-r}(1+z)^{2r} + (-z)^{-r}(1-z)^{2r}\big).$$

The values of the spline at odd points are

$$o_k^{2r} = \sum_{l=-\infty}^{\infty} c_l M_{2(k-l)+1}^{2r} \iff o^{2r}(z) = c(z) M_o^{2r}(z), \tag{27}$$

where

$$M_o^{2r}(z^2) := \sum_{l=-\infty}^{\infty} z^{2l} M^{2r}(2l+1) = \frac{z^{-1}}{2}\big(M^{2r}(z) - M^{2r}(-z)\big)$$
$$= \frac{z^{-1}}{2}\big(z^{-r}(1+z)^{2r} - (-z)^{-r}(1-z)^{2r}\big).$$

Finally, we have

$$o^{2r}(z) = U_d^{2r}(z) e(z), \qquad U_d^{2r}(z^2) := z^{-1}\frac{(1+z)^{2r} - (-1)^r(1-z)^{2r}}{(1+z)^{2r} + (-1)^r(1-z)^{2r}}.$$

Hence, the symbol of the ISS based on Discrete Spline Insertion Rule is

$$a_d^{2r}(z) = 1 + zU_d^{2r}(z^2) = \frac{2(1+z)^{2r}}{(1+z)^{2r} + (-1)^r(1-z)^{2r}}.$$

The following proposition, which was established in [3], characterizes the structure of the denominator $D_r(z) := (1+z)^{2r} + (-1)^r\,(1-z)^{2r}$ of the symbol $a_d^{2r}(z)$.

Proposition 2.4 ([3]). If $r = 2p+1$, then the following representation holds:

$$D_r(z) = 4rz^r \prod_{k=1}^{p} \frac{1}{\gamma_k^r}\left(1+\gamma_k^r z^{-2}\right)\left(1+\gamma_k^r z^2\right),$$
$$\text{where } \gamma_k^r = \cot^2\frac{(p+k)\pi}{2r} < 1,\ k = 1,\dots,p.$$

If $r = 2p$ then

$$D_r(z) = 2z^r \prod_{k=1}^{p} \frac{1}{\gamma_k^r}\left(1+\gamma_k^r z^{-2}\right)\left(1+\gamma_k^r z^2\right),$$
$$\text{where } \gamma_k^r = \cot^2\frac{(2p+2k-1)\pi}{4r} < 1,\ k = 1,\dots,p.$$

The proposition implies, in particular, that the mask of the devised ISS decays exponentially. The following relation

$$1 - zU_d^{2r}\left(z^2\right) = \frac{2(-1)^r(z-2+z^{-1})^r}{(1+z)^{2r} + (-1)^r(1-z)^{2r}}$$

guarantees that the presented ISS is exact on polynomials of degree $2r-1$.

The refinement mask $\{a_d^{2r}(k)\}_{k\in\mathbb{Z}}$ is closely related to the discrete-time Butterworth filter, which is commonly used in signal processing [3,12]. To be specific, application of this mask to a data array is equivalent to the subsequent forward and backward application of the Butterworth filter of order r.

Examples. (1) $r = 1$. In this case the mask is finite, $U_d^2(z) = (1+z^{-1})/2$, $a_d^2(z) = (1+z)^2/2z$.

(2) $r = 2$. In this case the mask coincides with the mask generated by the quadratic polynomial spline: $U_d^4(z) = U_I^3(z)$.

(3) $r = 3$.

$$U_d^6(z) = \frac{(z+14+z^{-1})(1+z^{-1})}{6z^{-1}+20+6z}, \qquad a_d^6(z) = \frac{(z+1)^6}{6\,z^5+20\,z^3+6\,z}.$$

3. Convergence and regularity of subdivision schemes with rational symbols

3.1. *Preliminary results*

For the investigation of convergence and regularity of the presented subdivision schemes we use the modified technique developed by Dyn, Gregory and Levin [8,9]. The difference is that, unlike these authors, we study subdivision schemes with infinite but exponentially decaying masks. Therefore, in the sequel we restrict the admissible initial data to the sequences of power growth. It means that for a sequence $\{f_k^0\}_{k\in\mathbb{Z}}$ positive constants A and M exist such that

$$\left|f_k^0\right| \leqslant Mk^A. \tag{28}$$

In this section we analyze subdivision schemes that have rational symbols $a(z) = T(z)/P(z)$ subject to the following requirements:

P1: The Laurent polynomials $P(z)$ and $T(z)$ are symmetric about inversion: $P(z^{-1}) = P(z)$, $T(z^{-1}) = T(z)$ thus they are real on the unit circle $|z| = 1$.

P2: The roots of the denominator $P(z)$ are simple and do not lie on the unit circle $|z| = 1$.

P3: The symbol $a(z)$ is factorized as follows:

$$a(z) = (1+z)q(z), \quad q(1) = 1. \tag{29}$$

In the sequel we will say that a subdivision scheme S_a belongs to class P if its symbol $a(z)$ possesses the properties P1–P3.

The above properties imply, in particular, that the coefficients a_k of the mask of the scheme S_a of class P are symmetric about zero. If P1 and P2 hold then $P(z)$ can be represented as follows:

$$P(z) = \prod_{n=1}^{r} \frac{1}{\gamma_n}(1+\gamma_n z)\left(1+\gamma_n z^{-1}\right),$$
$$0 < |\gamma_1| < |\gamma_2| < \cdots |\gamma_r| = \mathrm{e}^{-g} < 1,\, g > 0. \tag{30}$$

Note that all the subdivision schemes introduced in section 2 are the schemes of class P.

Proposition 3.1. If the symbol of a scheme S_a is $a(z) = T(z)/P(z)$ and equation (30) holds, then the mask satisfies the inequality

$$|a_k| \leqslant A\mathrm{e}^{-g|k|},$$

where A is a positive constant.

Proof. Assume the degree τ of $T(z)$ is less than the degree p of $P(z)$. If equation (30) holds, then the symbol can be represented as follows:

$$a(z) = \sum_{n=1}^{r}\left(\frac{A_n^+}{1+\gamma_n z} + \frac{A_n^- z}{1+\gamma_n z^{-1}}\right) = \sum_{n=1}^{r}\left(A_n^+ \sum_{k=0}^{\infty}(-\gamma_n)^k z^k + zA_n^- \sum_{k=0}^{\infty}(-\gamma_n)^k z^{-k}\right)$$
$$= \sum_{k=0}^{\infty}\left(a_k^+ z^k + a_k^- z^{1-k}\right), \quad a_k^+ = \sum_{n=1}^{r} A_n^+(-\gamma_n)^k,\, a_k^- = \sum_{n=1}^{r} A_n^-(-\gamma_n)^k,$$
$$\left|a_k^+\right| \leqslant |\gamma_r|^k \sum_{n=1}^{r}\left|A_n^+\right| \leqslant A\mathrm{e}^{-g|k|},\, \left|a_k^-\right| \leqslant |\gamma_r|^k \sum_{n=1}^{r}\left|A_n^-\right| \leqslant A\mathrm{e}^{-g|k|}. \tag{31}$$

If $p \geqslant \tau$ then a polynomial of degree $p - t$ is added to the expansion (31). Obviously, this addition does not affect the decay of the mask $a(k)$ as k tends to infinity. $\square$

