Computation of interpolatory splines via triadic subdivision

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Abstract

We present an algorithm for computation of interpolatory splines of arbitrary order at triadic rational points. The algorithm is based on triadic subdivision of splines. Explicit expressions for the subdivision symbols are established. These are rational functions. The computations are implemented by recursive filtering.

1 Introduction

Denote by $\mathfrak{S}^p$ the space of polynomial splines $S^p(x)$ of order p defined on the uniform grid $\mathbf{g}^0 \stackrel{\Delta}{=} \{k\}, \ k \in \mathbb{Z}$, such that the arrays $\{S^p(k)\}, \ k \in \mathbb{Z}$, belong to l_1. We propose to compute values of the splines via a triadic subdivision. The insertion rule for a spline $S(x) \in \mathfrak{S}^p$ is

Triadic Insertion Rule: Let $\mathbf{f}^0 \stackrel{\Delta}{=} \left\{f_k^0 = S^p(k)\right\}, \ k \in \mathbb{Z}$. For $j = 0, 1, \ldots$, we construct on the grid $\mathbf{g}^j \stackrel{\Delta}{=} \left\{k3^{-j}\right\}, \ k \in \mathbb{Z}$, a spline $S_j^p(x)$, which interpolates the sequence $\mathbf{f}^j \stackrel{\Delta}{=} \left\{f_k^j\right\}$ on the grid $\mathbf{g}^j$. Then, $f_k^{j+1} = S_j^p\left(k3^{-j+1}\right), \quad k \in \mathbb{Z}$.

Note that the value of a spline at any point can be expressed as a linear combination of its values at grid points. In other words, any value f_k^{j+1} can be derived by some filtering of the sequence $\mathbf{f}^j$. We present explicit expressions for these filters for splines of arbitrary order. Their transfer functions are rational functions. Computations are implemented by recursive filtering. Moreover, we prove that for any $j \in \mathbb{N}$, $f_k^j = S^0\left(k3^{-j}\right), \quad k \in \mathbb{Z}$. Thus, we obtain a fast algorithm that computes the values of a spline $S_0^p(x)$ from the space $\mathfrak{S}^p$, which interpolates the sequence $\mathbf{f}^0$ on the grid $\mathbf{g}^0$, at the triadic rational points $\left\{k3^{-j}\right\}, \ k \in \mathbb{Z}$.

2 Spline filters

2.1 B-splines

The centered B-spline of first order is the characteristic function of the interval $[-1/2, 1/2]$. The centered B-spline of order p can be expressed as the convolution $M^p(x) = M^{p-1}(x) * M^1(x), \quad p \geq 2$. Note that the B-spline of order p is supported on the interval $(-p/2, p/2)$. It is positive within its support and symmetric about zero. The B-spline M^p consists of pieces of polynomials of degree $p-1$ that are linked to each other at the nodes, such that $M^p \in C^{p-2}$. Nodes of B-splines of even order are located at points $\{k\}$ and of odd order at points $\{k+1/2\},\ k \in \mathbb{Z}$.

The Fourier transform of the B-spline of order p is

$$\widehat{M^p}(\omega) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} e^{-i\omega x} M^p(x)\,dx = \left(\frac{\sin \omega/2}{\omega/2}\right)^p. \tag{2.1}$$

The time domain representation of the B-spline is

$$M^p(x) = \frac{1}{(p-1)!}\sum_{k=0}^{p-1}(-1)^k \binom{p}{k}\left(x+\frac{p}{2}-k\right)_+^{p-1}, \quad x_+ \stackrel{\Delta}{=} (x+|x|)/2.$$