Lemma 3.1. Let S_a be the subdivision scheme, whose symbol is $a(z) = T(z)/P(z)$ and the Laurent polynomial $P(z)$ satisfies properties P1 and P2. If equation (30) holds then for any finite initial data f^0 the following inequalities hold:

$$\left|f_k^j\right| \leqslant A_j \mathrm{e}^{-g|k|2^{-j+1}}. \tag{32}$$

Proof. The mask of the scheme S_a decays exponentially: $|a_k| \leqslant A\mathrm{e}^{-g|k|}$. Due to (2)

$$f^1(z) = a(z) f^0\left(z^2\right) = \frac{T_1(z)}{P_1(z)},$$

where $T_1(z) := T(z) f^0(z^2)$ and $P_1(z) = P(z)$. Hence, the roots of $P_1(z)$ are: $\rho_n^1 = -\gamma_n$, $1 \leqslant n \leqslant r$, and, therefore, $|f_k^1| \leqslant A_1 \mathrm{e}^{-g|k|}$. The next refinement step produces the following z-transform:

$$f^2(z) = a(z) f^1\left(z^2\right) = \frac{T_2(z)}{P_2(z)}, \quad P_2(z) = P(z) P\left(z^2\right).$$

The roots of $P_2(z)$ satisfy the inequality $|\rho_n^2| \leqslant \sqrt{|\gamma_r|} = \mathrm{e}^{-g/2}$. Hence, $|f_k^2| \leqslant A_2 \mathrm{e}^{-g|k|/2}$. Then (32) is derived by induction. $\square$

Let S_a be a subdivision scheme of class P and S_q be the scheme with the symbol $q(z)$, which is defined in (29). Since the denominator of the symbol $q(z)$ is the same as the denominator of $a(z)$, the mask $\{q_k\}$ of the scheme S_q satisfies the inequality

$$|q_k| \leqslant Q \mathrm{e}^{-g|k|}. \tag{33}$$

Denote by Δ the difference operator: $\Delta f_k = f_{k+1} - f_k$.

Proposition 3.2 ([9]). If the scheme S_a is of class P, then

$$\Delta(S_a f) = S_q \Delta f$$

for any data set $f \in l_1$.

Proof. Obviously, $(\Delta f)(z) = (z-1) f(z)$ and using (2) we have

$$\begin{aligned}(\Delta S_a f)(z) &= (z-1)(S_a f)(z) = (z-1)\, a(z) f\left(z^2\right) \\ &= q(z)\left(z^2 - 1\right) f\left(z^2\right) = q(z)(\Delta f)\left(z^2\right) \Leftrightarrow \Delta(S_a f) = S_q \Delta f. \qquad \square\end{aligned}$$

Denote $\|f^j\|_\infty := \max_{k\in\mathbb{Z}} |f_k^j|$. Equation (1) implies that

$$f_{2k}^{j+1} = \sum_{l\in\mathbb{Z}} a_{2k-2l} f_l^j, \qquad f_{2k+1}^{j+1} = \sum_{l\in\mathbb{Z}} a_{2k+1-2l} f_l^j.$$

Hence, it follows

$$\left\| f^{j+1} \right\|_\infty \leqslant \|S_a\| \|f^j\|_\infty, \quad \text{where } \|S_a\| := \max\left\{\sum_{k\in\mathbb{Z}} |a_{2k}|,\ \sum_{k\in\mathbb{Z}} |a_{2k+1}|\right\}.$$

Similarly, after L refinement steps we have

$$\left\| f^{j+L} \right\|_\infty \leqslant \left\| S_a^L \right\| \left\| f^j \right\|_\infty, \quad \text{where } \left\| S_a^L \right\| := \max_n \left\{\sum_k |a^{[L]}_{n+2^L k}|\colon\ 0 \leqslant n \leqslant 2^L - 1\right\}$$

and $\{a_k^{[L]}\}$ is the mask of the operator S_a^L.

3.2. Existence and regularity of basic limit function

Let S_a be a subdivision scheme whose mask is $a = \{a_k\}_{k\in\mathbb{Z}}$.

Definition 3.1. Let the initial data set be the Kronecker delta $f^0 = \{\delta(k)\}_{k\in\mathbb{Z}}$ and $f^j(t)$ be the sequence of polygonal lines (second-order splines) that interpolates the data generated by S_a at the corresponding refinement level: $\{f^j(2^{-j}k) = f_k^j = (S_a^j f^0)_k\}_{k\in\mathbb{Z}}$. If $\{f^j(t)\}$ converges uniformly to a continuous function $\phi_a(t)$ then this function is called the basic limit function (BLF) of the scheme S_a.

Remark. This definition is equivalent to more common definition via the difference between the limit function and the refined data at dyadic points ([6], for example).

We single out a particular class of spline-based ISS's when the BLF exists and can be explicitly presented.

Theorem 3.1. If the symbol of the ISS S_a: $a_I^{2r}(z) = 1 + zU_I^{2r}(z^2)$ is derived from a polynomial interpolatory spline of order $2r$ then there exists the BLF $\phi_a(t)$ of the scheme, which is equal to the fundamental spline $L^{2r}(t)$. Thus, the BLF $\phi_a \in C^{2r-2}$ and decays exponentially as t tends to infinity.

The theorem is a straightforward consequence of theorem 2.2.

In the rest of the section we establish the conditions for a subdivision scheme of class P to have BLF, which decays exponentially.

Proposition 3.3. Let S_a be a subdivision scheme of class P and S_q be the scheme, whose symbol is $q(z)$ and the mask is $\{q_k\}_{k\in\mathbb{Z}}$. If for some $L \in \mathbb{N}$ the following inequality holds

$$\left\| S_q^L \right\| := \max_n \left\{\sum_k |q^{[L]}_{n+2^L k}|\colon\ 0 \leqslant n \leqslant 2^L - 1\right\} = \mu < 1, \tag{34}$$

then there exists a continuous BLF $\phi_a(t)$ of the scheme S_a.

If the condition (34) holds, then the scheme S_q is called *contractive*.

The proof of this proposition is a slightly modified version of the proof of a related assertion in [9].

Proof. We recall that due to lemma 3.1, the sequences $f^j = S_a^j f^0$ belong to l_1 $\forall j \in \mathbb{Z}_+$. We have to show that the sequence of the second-order splines $\{f^j(t)\}_{j\in\mathbb{Z}_+}$, which interpolate the subsequently refined data $f^j(2^{-j}k) = f_k^j$, $k \in \mathbb{Z}$, where $\{f_k^0 = \delta(k)\}_{k\in\mathbb{Z}}$, converges to a continuous function as $j \to \infty$. Denote

$$D^{j+1}(t) := f^{j+1}(t) - f^j(t). \tag{35}$$

The maximum absolute value of this piecewise linear function is reached at its breakpoints. Therefore, if $t = 2^{-j}(k + \tau)$, $0 \leqslant \tau \leqslant 1$, then

$$\left|D^{j+1}(t)\right| \leqslant \max\left\{\sup_{k\in\mathbb{Z}}\left|f_{2k}^{j+1} - f_k^j\right|, \sup_{k\in\mathbb{Z}}\left|f_{2k+1}^{j+1} - \frac{f_k^j + f_{k+1}^j}{2}\right|\right\}. \tag{36}$$

Let

$$m_{2k}^{j+1} = f_k^j, \qquad m_{2k+1}^{j+1} = \frac{f_k^j + f_{k+1}^j}{2}, \quad k \in \mathbb{Z}.$$

Then, the z-transform $m^{j+1}(z)$ is equal to

$$m^{j+1}(z) = l(z) f^j\left(z^2\right), \quad l(z) := \frac{1}{2}\left(z^{-1} + 2 + z\right) = \frac{z^{-1}}{2}(1+z)^2,$$

and we obtain

$$\sup_{t\in\mathbb{R}}\left|D^{j+1}(t)\right| = \left\| f^{j+1} - m^{j+1}\right\|_\infty. \tag{37}$$

Since $q(1) = 1$, the function $q(z) - (1+z)/(2z)$ can be represented as $(1-z)r(z)$, where $r(z) = \sum_{k\in\mathbb{Z}} r_k z^{-k}$ is a rational function with the same denominator $P(z)$ as the symbol $a(z)$ has. Hence,

$$|r_k| \leqslant R\mathrm{e}^{-g|k|}. \tag{38}$$

Equation (29) implies

$$\begin{aligned} f^{j+1}(z) - m^{j+1}(z) &= \big((1+z)q(z) - l(z)\big) f^j\left(z^2\right) \\ &= (1+z)\left(q(z) - \frac{1+z}{2z}\right) f^j\left(z^2\right) = (1+z)(1-z)r(z) f^j\left(z^2\right) \\ &= r(z) h^j\left(z^2\right), \end{aligned} \tag{39}$$

where $h^j(z) = (\Delta f^j)(z)$. Combining (37) and (39) we derive

$$\begin{aligned} \sup_{t\in\mathbb{R}}\left|f^{j+1}(t) - f^j(t)\right| &= \left\|f^{j+1} - m^{j+1}\right\|_\infty \leqslant \rho \max_k\left|f_{k+1}^j - f_k^j\right| \\ &= \rho\left\|\Delta\left(f^j\right)\right\|_\infty \leqslant \rho\left\|S_q^j\left(\Delta f^0\right)\right\|_\infty, \end{aligned}$$

where $\rho = \sum_{k\in\mathbb{Z}} |r_k|$. If (34) holds then

$$\sup_{t\in\mathbb{R}}\left| f^{j+1}(t) - f^{j}(t)\right| \leqslant \rho\mu^{[j/L]} \max_{0\leqslant n\leqslant L}\left\| \left(\Delta f^0\right)^n \right\|_\infty \leqslant C\eta^j, \quad \eta := \mu^{1/L} < 1. \tag{40}$$

Equation (40) implies that the sequence of the second-order splines $\{f^j(t)\}$ converges uniformly to a continuous function $f^\infty(t)$. □

Proposition 3.4 ([8]). Let S_a be a subdivision scheme of class P and in addition the symbol factorizes as follows:

$$a(z) = \frac{(1+z)^m}{2^m} b(z).$$

If there exists the continuous BLF $\phi_b(t)$ of the subdivision scheme S_b, whose symbol is $b(z)$, then there exists the BLF $\phi_a(t)$ of the subdivision scheme S_a. The function $\phi_a(t)$ has m continuous derivatives

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\phi_a(t) = \sum_{n=0}^{m} (-1)^{m-n} \binom{n}{m} \phi_b(t+n)$$

as $t \in \mathbb{R}$.

3.3. Exponential decay of the BLF

In theorem 3.1 we established an exponential decay of BLF of the ISS derived from the polynomial interpolatory splines of even order. We prove that all the convergent schemes of class P possess such a property.

Theorem 3.2. Let S_a be a subdivision scheme of class P and S_q be the scheme, whose symbol is $q(z)$ and the mask is $\{q_k\}_{k\in\mathbb{Z}}$. If for some $L \in \mathbb{N}$ the inequality (34) holds then there exists a continuous BLF $\phi_a(t)$ of the scheme S_a, which decays exponentially as $|t| \to \infty$. Namely, if (30) holds then for any $\varepsilon > 0$ a constant $\Phi_\varepsilon > 0$ exists such that the following inequality

$$\left|\phi_a(t)\right| \leqslant \Phi_\varepsilon\, \mathrm{e}^{-(g-\varepsilon)|t|}$$

is true.

Proof. To simplify the calculations, we assume that in (34) $L = 1$ (the case $L > 1$ is treated similarly). Thus

$$\|S_q\| = \max\left\{ \sum_k \left|q_k^e\right|, \sum_k \left|q_k^o\right| \right\} = \mu < 1, \tag{41}$$

where $q_k^e := q_{2k}$ and $q_k^o := q_{2k+1}$.

We apply the subdivision to the initial data $f^0 = \{\delta(k)\}_{k\in\mathbb{Z}}$. Due to lemma 3.1, each second-order spline, which interpolates the refined data $\{f_k^j\}_{k\in\mathbb{Z}}^{j\in\mathbb{Z}^+}$, decays exponentially as $|t| \to \infty$. Let us fix an index $J \in \mathbb{Z}^+$. Equation (32) implies that if $t = 2^{-J}(k+\tau), 0 \leqslant \tau \leqslant 1$, then

$$\left|f^J(t)\right| \leqslant \max\left\{\left|f_k^J\right|, \left|f_{k+1}^J\right|\right\} \leqslant A_J \mathrm{e}^{-g|k|2^{-J+1}} \leqslant \alpha_J \mathrm{e}^{-gt}, \quad \alpha_J := A_J \mathrm{e}^{g2^{-J+1}}.$$

We prove that the difference $d^J(t) = \phi_a(t) - f^J(t)$ decays exponentially as $t \to \infty$. For this purpose we analyze the local behavior of the function $D^{j+1}(t)$ that was defined in (35). Due to (36) it reduces to evaluation of the sequence y^{j+1}:

$$y_l^{j+1} := \begin{cases} f_{2k+1}^{j+1} - \dfrac{f_k^j + f_{k+1}^j}{2}, & l = 2k+1; \\ f_{2k}^{j+1} - f_k^j, & l = 2k, \end{cases} \qquad k \in \mathbb{Z}.$$

Equation (39) implies that

$$y^{j+1}(z) = r(z)h^j\left(z^2\right), \qquad h^j(z) = \left(S_q^j\left(\Delta f^0\right)\right)(z). \tag{42}$$

Denote

$$h^j := \left\{h_k^j\right\}_{k\in\mathbb{Z}} = S_q^j h^0, \qquad h_k^0 = \delta(k+1) - \delta(k) \quad \text{and} \quad H_J := \max_k \left|h_k^J\right|.$$

The rest of the proof is split into four major steps;

1. Analysis of the sequence h^j. Due to lemma 3.1 and inequality (41) we have

$$\left|h_k^J\right| \leqslant B_J \mathrm{e}^{-g|k|2^{-J+1}}, \qquad H_J \leqslant \left\|S_q^J\right\| \left\|h^0\right\|_\infty \leqslant \mu^J, \tag{43}$$

where B_J is a positive constant. Let $k \in \mathbb{Z}_+$. Then,

$$h_k^{J+1} = \sum_{l=-\infty}^{\infty} q_{k-2l} h_l^J \iff h_{2k}^{J+1} = \sum_{l=-\infty}^{\infty} q_l^e h_{k-l}^J, \quad h_{2k+1}^{J+1} = \sum_{l=-\infty}^{\infty} q_l^o h_{k-l}^J. \tag{44}$$