Shifts of B-splines form a basis in the space $\mathfrak{S}^p$. Namely, any spline $S^p(x) \in \mathfrak{S}^p$ has the following representation:

$$S^p(x) = \sum_{k\in\mathbb{Z}} q_k\, M^p(x-k).$$

Equation (2.1) implies that for any $p \ge 2$

$$M^p(x-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-k)}\left(\frac{\sin\omega/2}{\omega/2}\right)^p d\omega = \sum_{l\in\mathbb{Z}} e^{2\pi i l x}\int_0^1 e^{2\pi i\omega(x-k)}\frac{(\sin\pi\omega)^p(-1)^{lp}}{\pi(l+\omega))^p}\,d\omega$$

$$= \int_0^1 e^{-2\pi i\omega k} m_x^p(\omega)\,d\omega, \quad \text{where } m_x^p(\omega) \stackrel{\Delta}{=} e^{2\pi i\omega x}(\sin\pi\omega)^p \sum_{l\in\mathbb{Z}} e^{2\pi i l x}\frac{(-1)^{lp}}{(\pi(l+\omega))^p}. \tag{2.2}$$

The relation (2.2) means that $M^p(x-k)$ is a Fourier coefficient of the 1-periodic function $m_x^p(\omega)$ and this function can be represented as the sum: $m_x^p(\omega) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\omega}\, M^p(x-k)$.

The functions $m_x^p(\omega)$ are 1-periodic with respect to the variable ω and are splines from $\mathfrak{S}^p$ with respect to x. Their shifts along the grid are

$$m_{x+l}^p(\omega) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\omega}\, M^p(x-(k-l)) = e^{2\pi i l\omega}\sum_{k\in\mathbb{Z}} e^{2\pi i k\omega}\, M^p(x-k) = e^{2\pi i l\omega} m_x^p(\omega).$$

Following Schoenberg [4], we call the splines $m_x^p(\omega)$ the exponential splines.

2.2 Interpolatory splines

Equation (2.2) implies that a spline $S^p(x) \in \mathfrak{S}^p$ can be represented via the exponential splines:

$$S^p(x) = \sum_{k\in\mathbb{Z}} q_k\, M^p(x-k) = \sum_{k\in\mathbb{Z}} q_k \int_0^1 e^{-2\pi i\omega k} m_x^p(\omega)\,d\omega = \int_0^1 \hat{q}(\omega) m_x^p(\omega)\,d\omega, \tag{2.3}$$

where $\hat{q}(\omega) \stackrel{\Delta}{=} \sum_{k\in\mathbb{Z}} e^{-2\pi i\omega k}\, q_k$ is the discrete Fourier transform (DFT) of the coefficient sequence $\mathbf{q}^0 \stackrel{\Delta}{=} \{q_k\},\ k \in \mathbb{Z}$. Integral representation (2.3) was established in [5].

Denote

$$u^p(\omega) \stackrel{\Delta}{=} m_0^p(\omega) = \sum_{k\in\mathbb{Z}} e^{-2\pi i k\omega}\, M^p(k). \tag{2.4}$$

Due to the compact support and symmetry of the B-splines, the functions $u^p(\omega)$ are cosine polynomials. They were extensively studied in [1]. In particular, [1] established that $u^p(\omega) > 0$ for all $\omega \in \mathbb{R}$ and $p \in \mathbb{N}$. In addition, recurrent relations for computation of the polynomials of any order were presented in [1]. However, $u^p(\omega)$ can be easily calculated directly from Eq. (2.4). Equation (2.2) provides the following representation of the functions $u^p(\omega)$:

$$u^p(\omega) = (\sin \pi\omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l+\omega))^p}.$$

The values of the spline $S^p(x)$ at the grid points and their DFT are

$$S^p(n) = \int_0^1 \hat{q}(\omega) m_n^p(\omega)\, d\omega = \int_0^1 e^{2\pi i \omega n} \hat{q}(\omega) u^p(\omega)\, d\omega \Longleftrightarrow \widehat{S^p}(\omega) = \hat{q}(\omega) u^p(\omega).$$

Assume that the spline $S^p(x)$ interpolates a sequence $\mathbf{f}^0$ at the grid points

$$S^p(n) = f_n^0 \Longleftrightarrow \widehat{f^0}(\omega) = \widehat{S^p}(\omega) \Longleftrightarrow \hat{q}(\omega) = \frac{\widehat{f^0}(\omega)}{u^p(\omega)}. \tag{2.5}$$

According to Triadic Insertion Rule, which was formulated in Section 1, $f_n^1 = S^p(n/3)$. Then,

$$\widehat{f^1}(\omega) = \sum_{n \in \mathbb{Z}} e^{-2\pi i \omega n} S^p(n/3) = e^{2\pi i \omega} \sigma_{-1}(3\omega) + \sigma_0(3\omega) + e^{-2\pi i \omega} \sigma_1(3\omega), \tag{2.6}$$

$$\sigma_0(\omega) \stackrel{\Delta}{=} \sum_{n \in \mathbb{Z}} e^{-2\pi i \omega n} S^p(n) = \hat{q}(\omega) u^p(\omega) = \widehat{f^0}(\omega), \quad \sigma_{\pm 1}(\omega) \stackrel{\Delta}{=} \sum_{n \in \mathbb{Z}} e^{-2\pi i \omega n} S^p\left(n \pm \frac{1}{3}\right).$$

Due to Eq. (2.2),

$$S^p\left(n \pm \frac{1}{3}\right) = \int_0^1 \hat{q}(\omega) m_{n \pm 1/3}^p(\omega)\, d\omega = \int_0^1 e^{2\pi i \omega n} \hat{q}(\omega) v_{-1}^p(\omega)\, d\omega,$$

$$v_{\pm 1}^p(\omega) \stackrel{\Delta}{=} m_{\pm 1/3}^p(\omega) = e^{\pm 2\pi i \omega/3} (\sin \pi\omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp} e^{\pm 2\pi i l/3}}{(\pi(l+\omega))^p}. \tag{2.7}$$

Hence, using Eq. (2.5), we obtain

$$\sigma_{\pm 1}(\omega) = \frac{v_{\pm 1}^p(\omega)}{u^p(\omega)} \widehat{f^0}(\omega). \tag{2.8}$$

3 Triadic subdivision

3.1 Structure of filters

Substituting Eq. (2.8) into Eq. (2.6), we obtain

$$\widehat{f^1}(\omega) = \widehat{T^p}(\omega) \widehat{f^0}(3\omega), \quad \widehat{T^p}(\omega) \stackrel{\Delta}{=} e^{2\pi i \omega} \frac{v_{-1}^p(3\omega)}{u^p(3\omega)} + 1 + e^{-2\pi i \omega} \frac{v_1^p(3\omega)}{u^p(3\omega)}. \tag{3.9}$$

Similarly, $\widehat{f^{j+1}}(\omega) = \widehat{T^p}(\omega) \widehat{f^j}(3\omega),\ j \in \mathbb{N}$. Thus, in order to produce the array $\mathbf{f}^{j+1}$, we need to filter the upsampled array $\mathbf{f}^j$ with the filter $\mathbf{T}^p$, whose DFT (the frequency response) $\widehat{T^p}(\omega)$ is given in Eq. (3.9). We establish now the structure of the filter $\mathbf{T}^p$.

Theorem 3.1 *The DFT of the filter* $\mathbf{T^p}$*, which originates from the spline of order* p*, is*

$$\widehat{T^p}(\omega) = \frac{(1+2\cos\pi\omega)^p \ u^p(\omega)}{3^{p-1}u^p(3\omega)}. \tag{3.10}$$