We split the even subsequence into two sums:

$$h_{2k}^{J+1} = \chi_1(s) + \chi_2(s), \quad \chi_1(s) := \sum_{l=-s}^{s} q_l^e h_{k-l}^J, \ \chi_2(s) := \sum_{l=-\infty}^{-s-1} q_l^e h_{k-l}^J + \sum_{l=s+1}^{\infty} q_l^e h_{k-l}^J.$$

It follows from (43) and (33) that

$$\left|\chi_1(s)\right| \leqslant B_J \mathrm{e}^{-g(k-s)2^{-J+1}} \sum_{l=-\infty}^{\infty} \left|q_l^e\right| \leqslant B_J \mu \mathrm{e}^{-g(k-s)\,2^{-J+1}},$$

$$\left|\chi_2(s)\right| \leqslant 2H_J \sum_{l=s+1}^{\infty} \left|q_l^e\right| \leqslant 2\mu^J Q \frac{\mathrm{e}^{-2gs}}{1-\mathrm{e}^{-g}}.$$

Let $s = k2^{-J}$. Then we have

$$\big|\chi_2(s)\big| \leqslant \eta_J \mu \mathrm{e}^{-g|k|2^{-J+1}}, \quad \text{where } \eta_J := \frac{2\mu^{J-1}Q}{1-\mathrm{e}^{-g}},$$
$$\big|\chi_1(s)\big| \leqslant B_J \mu \mathrm{e}^{-g(1-2^{-J})k2^{-J+1}} = B_J \mu \mathrm{e}^{gk2^{-2J+1}} \mathrm{e}^{-gk2^{-J+1}}.$$

Combining the estimates, we obtain

$$\big|h_{2k}^{J+1}\big| \leqslant \mu\big(B_J \mathrm{e}^{gk2^{-2J+1}} + \eta_J\big)\, \mathrm{e}^{-gk2^{-J+1}}.$$

The same estimate is true for the odd subsequence. Finally, we have

$$\big|h_k^{J+1}\big| \leqslant \mu B_J \beta_J\, \mathrm{e}^{-gk2^{-J}}, \quad \text{where } \beta_J := \big(\mathrm{e}^{gk2^{-2J}} + \eta_J\big).$$

Similarly, we derive the inequality

$$\big|h_k^{J+2}\big| \leqslant \mu^2\, B_J \beta_J \beta_{J+1}\, \mathrm{e}^{-gk2^{-J-1}},$$

and after j iterations we get

$$\big|h_k^{J+j}\big| \leqslant \mu^j B_J\, \mathrm{e}^{-gk2^{-J-j+1}} \prod_{l=0}^{j-1} \beta_{J+l}.$$

2. Evaluation of the sequence y^{j+1}. As it follows from equation (42), the odd terms of the sequence y^{J+j+1} are

$$y_{2k+1}^{J+j+1} = \sum_{l\in\mathbb{Z}} r_{2l+1}\, h_{k-l}^{J+j}. \tag{45}$$

Recall that the filter $r = \{r_l\}_{l\in\mathbb{Z}}$ satisfies the inequality (38). Thus, it is obvious that equation (45) is similar to (44) and by similar means we obtain the estimate

$$\big|y_{2k+1}^{J+j+1}\big| \leqslant \rho\mu^j B_J\, \mathrm{e}^{-gk2^{-J-j+1}} \epsilon_j \prod_{l=0}^{j-1} \beta_{J+l}, \tag{46}$$

where

$$\rho = \sum_{k\in\mathbb{Z}} |r_k|, \qquad \epsilon_j := \left(\mathrm{e}^{gk2^{-2(J+j)+1}} + \frac{2\mu^{J+j-1}R}{1-\mathrm{e}^{-g}}\right) \leqslant C\mathrm{e}^{gk2^{-2(J+j)+1}}$$

and R is some positive constant. The even terms are subject to the same inequality

$$\big|y_{2k}^{J+j+1}\big| \leqslant \rho\mu^j B_J \mathrm{e}^{-gk2^{-J-j+1}}\, \epsilon_j \prod_{l=0}^{j-1} \beta_{J+l}. \tag{47}$$

3. Estimation of the difference $D^{J+j}(t) = f^{J+j}(t) - f^{J+j-1}(t)$ for $t = 2^{-J}(k+\tau), 0 \leqslant \tau \leqslant 1$. Denote $Y_k^J := \max\{|y_{2k+1}^{J+1}|, |y_{2k}^{J+1}|\}$.

$$\left|D^{J+1}(t)\right| \leqslant Y_k^J \leqslant \rho B_J \epsilon_1 \mathrm{e}^{-g|k|2^{-J}} \leqslant C\rho B_J \mathrm{e}^{gk2^{-2J}} \mathrm{e}^{-g|k|2^{-J}}$$
$$\leqslant C\rho B_J \mathrm{e}^{g2^{-J}} \mathrm{e}^{gt2^{-J}} \mathrm{e}^{-gt}. \tag{48}$$

At the half-interval $t = 2^{-J-1}(2k + \tau_1)$, $0 \leqslant \tau_1 \leqslant 1$, we have

$$\left|D^{J+2}(t)\right| \leqslant Y_{2k}^{J+2} \leqslant \mu\rho B_J \beta_J \epsilon_2 \mathrm{e}^{-g2k2^{-J-1}} \leqslant C\mu\rho B_J \mathrm{e}^{g2k2^{-2(J+1)}} \mathrm{e}^{-g2k2^{-(J+1)}}$$
$$\leqslant C\mu\rho B_J(1+\eta_J)\mathrm{e}^{2g(t+1)2^{-J-1}} \mathrm{e}^{-gt}. \tag{49}$$

Employing Y_{2k+1}^{J+2} instead of Y_{2k}^{J+2} we obtain a similar estimate for the second half-interval $t = 2^{-J-1}(2k + 1 + \tau_2)$, $0 \leqslant \tau_2 \leqslant 1$. So, inequality (49) is true on the whole interval $[k/2^J, (k+1)/2^J]$. Denote the converging infinite product by

$$N_J(\mu) := \prod_{j=0}^{\infty}(1+\eta_{J+j}) = \prod_{j=0}^{\infty}\left(1 + \frac{2\mu^{J-1+j}Q}{1-\mathrm{e}^{-g}}\right) > 1$$

and note that

$$\prod_{j=0}^{\infty} \mathrm{e}^{g(t+1)2^{-J-j}} = \exp\left(g(t+1)\sum_{j=0}^{\infty} 2^{-J-j}\right) = \mathrm{e}^{g(t+1)2^{-J+1}}.$$

Then the estimates (48) and (49) can be combined as follows

$$\left|D^{J+1+j}(t)\right| \leqslant C\mu^j N_J \rho B_J \mathrm{e}^{g(t+1)3\cdot 2^{-J}} \mathrm{e}^{-gt}, \quad j = 0, 1. \tag{50}$$

One can observe that due to (46) and (47), inequality (50) is true for any $j \in \mathbb{N}$.