Proof: We have from Eqs. (2.7) and (3.9) $\widehat{T^p}(\omega) = G^p(\omega)/u^p(3\omega)$, where

$$G^p(\omega) = e^{2\pi i\omega}v^p_{-1}(3\omega) + u^p(3\omega) + e^{-2\pi i\omega}v^p_{-1}(3\omega) = (\sin 3\pi\omega)^p \sum_{l\in\mathbb{Z}} \frac{(-1)^{lp}\left(1+e^{2\pi il/3}+e^{-2\pi il/3}\right)}{(\pi(l+3\omega))^p}$$

$$= (1+2\cos\pi\omega)^p\,(\sin\pi\omega)^p \sum_{l\in\mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l+3\omega))^p}\,(1+2\cos 2\pi l/3)\,.$$

We split the sum in the last equation into three sub-sums along $l=3m$, $l=3m+1$ and $l=3m-1$. Respectively, the function $G^p(\omega)$ splits as $G^p(\omega) = (1+2\cos\pi\omega)^p\,(\gamma_0(\omega)+\gamma_{-1}(\omega)+\gamma_1(\omega))$, where

$$\gamma_0(\omega) \triangleq \frac{(\sin\pi\omega)^p}{3^p} \sum_{m\in\mathbb{Z}} \frac{(-1)^{mp}}{(\pi(m+\omega))^p}\,(1+2\cos 2\pi m) = 3^{-p+1}u^p(\omega),$$

$$\gamma_{\pm1}(\omega) \triangleq \frac{(\sin\pi\omega)^p}{3^p} \sum_{m\in\mathbb{Z}} \frac{(-1)^{(m+1)p}}{(\pi(m+\omega)\pm\pi/3)^p}\,(1+2\cos 2\pi(m\pm 1/3)) = 0.$$

Hence, Eq. (3.10) follows. ■

3.2 Convergence

The spline $L^p(x) \in \mathfrak{S}^p$, which interpolates the Kronecker delta sequence δ^0_k, is called the fundamental spline of the space $\mathfrak{S}^p$. The spline $S^p(x) \in \mathfrak{S}^p$, which interpolates the data sequence $\mathbf{f}^0 = \left\{f^0_k\right\}$, can be represented as

$$S^p(x) = \sum_{k\in\mathbb{Z}} f^0_k L^p(x-k). \tag{3.11}$$

$L^p(x)$ decays exponentially when x grows. Therefore, Eq. (3.11) represents the spline, which interpolates the sequence $\left\{f^0_k\right\}$, $k\in\mathbb{Z}$, even in the case when the sequence has a power growth. In this case, the spline $S^p(x)$ also has a power growth. It was proved in [2] that the spline of power growth, interpolating the data of power growth, is unique.

We prove that the values of the spline $S^p(x)$ at any set of triadic rational points $S^p(k3^{-j})$ can be calculated by successive application of Triadic Insertion Rule, which was formulated in Section 1, to the array $\mathbf{f}^0$. A similar fact holds for Dyadic Insertion Rule when order p of splines is even. But this is not true for Dyadic Insertion Rule when p is odd. In this case, when p is odd, the spline-based subdivision no longer converges to a spline but rather to a function, which is smoother than the generating spline (see [6]).

Theorem 3.2 *Assume the spline $S_0^p(x)$ belongs to $\mathfrak{S}^p$, $S_0^p(k) = f_k^0$ and $\mathbf{f}^0 = \left\{f_k^0\right\} \in l_1$. Let $\boldsymbol{T}^p$ be the filter defined by Eq. (3.10). Assume that for $j = 1, 2, \ldots$, $\mathbf{f}^j = \left\{f_k^j\right\}$ is the array, whose DFT is found from the relation $\widehat{f^{j+1}}(\omega) = \widehat{T^p}(\omega)\widehat{f^j}(3\omega)$. Then,*

$$S_0^p(k3^{-j}) = f_k^j, \quad j = 1, 2, \ldots, . \tag{3.12}$$

Proof: We have $S_0^p(k3^{-1}) = f_k^1$ from the definition of Triadic Insertion Rule. Next step of subdivision consists in the construction of the spline $S_1^p(x)$ on the grid $\mathbf{g}^1 = \left\{k3^{-1}\right\}$, such that $S_1^p(k3^{-1}) = f_k^1$. Then, $f_k^2 = S_1^p(k3^{-2})$. We prove that $f_k^2 = S_0^p(k/9)$. The subsequent relations in Eq. (3.12) are derived by a simple induction.