4. Completion of the proof. Inequality (50) enables us to evaluate the difference $d^J(t) = \phi_a(t) - f^J(t)$:

$$\left|d^J(t)\right| \leqslant \sum_{j=0}^{\infty}\left|D^{J+1+j}(t)\right|$$
$$\leqslant N_J \rho B_J \mathrm{e}^{g(t+1)3\cdot 2^{-J}} \mathrm{e}^{-gt} \sum_{j=0}^{\infty} C\mu^j = \frac{C_J}{1-\mu}\mathrm{e}^{-g(1-2^{-J})t}. \tag{51}$$

Hence we derive that the BLF

$$\left|\phi_a(t)\right| \leqslant \left|f^J(t)\right| + \left|d^J(t)\right| \leqslant B_J \mathrm{e}^{-gt} + \frac{C_J}{1-\mu}\mathrm{e}^{-g(1-2^{-J})t} \leqslant \Phi_J \mathrm{e}^{-g(1-2^{-J})t}.$$

For any $\varepsilon > 0$ we can choose $J(\varepsilon) \in \mathbb{N}$ such that $g2^{-J} < \varepsilon$. Then we have

$$\left|\phi_a(t)\right| \leqslant \Phi_\varepsilon \mathrm{e}^{-(g-\varepsilon)t}.$$ □

3.4. Convergence of subdivision schemes

Now we are in a position to discuss the convergence of subdivision schemes with rational symbols. As it was mentioned above, the initial data sequences are of power growth (see (28)).

Definition 3.2. Assume that the initial data $f^0 = \{f_k^0\}_{k\in\mathbb{Z}}$ is of power growth. Let $f^j(t)$ be the sequence of polygonal lines that interpolates the data generated by S_a at the corresponding refinement level: $\{f^j(2^{-j}k) = f_k^j = (S_a^j f^0)_k\}_{k\in\mathbb{Z}}$. If $\{f^j(t)\}$ converges uniformly at any finite interval to a continuous function $f^\infty(t)$ as $j \to \infty$, then we say that the subdivision scheme S_a converges on the initial data f^0 and $f^\infty(t)$ is called its limit function.

Theorem 3.3. Let S_a be a subdivision scheme of class P and S_q be the scheme, whose symbol is $q(z)$ and the mask is $\{q_k\}_{k\in\mathbb{Z}}$. If for some $L \in \mathbb{N}$ inequality (34) holds then the scheme S_a converges on any initial data $f^0 = \{f_k^0\}_{k\in\mathbb{Z}}$ of power growth. The limit function $f^\infty(t)$ is of power growth and can be represented by the sum

$$f^\infty(t) = \sum_{l\in\mathbb{Z}} f_l^0 \phi_a(t-l), \tag{52}$$

where $\phi_a(t)$ is the BLF of the scheme S_a.

Proof. We denote by $\{\phi_k^0 = \delta(k)\}_{k\in\mathbb{Z}}$ the delta sequence and by $\phi^j := S_a^j \phi^0$. The function $\phi^j(t)$ is the second-order spline, which interpolates the data ϕ^j. The set of splines $\phi^j(t)$ converges uniformly to the continuous BLF $\phi_a(t)$. Equation (40) implies that

$$\sup_{t\in\mathbb{R}}\left|\phi^{j+1}(t) - \phi^j(t)\right| \leqslant C\mu^j \Rightarrow \sup_{t\in\mathbb{R}}\left|d^j(t)\right| \leqslant C_1\mu^j, \quad 0<\mu<1, \tag{53}$$

where $d^j(t) := \phi_a(t) - \phi^j(t)$. On the other hand, since both $\phi_a(t)$ and $\phi^j(t)$ decay exponentially, we have

$$\left|d^j(t)\right| \leqslant C_2 \mathrm{e}^{-\gamma|t|} \quad \forall j \in \mathbb{Z}^+, \ 0<\gamma<1. \tag{54}$$

We can represent the initial data sequence as

$$f_k^0 = \sum_{l\in\mathbb{Z}} f_l^0 \phi_{k-l}^0.$$

Hence, the spline, which interpolates the refined data f^j, is

$$f^j(t) = \sum_{l\in\mathbb{Z}} f_l^0 \phi^j(t-l). \tag{55}$$

The series in (55) converges for any t due to the exponential decay of the spline $\phi^j(t)$ as $t \to \infty$. The series

$$F(t) := \sum_{l\in\mathbb{Z}} f_l^0 \phi_a(t-l)$$

also converges due to the exponential decay of the BLF $\phi_a(t)$ and its sum $F(t)$ is of power growth. We evaluate the difference $X^j(t) := F(t) - f^j(t)$ as $|t| \leqslant T$. We have

$$X^j(t) = \sum_{l=-\infty}^{\infty} f_l^0\, d^j(t-l) = Y_s^j(t) + Z_s^j(t),$$

$$\text{where } Y_s^j(t) := \sum_{l=-s}^{s} f_l^0\, d^j(t-l),\ \ Z_s^j(t) := \sum_{l=-\infty}^{-s-1} f_l^0 d^j(t-l) + \sum_{l=s+1}^{\infty} f_l^0\, d^j(t-l).$$

Given a value $\varepsilon > 0$ we can, using inequality (54), choose the numbers $s = s(T)$ such that $|Z_s^j(t)| \leqslant \varepsilon/2,\ \forall t \in [-T, T]$. Then, using (53), we choose $J = J(s)$ such that $|Y_s^j(t)| \leqslant \varepsilon/2,\ \forall j \geqslant J,\ \forall t \in [-T, T]$. Thus $|X^j(t)| \leqslant \varepsilon,\ \forall j \geqslant J\ \forall t \in [-T, T]$. This means that the sequence of second-order splines $f^j(t)$ converges uniformly on $[-T, T]$ to the continuous function $F(t) = f^\infty(t)$. □

Remark. Equation (52) implies that, provided the initial data belongs to l_1, the limit function is absolutely integrable.

Similarly, proposition 3.4 can be extended to the case when the initial data is of power growth and the following representation

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} f^\infty(t) = \sum_{l\in\mathbb{Z}} f_l^0 \phi_a(t-l)^{(m)},$$

holds.

3.5. Evaluation of coefficients of subdivision masks via the discrete Fourier transform

The above propositions yield a practical algorithm for establishing the convergence of a subdivision scheme and analyzing its regularity. The key operation is evaluation of sums of coefficients of type (34) of the coefficients of masks. These sums can be calculated directly for subdivision schemes with finite masks. But for infinite masks different methods of evaluation of the coefficients are required.

Again we consider the case when the number L in equation (34) is equal to 1. The cases with $L > 1$ are similarly treated. We assume that $N = 2^p$, $p \in \mathbb{N}$, and $\sum_k^p$ stands for $\sum_{k=-N/2}^{N/2-1}$. The discrete Fourier transform (DFT) of an array $x^p = \{x_k^p\}_{k=-N/2}^{N/2-1}$ and its inverse (IDFT) are

$$\hat{x}_n^p = \sum_k^p \mathrm{e}^{-2\pi \mathrm{i} kn/N} x_k^p \quad \text{and} \quad x_k^p = \frac{1}{N}\sum_n^p \mathrm{e}^{2\pi \mathrm{i} kn/N} \hat{x}_n^p.$$

As before, $y(z)$ denotes the z-transform of a sequence $\{y_k\} \in l_1$. We assume that $z = \mathrm{e}^{-\mathrm{i}\omega}$.

The coefficients of the masks that we deal with are evaluated as follows:

$$|a_k| \leqslant a\gamma^k \Rightarrow \sum_{k=N}^{\infty} |a_k| \leqslant B\gamma^N, \quad B = \frac{a}{1-\gamma}, \tag{56}$$

where $0 < \gamma < 1$ and a is some positive constant.

We need to evaluate the sums $S_e(a) = \sum_{k=-\infty}^{\infty} |a_{2k}|$, $S_o(a) = \sum_{k=-\infty}^{\infty} |a_{2k+1}|$. We denote

$$A(\omega) = a\bigl(\mathrm{e}^{-\mathrm{i}\omega}\bigr) = \frac{Q(\mathrm{e}^{-\mathrm{i}\omega})}{P(\mathrm{e}^{-\mathrm{i}\omega})} = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega k} a_k.$$

Let us calculate the function A at the discrete set of points

$$\hat{a}_n = A\left(\frac{2\pi n}{N}\right) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}kn/N} a_k = \sum_{r=-N/2}^{N/2-1} \mathrm{e}^{-2\pi\mathrm{i}rn/N} \varphi_r,$$

$$\varphi_r = \sum_{l=-\infty}^{\infty} a_{r+lN} = a_r + \psi_r, \ \psi_r = \sum_{l\in\mathbb{Z}/0} a_{r+lN}.$$

It follows from (56) that

$$|\psi_r| \leqslant 2B\alpha^N \Rightarrow |a_r| = |\varphi_r| + \alpha_r^N, \quad \left|\alpha_r^N\right| \leqslant 2B\gamma^N. \tag{57}$$

The samples φ_k are available via IDFT: $\varphi_k = \frac{1}{N}\sum_n^p \mathrm{e}^{2\pi\mathrm{i}kn/N} \hat{a}_n$. Using (57) we can evaluate the sums we are interested in as follows:

$$S_e(a) = \sum_{k=-N/4}^{N/4-1} |a_{2k}| + 2\sum_{k=N/4}^{\infty} |a_{2k}| = \sum_{r=-N/4}^{N/4-1} |\varphi_{2k}| + \rho_N,$$

$$\rho_N = \sum_{r=-N/4}^{N/4-1} \left|\alpha_N(2k)\right| + 2\sum_{k=N/4}^{\infty} |a_{2k}|, \quad |\rho_N| \leqslant B(N+2)\gamma^N.$$

Hence, it follows that, doubling N, we can approximate the infinite series $S_e(a)$ by the finite sum $\sigma_e^N(a) = \sum_{r=-N/4}^{N/4-1} |\varphi_{2k}|$, whose terms are available via DFT. An appropriate value of N can be found theoretically using estimations of the roots of the denominator $P(z)$. But practically, we can iterate calculations by gradually doubling N until the result of calculation $\sigma_e^{2N}(a)$ becomes identical to $\sigma_e^N(a)$ (up to machine precision). The same approach is valid for evaluation of the sum $S_o(a)$ and of the sums $\sum_k |q_{i-2^L k}^{[L]}|$ with any L.

3.6. Approximation order of ISS's

Definition 3.3. A convergent subdivision scheme S has an approximation order n if for any sufficiently smooth function F and the initial data $f^0 = \{f_k^0 = F(kh)\}_{k\in\mathbb{Z}}$

$$\left|\left(F - S^\infty f^0\right)(x)\right| \leqslant Ch^n, \quad x \in \mathbb{R},$$

where the constant C may depend on F, n, x, and S but not on h.

Proposition 3.5 ([9]). The approximation order of a convergent subdivision scheme, which is exact for polynomials of degree $n-1$ is n.

Theorem 3.4. The approximation order of the subdivision schemes derived from the polynomial interpolatory splines of orders $2r-1$ and $2r$ is $2r$. The approximation order of the subdivision schemes derived from the discrete interpolatory splines of order $2r$ is $2r$.

4. Examples of spline-based subdivision schemes

In this section we describe in details properties of three interpolatory subdivision schemes that are based on splines. We compare these properties with the properties of the Dubuc and Deslauriers ISS's.

4.1. Convergence and smoothness

Quadratic interpolatory spline. We label this scheme by PS3. The symbol of the scheme is

$$a_I^3(z) = 1 + zU_I^3\left(z^2\right) = \frac{(1+z)^4}{z^4+6z^2+1} = (1+z)q(z), \quad q(z) = \frac{(1+z)^3}{z^4+6z^2+1}.$$

To establish the convergence we have to prove that the scheme S_q with the rational symbol $q(z)$ and the infinite mask $\{q_k\}_{k\in\mathbb{Z}}$ is contractive. For this purpose we evaluate the norms $\|S_q^L\| = \max\{\sum_k |q_{i-2^L k}^{[L]}|\}$ using DFT as it is described in section 3.5. Let us begin with $L=1$. In this case

$$\hat{q}_n^{[1]} = q\left(\mathrm{e}^{-2\pi \mathrm{i}n/N}\right) = \frac{2\mathrm{e}^{\mathrm{i}\pi n/N}\cos^3 \mathrm{i}\pi n/N}{1+\cos^2 2\pi n/N}.$$

The sums $\sum_{k=-\infty}^{\infty} |q_{i-2k}^{[1]}| \simeq \sum_{r=-N/4}^{N/4-1} |\varphi_{2k+i}|$, $i = 0, 1$, provided N is sufficiently large. The values φ_k are calculated via IDFT: $\varphi_k = N^{-1}\sum_n^p \mathrm{e}^{2\pi \mathrm{i}kn/N}\,\hat{q}_n^{[1]}$. Direct calculation yields the estimate: $\|S_q^1\| \leqslant 0.7071$. Thus, the scheme converges.

To establish the differentiability of the limit function f^∞ of the scheme S_a we have to prove that the scheme S_{b^1} with the symbol $b^1(z) = (1+z)^{-1}\,a_I^3(z)$ converges. For this purpose we have to prove that the scheme S_{q^1} with the symbol $q^1(z) = 2(1+z)^{-2}\,a_I^3(z)$ is contractive. The norm of the operator S_{q^1} does not meet the requirement $\|S_{q^1}\| < 1$. But we succeed in proving that $\|S_{q^1}^2\| \leqslant 0.6667$. Hence the limit function $f^\infty \in C^1$.

But even a stronger assertion is true: the limit function $f^\infty \in C^2$. To establish it, we prove that the scheme S_{q^2} with the symbol $q^2(z) = 4(1+z)^{-3} a_I^3(z)$ is contractive. As in the previous case our calculations lead to the estimation: $\|S_{q^2}^2\| \leqslant 0.6667$, which proves the statement.

Interpolatory spline of fifth order (fourth degree): We label this scheme by PS5. The symbol of the scheme is

$$a_I^5(z) = 1 + zU_I^5(z^2) = \frac{(1+z)^6(z^2+10z+1)}{z^8+76(z^6+z^2)+230z^4+1} = (1+z)q(z),$$
$$\hat{q}_n = q\left(\mathrm{e}^{-2\pi \mathrm{i}n/N}\right) = \frac{8\cos^6 \pi n/N(5+\cos 2\pi n/N)}{5+18\cos^2 2\pi n/N + \cos^4 2\pi n/N}.$$

As in the previous case, we find that the scheme with the symbol $a_I^5(z)$ converges and, moreover, the limit function $f^\infty \in C^4$.

Discrete interpolatory spline of sixth order: We label this scheme by DS6. The symbol of the scheme is

$$a_d^6(z) = 1 + zU_d^6(z^2) = \frac{(1+z)^6}{2z(3z^4+10z^2+3)}.$$

This scheme also converges and the limit function $f^\infty \in C^4$.