The array $\mathbf{f}^2$ is obtained by repeated application of the filter $\mathbf{T}^p$ to the array $\mathbf{f}^0$:

$$\begin{aligned}
&\widehat{f^2}(\omega) = \widehat{T^p}(\omega)\widehat{f^1}(3\omega) = \widehat{T^p}(\omega)\widehat{T^p}(3\omega)\widehat{f^0}(9\omega) \\
&= \frac{(1+2\cos\pi\omega)^p\ u^p(\omega)}{3^{p-1}u^p(3\omega)}\ \frac{(1+2\cos 3\pi\omega)^p\ u^p(3\omega)}{3^{p-1}u^p(9\omega)}\ \widehat{f^0}(9\omega) = \frac{H(\omega)}{u^p(9\omega)}\ \widehat{f^0}(9\omega),
\end{aligned}$$

where

$$H(\omega) \triangleq 9^{1-p}\ ((1+2\cos\pi\omega)(1+2\cos 3\pi\omega))^p\, u^p(\omega).$$

Denote $s_k = S_0^p(k/9),\ k \in \mathbb{Z}$. Then, the DFT of this array is $\widehat{s}(\omega) = \widetilde{H}\widehat{f^0}(9\omega)(\omega)/u^p(9\omega)$, where

$$\widetilde{H}(\omega) \triangleq \sum_{\nu=1}^{4} \left(e^{-2\nu\pi i\omega} w_\nu(9\omega) + e^{2\nu\pi i\omega} w_{-\nu}(9\omega)\right) + u^p(9\omega), \quad w_{\pm\nu}(\omega) \triangleq m^p_{\pm\nu/9}(\omega).$$

We can write, using Eq. (2.7),

$$\begin{aligned}
\widetilde{H}(\omega) &= (\sin 9\pi\omega)^p \sum_{l\in\mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l+9\omega))^p} \left(1 + \sum_{\nu=1}^{4} e^{2\nu\pi il/9} + e^{-2\nu\pi il/9}\right) \\
&= ((1+2\cos 3\pi\omega)(1+2\cos\pi\omega))^p\, (\sin\pi\omega)^p \sum_{l\in\mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l+9\omega))^p} \left(1 + 2\sum_{\nu=1}^{4} \cos 2\pi\nu l/9\right) \\
&= ((1+2\cos 3\pi\omega)(1+2\cos\pi\omega))^p \sum_{\mu=-4}^{4} \kappa_\mu(\omega),
\end{aligned}$$

where

$$\kappa_\mu(\omega) \triangleq \left(1 + 2\sum_{\nu=1}^{4} \cos 2\pi\nu\mu/9\right) \frac{(\sin\pi\omega)^p}{9^p} \sum_{m\in\mathbb{Z}} \frac{(-1)^{(m+1)p}}{(\pi(m+\omega)+\mu\pi/9)^p}.$$

It is easily verified that for $\mu = \pm 1, ..., \pm 4$ $\kappa_\mu(\omega) \equiv 0$, while $\kappa_0(\omega) = 9^{1-p}u^p(\omega)$. Hence, it follows that

$$\widetilde{H}(\omega) = H(\omega) \Longleftrightarrow f_k^2 = S_0^p(k3^{-2}), \quad k \in \mathbb{Z}.$$

Repeating our reasoning with the initial data set $\mathbf{f}^1$ instead of $\mathbf{f}^0$, we prove that $f_k^3 = S_1^p(k3^{-3})$ and so on. ■

Remark If the initial data is the delta sequence $f_k^0=\delta_k^0$ then we get $f_k^j=L^p\left(k3^{-l}\right)$, where $L^p(x)$ is the fundamental spline of the space $\mathfrak{S}^p$. Therefore, we can extend the assertion of Theorem 3.2 from the splines belonging to $\mathfrak{S}^p$ to the splines that interpolate sequences of power growth.