4.2. *Implementation of subdivision s with rational symbols*

Although the masks of the presented spline subdivision schemes are infinite, the rational structure of their symbols enables to implement the refinement via the so-called recursive filtering, which is commonly used in signal processing. We illustrate the procedure on the example of the ISS PS3. Due to equation (24), in order to derive the refined data $\{f_{2k+1}^{j+1}\}_{k\in\mathbb{Z}}$ we have to perform the following filtering:

$$f_o^{j+1}(z) = U_I^3(z) f^j(z), \quad \text{where } U_I^3(z) = \frac{4(1+z^{-1})}{z^{-1}+6+z} = 4\alpha \frac{1+z^{-1}}{(1+\alpha z)(1+\alpha z^{-1})},$$

and $\alpha = 3 - 2\sqrt{2} \approx 0.172$. In time domain filtering is conducted as follows: $f_{2k}^{j+1} = f^j(k)$, $f_{2k+1}^{j+1} = s_k^j$, where the values s_k^j are derived from f^j by a cascade of elementary recursive filters:

$$x_k = 4\alpha\left(f_k^j + f_{k+1}^j\right), \qquad x_k^1 = x_k - \alpha x_{k-1}^1, \qquad s_k^j = x_k^1 - \alpha s_{k+1}^j.$$

The cost to compute a value f_{2k+1}^{j+1} is 3 multiplications (M) and 3 additions (A). For comparison, the 4-point Dubuc and Deslauriers ISS based on cubic polynomials, which we label by DD3, requires 2M + 3A operations, but the regularity of the limit function is inferior to the regularity of the limit function of the above scheme. The 6-point Dubuc and Deslauriers ISS based on quintic polynomials, which we label by DD5, produces the limit function of approximately the same regularity as the spline ISS PS3. It requires 3M + 5A operations. The scheme DS6, which is based on the discrete splines, of sixth

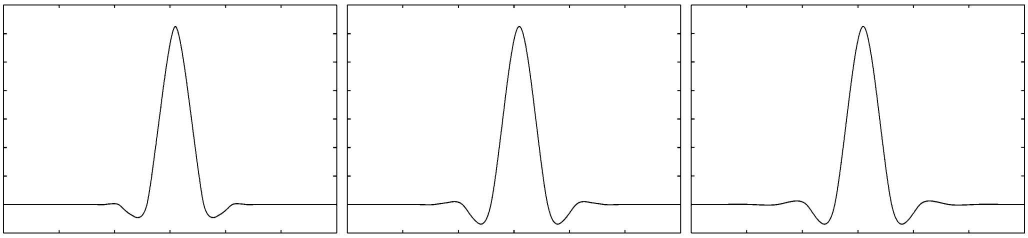

Figure 1. The basic limit functions: Left: 4-point ISS DD3. Center: 6-point ISS DD5. Right: quadratic spline ISS PS3.

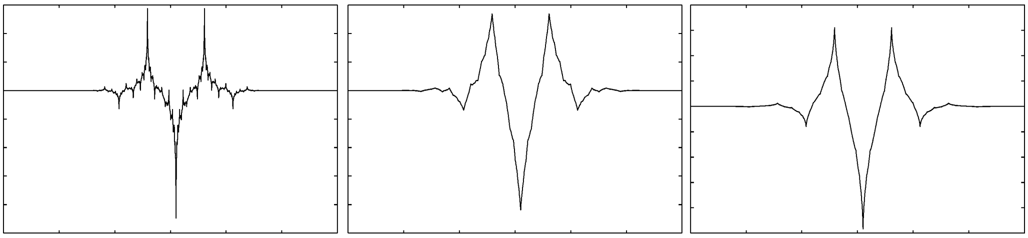

Figure 2. Second derivatives of the BLF's: Left: 4-point ISS DD3. Center: 6-point ISS DD5. Right: quadratic spline ISS PS3.

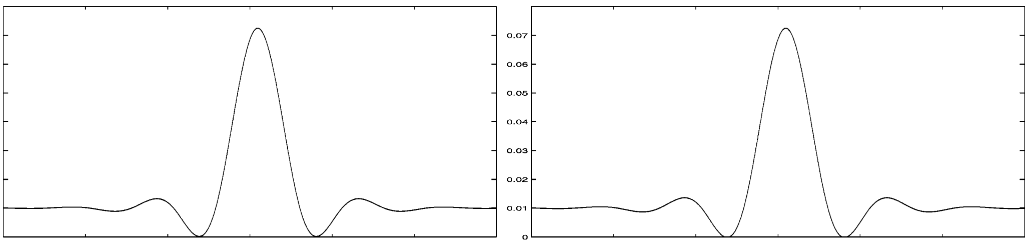

Figure 3. The basic limit functions of the sixth-order discrete splines ISS DS6 (left) and of the fifth-order splines ISS PS5 (right).

order requires 4M + 5A operations. However, it produces limit functions that belong to C^4. The scheme PS5 based on the polynomial splines of fifth order also produces limit functions belonging to C^4 but its computational cost – 6M + 7A operations – is higher than the cost of the implementation of DS6.

In figure 1 we display the basic limit functions of the Dubuc and Deslauriers 4-point ISS DD3, of the Dubuc and Deslauriers 6-point ISS DD5 and of the ISS PS3 based on quadratic splines (right picture). The second derivatives of the BLF's are displayed in figure 2.

It is well known that the second derivative of the BLF of the 4-point ISS DD3 does not exist. The BLF of the 6-point ISS DD5 belongs to C^α, $\alpha < 2.830$. The second derivative of the BLF of the quadratic spline scheme PS3 in figure 2 looks smoother than BLF of the DD5 ISS. Thus, we conjecture that the BLF of PS3 belongs to C^β, $\beta > \alpha$.

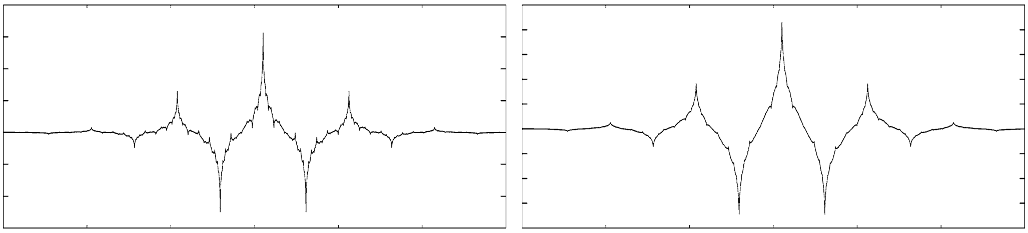

Figure 4. Fourth derivatives of BLF of the sixth-order discrete splines ISS DS6 (left) and of the fifth-order splines ISS PS5 (right).

Table 1
Properties of the ISS's. Left column contains the names of the ISS. The other five columns C^k, $k = 0, \dots, 4$, describe the smoothness of the ISS's. The column C^k comprises the norm of the operator S_q^L (see (34)) and the number of iterations L required to achieve the inequality $\|S_q^L\| = \mu < 1$. The last column is the number of operations required to derive f_{2k+1}^{j+1} from the array f^j. Left: the number of additions, right: the number of multiplications.