4 Remarks on the computational aspects of the spline subdivision

It was stated in Section 3.1 that in order to produce the array $\mathbf{f}^{j+1}$ we need to filter the upsampled array $\mathbf{f}^j$ with the filter $\mathbf{T}^p$. In the frequency domain, filtering is presented by Eq. (3.9). The frequency response of the filter is given by Eq. (3.10). For computational purposes, we present filtering using the z-transform. We recall that the the z-transform of a sequence $\mathbf{a}=\{a_k\}\in l_1$ is $A(z)\stackrel{\Delta}{=}\sum_{k\in\mathbb{Z}}z^{-k}a_k$. The z-transform of the shifted sequence is $\sum_{k\in\mathbb{Z}}z^{-k}a_{k+l}=z^lA(z)$.

Denote $z\stackrel{\Delta}{=}e^{2\pi i\omega}$. Then, the DFT of a sequence $\mathbf{a}$ becomes its z-transform$\widehat{a}(\omega)=A(z)$. The DFT of a filter $\mathbf{T}$ becomes its z-transform$\widehat{T}(\omega)=T(z)$ and it is called the transfer function. The subdivision is represented as

$$F^{j+1}(z)=T^p(z)F^j(z^3),\quad T^p(z)=\frac{(z^{-1}+1+z)^pU^p(z)}{U^p(z^3)},\ U^p(z)\stackrel{\Delta}{=}\sum_{k\in\mathbb{Z}}z^{-k}M^p(k). \tag{4.13}$$

The function $U^p(z)$ is a Laurent polynomial and the transfer function $T^p(z)$ is a rational function. Such filters are called the infinite impulse response (IIR) filters as opposite to the finite impulse response (FIR) filters, whose transfer functions are Laurent polynomials.

Proposition 4.1 ([2]) *The roots of the Laurent polynomials $U^p(z)$ are all simple and negative. Each root ζ can be paired with a dual root θ such that $\zeta\,\theta=1$. Thus, if $p=2r-1,\ p=2r$ then $U^p(z)$ can be represented by:*

$$U^p(z)=\prod_{n=1}^{r}\frac{1}{\gamma_n}(1+\gamma_nz)(1+\gamma_nz^{-1}),\quad 0<|\gamma_1|<|\gamma_2|<\ldots|\gamma_r|=e^{-g}<1,\ g>0.$$

The above properties of the polynomials $U^p(z)$ enable to implement filtering via a recursive algorithm, which is widely used in signal processing.

There are two ways to implement the described subdivision scheme.

Polyphase filtering: One way is to implement the filter $\mathbf{T}^p=\{T_k\},\ k\in\mathbb{Z}$, using the so-called polyphase representation of the filter:

$$T^p(z)=zT^p_{-1}(z^3)+T^p_0(z^3)+z^{-1}T^p_1(z^3),\quad T^p_\nu(z)\stackrel{\Delta}{=}\sum_{k\in\mathbb{Z}}z^{-k}T^p_{3k+\nu},\ \nu=-1,0,1.$$

Then, the polyphase representation of the array $\mathbf{f}^{j+1}$ is

$$F^{j+1}(z)=zF^{j+1}_{-1}(z^3)+F^{j+1}_0(z^3)+z^{-1}F^{j+1}_1(z^3),$$
$$F^{j+1}_\nu(z)\stackrel{\Delta}{=}\sum_{k\in\mathbb{Z}}z^{-k}f^{j+1}_{3k+\nu}=T^p_\nu(z)F^j(z),\ \nu=-1,0,1.$$

Thus, in order to retrieve the sub-arrays $\left\{f_{3k\pm1}^{j+1}\right\}$ we have to filter the array $\left\{f_k^j\right\}$ with the filters, whose transfer functions are $T_{\pm1}^p(z)$, respectively. Recall that $\left\{f_{3k}^{j+1}=f_k^j\right\}$. The polyphase components of the transfer function $T^p(z)$ can be derived from the expression for $T^p(z)$ given in Eq. (4.13). Obviously, $T_1^p(z)=T_{-1}^p(z^{-1})$. However, it is easily seen from Eq. (3.9) that $T_{\pm1}^p(e^{2\pi i\omega})=v_{\pm1}^p(\omega)/u^p(\omega)$.

Direct filtering: Equation (4.13) suggests that, when several steps of the subdivision must be carried out, a direct application of the filter $\mathbf{T}^p$ is preferable. It follows from Eq. (4.13) that

$$F^{j+1}(z)=T^p(z)\cdot F^j(z^3)=\prod_{l=1}^{j-1}T^p(z^{3^l})\cdot F^0(z^{3^j})=\frac{U^p(z)\prod_{l=0}^{j-1}(z^{-3^l}+1+z^{3^l})^p}{U^p(z^{3^j})}\cdot F^0(z^{3^j}).$$

For example,

$$F^3(z)=\frac{U^p(z)(z^{-1}+1+z)^p(z^{-3}+1+z^3)^p(z^{-9}+1+z^9)^p}{U^p(z^{27})}\cdot F^0(z^{27}).$$

Thus, the subdivision is implemented via the following steps:

1. The IIR filter with the transfer function $1/U^p(z)$ is applied to the data array $\mathbf{f}^0$.
2. The produced array is upsampled[1] and filtered with FIR filter, whose transfer function is $(z^{-1}+1+z)^p$ (repeated j times).
3. The produced array is filtered with FIR filter whose transfer function is $U^p(z)$.

Note that the IIR filtering is applied only once.

Examples of filters

Linear spline: $T^2(z)=\left(z+1+z^{-1}\right)^2/3,\quad T_1^2(z)=T_{-1}^2(z^{-1})=(z+2)/3$. This is a single FIR filter in the family. All the filters derived from splines of higher orders are IIR.

Quadratic spline:

$$T^3(z)=\frac{\left(z+6+z^{-1}\right)\left(z+1+z^{-1}\right)^3}{9\left(z^3+6+z^{-3}\right)},\quad T_1^3(z)=T_{-1}^3(z^{-1})=\frac{25z+46+z^{-1}}{9\left(z+6+z^{-1}\right)}. \tag{4.14}$$

Cubic spline:

$$T^4(z)=\frac{\left(z+4+z^{-1}\right)\left(z+1+z^{-1}\right)^4}{27\left(z^3+4+z^{-3}\right)},\quad T_1^4(z)=T_{-1}^4(z^{-1})=\frac{z^2+60z+93+8z^{-1}}{27\left(z+4+z^{-1}\right)}.$$

[1]Upsampling means replacing an array $\{a_k\}$ by the array $\{\tilde a_k\}$ such that $\tilde a_{3k}=a_k$ and $\tilde a_{3k\pm1}=0$.

Spline of fourth degree :

$$T^5(z) = \frac{\left(z^2 + 76z + 230 + 76z^{-1} + z^{-2}\right)\left(z + 1 + z^{-1}\right)^5}{81\left(z^6 + 76z^3 + 230 + 76z^{-3} + z^{-6}\right)},$$

$$T_1^5(z) = T_{-1}^5(z^{-1}) = \frac{625z^2 + 11516z + 16566 + 2396z^{-1} + z^{-2}}{81\left(z^2 + 76z + 230 + 76z^{-1} + z^{-2}\right)}.$$