	C^0		C^1		C^2		C^3		C^4		Comp. cost	
ISS	$\|S_q^L\|$	L	$\|S_q^L\|$	L	$\|S_q^L\|$	L	$\|S_q^L\|$	L	$\|S_q^L\|$	L	Add.	Mult
DD3	0.625	1	0.75	2	–	–	–	–	–	–	3	2
DD5	0.6953	1	0.6584	2	0.7109	2	–	–	–	–	5	3
PS3	0.7071	1	0.6667	2	0.6667	2	–	–	–	–	3	3
DS6	0.8333	1	0.6	2	0.8	2	0.6830	4	0.9512	11	5	4
PS5	0.8532	1	0.5965	2	0.8070	2	0.9962	3	0.8902	5	7	6

In figure 3 we display the basic limit functions of the ISS DS6 that are based on discrete splines of sixth-order and of the ISS PS5 based on polynomial splines of fifth order. The fourth derivatives of the BLF's are displayed in figure 4. We observe that the fourth derivative of the BLF of the sixth-order discrete splines ISS is of near-fractal appearance. Nevertheless, it is proved that it is continuous.

Table 1 summarizes the properties of the presented interpolatory subdivision schemes PS3, PS5 and DS6. For comparison we cite also the properties of the Dubuc and Deslauriers ISS's DD3 and DD5.

From (40) we see that the convergence speed of a subdivision scheme to a continuous limit function is determined by the number of iterations L that are needed to achieve the inequality $\|S_q^L\| = \mu < 1$ and by the value of μ. The smaller are L and μ, the faster is the convergence. We conjecture that these two parameters determine the Hölder class of limit functions. Our examples provide some support to this conjecture. Namely, for the second derivative of the BLF of the scheme DD5, $L = 2$ and $\mu_1 = 0.7109$ and for the PS3, $L = 2$ and $\mu_2 = 0.6667$. The value $\mu_2 < \mu_1$ and the graph of the second derivative of the PS3 is smoother than the graph of DD5. For the fourth derivative of the BLF of the scheme DS6 $L = 11$ and for the PS5 $L = 5$. The graph of the fourth derivative of the PS5 is much smoother than the graph for the DS6.

5. Conclusions

A generic technique for the construction of diversity of interpolatory subdivision schemes on the base of polynomial and discrete splines is presented in the paper. Although the masks of the schemes are infinite, the refinement can be implemented in a fast way using recursive filtering. The devised schemes are competitive (regularity, speed of convergence, computational complexity) with the schemes that have finite masks, such as the popular Dubuc and Deslauriers schemes. We prove that the basic limit functions of schemes with rational symbols decay exponentially and establish conditions, which guaranty the convergence of these schemes on initial data of power growth. We find that due to the super-convergence property, the approximation order of the ISS based on a spline of even degree is higher than the approximation order of the spline. Moreover, the limit functions of the ISS are smoother than the spline itself. Actually, these limit functions form a new class of functions, which deserves a thorough investigation. On the other hand, the basic limit function of the scheme derived from a spline of odd degree (even order) coincides with the fundamental spline.

The approach to construction of subdivision schemes that is developed in the paper for the equally spaced initial data can be extended to a data that is defined on an irregular grid. An actual problem is to evaluate the Hölder exponents of limit functions of the designed schemes.

Appendix A

Proof of lemma 2.1. Due to (11) and (12) we have

$$1 - R^{2r-1}\left(\cos\frac{\omega}{2}\right) = \frac{P^{2r-1}(\cos\omega/2) - Q^{2r-1}(\cos\omega/2)}{P^{2r-1}(\cos\omega/2)} = \frac{-2(\sin\omega/2)^{2r-1}T_{2r-1}(\omega)}{P^{2r-1}(\cos\omega/2)},$$

$$T_{2r-1}(\omega) := \sum_{l=-\infty}^{\infty}\frac{1}{(\pi(2l+1)+\omega/2)^{2r-1}}.$$

The function $T_{2r-1}(\omega)$ is infinitely differentiable at the point $\omega = 0$ and in its vicinity and the Taylor expansion holds

$$T_{2r-1}(\omega) = \sum_{n=0}^{\infty}\frac{T_{2r-1}^{(n)}(0)}{n!}\omega^n.$$

We can write

$$T_{2r-1}^{(n)}(0) = (-1)^n\sum_{l=-\infty}^{\infty}\frac{2^{2r-1}(2r-1)\cdots(2r+n-2)}{(2\pi(2l+1))^{2r-1+n}}.$$

Hence we see that

$$T_{2r-1}(0) = \sum_{l=-\infty}^{\infty} \frac{1}{(\pi(2l+1))^{2r-1}} = 0.$$

Similarly $T_{2r-1}^{(2k)}(0) = 0 \ \forall k \in \mathbb{N}$. This is not the case for the derivatives of odd orders:

$$T_{2r-1}^{(2k+1)}(0) = -\frac{(2r-1)\cdots(2(r+k)-1)}{2^{2k}(\pi)^{2(r+k)}} \sum_{l=0}^{\infty} \frac{1}{(2l+1))^{2(r+k)}}.$$

Using a known formula [1]

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1))^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}|b_{2n}|,$$

we get

$$\begin{aligned} T_{2r-1}^{(2k+1)}(0) &= -\frac{(2r-1)\cdots(2(r+k)-1)}{2^{2k}} \frac{(2^{2(r+k)}-1)}{2(2(r+k))!}|b_{2(r+k)}| \\ &= -\frac{(2^{2(r+k)}-1)}{2^{2(k+1)}(r+k)(2r-2)!}|b_{2(r+k)}|. \end{aligned}$$

Finally, in the neighborhood of $\omega = 0$ we have

$$\begin{aligned} T_{2r-1}(\omega) &= \sum_{k=0}^{\infty} \frac{T_{2r-1}^{(2k+1)}(0)}{(2k+1)!}\omega^{(2k+1)} \\ &= -\sum_{k=0}^{\infty} \frac{(2^{2(r+k)}-1)}{2^{2(k+1)}(r+k)(2r-2)!(2k+1)!}|b_{2(r+k)}|\omega^{(2k+1)} \\ &= -\frac{(4^r-1)}{4r(2r-2)!}|b_{2r}|\omega + \mathrm{O}(\omega^3) \\ &= \sin\frac{\omega}{2}\left[-\frac{(4^r-1)}{2r(2r-2)!}|b_{2r}| + \mathrm{O}\left(\sin^2\frac{\omega}{2}\right)\right]. \end{aligned} \tag{58}$$

Hence (15) follows. □

Appendix B

Table 2
Values of the sequences v^p and w^p .

	−4	−3	−2	−1	0	1	2	3	4
v^2	0	0	0	0	1	0	0	0	0
$v^3 \times 8$	0	0	0	1	6	1	0	0	0
$v^4 \times 6$	0	0	0	1	4	1	0	0	0
$v^5 \times 384$	0	0	1	76	230	76	1	0	0
$v^6 \times 120$	0	0	1	76	230	76	1	0	0
$v^7 \times 46080$	0	1	722	10543	23548	10543	722	1	0
$w^3 \times 2$	0	0	0	1	1	0	0	0	0
$w^4 \times 48$	0	0	1	23	23	1	0	0	0
$w^5 \times 24$	0	0	1	11	11	1	0	0	0
$w^6 \times 3840$	0	1	237	1682	1682	237	1	0	0
$w^7 \times 720$	0	1	57	302	302	57	1	0	0

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