Spline of fifth degree :

$$T^6(z) = \frac{\left(z^2 + 26z + 66 + 26z^{-1} + z^{-2}\right)\left(z + 1 + z^{-1}\right)^6}{243\left(z^6 + 26z^3 + 66 + 26z^{-3} + z^{-6}\right)},$$

$$T_1^6(z) = T_{-1}^6(z^{-1}) = \frac{z^3 + 1018z^2 + 10678z + 14498 + 29336z^{-1} + 32z^{-2}}{243\left(z^2 + 26z + 66 + 26z^{-1} + z^{-2}\right)}.$$

We now provide an example of a recursive implementation of an IIR filter. In [6] this is discussed in more details.

Implementation of the filter $\mathbf{T}_1^3$. The z-transform of this filter is given in Eq. (4.14):

$$T_1^3(z) = \frac{25z + 46 + z^{-1}}{9\left(z + 6 + z^{-1}\right)} = \frac{\alpha}{9}\frac{25z + 46 + z^{-1}}{(1 + \alpha z)(1 + \alpha z^{-1})},$$

where $\alpha = 3 - 2\sqrt{2} \approx 0.172$. Then, application of the filter $\mathbf{T}_1^3$ to an array $\mathbf{a} = \{a_k\}$, whose z-transform is $A(z)$, is implemented as a subsequent application of three filters: $\mathbf{b} = \mathbf{T}_1^3\mathbf{a} = \mathbf{R}_l\,\mathbf{R}_r\,\mathbf{F}\cdot\mathbf{a}$, which are are defined by their z-transforms:

$$F(z) = \frac{\alpha}{9}\left(25z + 46 + z^{-1}\right),\ R_r(z) = \frac{1}{1 + \alpha z^{-1}},\ R_l(z) = \frac{1}{1 + \alpha z}.$$

Thus, filtering is carried out in three steps:

$$A^1(z) = F(z)A(z) \Longleftrightarrow a_k^1 = \frac{\alpha}{9}\left(25a_{k+1} + 46a_k + a_{k-1}\right),$$

$$A^2(z) = R_r(z)A^1(z) \Longleftrightarrow \left(1 + \alpha z^{-1}\right)A^2(z) = A^1(z) \Longleftrightarrow a_k^2 = a_k^1 - \alpha a_{k-1}^2,$$

$$B(z) = R_l(z)A^2(z) \Longleftrightarrow (1 + \alpha z)\,B(z) = A^2(z) \Longleftrightarrow b_k = a_k^2 - \alpha b_{k+1}.$$

The filter $\mathbf{F}$ is FIR unlike the filters $\mathbf{R}_l$ and $\mathbf{R}_r$. Application of the filter $\mathbf{R}_r$ is called causal recursive filtering. Here, for the calculation of the term a_k^2, the previously derived term a_{k-1}^2 is used. Application of the $\mathbf{R}_l$ is called anti-causal recursive filtering. All these procedures are implemented in a fast way. Computation of splines of higher orders uses filters, which are factorized into longer cascades of the same structure.

References

[1] I. J. Schoenberg, *Contribution to the problem of approximation of equidistant data by analytic functions*, Quart.Appl. Math., **4** (1946), 45-99, 112-141.

[2] I. J. Schoenberg, *Cardinal interpolation and spline functions*, J. Approx. Th., **2**, (1969), 167-206.

[3] I. J. Schoenberg, *Cardinal interpolation and spline functions II,* J. Approx. Theory, **6** (1972), 404-420.

[4] I. J. Schoenberg, *Cardinal spline interpolation*, CBMS, **12**, SIAM, Philadelphia, 1973.

[5] V. A. Zheludev, *Integral representation of slowly growing equidistant splines*, Approximation Theory and Applications, **14**, no. 4, (1998), 66-88.

[6] V. A. Zheludev, Interpolatory subdivision schemes with infinite masks originated from splines, Advances in Comp. Math. **25**,(2006), 475–506.