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LOCAL SPLINES OF DEFECT 1 ON A UNIFORM MESH

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INTRODUCTION

Polynomial splines are comparatively new, intensely developing apparatus for the approximation of functions. This apparatus is being applied for solving varied problems of numerical analysis and of technology. There are a lot of monographs and papers which are devoted to spline-functions. A remarkable area in the theory of splines is the investigation of local splines. This paper is a survey of some results established by the author in this direction. As it is manifested in the title of the paper, the subject of the discussion will be local splines of arbitrary degrees and of defect 1 constructed on a uniform mesh.

The term "local spline" of the degree $m-1$ is to be understood throughout the paper as a polynomial spline of defect 1 constructed on a uniform mesh with the step $h - \{x_r\}$ which is a linear combination of shifts of the central B-spline $b^m(x)$ of the degree $m-1$:

$$S^m(f,x) = \sum_k l_k(f) b^m(x - x_k), \tag{1}$$

with coefficients $l_k(f)$ given explicitly as finite linear combinations of samples $\{f(x_r)\}$ of the function.

Almost all authors who write on local splines take note on their advantages in comparison with the global interpolating and smoothing splines such as easy and fast constructing, the possibility to keep in operating memory only several closest to the point x values $\{f(x_r)\}$ when the value $S^m(f,x)$ is being evaluated. Moreover, a local spline can provide the approximation of most po ssible order, although constants in the estimations of the remainder terms of approximation for local splines exceed those for interpolating splines. It should be pointed out that local splines permit to process an information on the real time scale. Moreover, local splines provide a certain smoothing effect which makes it possible to use these for recovering functions and their derivatives from discrete noised data.

First local scheme of the spline approximation had appeared, to the author's knowledge, in the famous paper of I.J.Schoenberg [19] where the author suggests, in particular, an algorithm for constructing local splines of arbitrary degree on a uniform mesh which reproduce precisely polynomials of corresponding degree from their discrete values. Later there had appeared local schemes of the spline approximation on arbitrary meshes with various coefficients $l_k(f)$ in B-spline expansion. We mention in this connection the papers [1], [2].

In [14], [25] general formulas are given for constructing local splines of arbitrary degree on arbitrary meshes which reproduce exactly polynomials of corresponding degree and the estimations of the remainders of approximation had been established. However, very little is known about a reasonable numerical values of constants in these estimations. The formulas suggested are very complicated and hardly can be used for constructing the splines of degree more then three. The various topics of local spline approximation were being investigated lately. In the papers [27], [28], [23], [24], local schemes were being elaborated to improve the preciseness of approximation.

Local splines on a uniform mesh are being regarded usually as a special case of splines constructed on arbitrary meshes. Therefore the theory of such splines, is advanced only a few further then the theory of splines on arbitrary meshes. In [12] the precise estimations of the remainders of approximation for splines of second degree and their derivatives have been obtained. In [8], [13], [23] the local splines of second and third degree have been appeared which quasiinterpolate functions. In [9] there were studied local splines of the fifth degree near boundaries of the interval of definition. In [25] the asymptotic expansions and the exact estimations of remainder terms of approximation for cubic local splines on a uniform mesh were established. It should be mentioned the paper [4], where local schemes of defect 1 and 2 were presented and linear combinations of B-splines were used as basic functions.

The distinctive feature of the presented work is that local splines on a uniform mesh are being considered as an independent subject of investigation. To study these splines special techniques have been developed which exploit essentially the uniformness of the mesh. This approach yields the remarkable progress in the study of known classes of local splines and makes it possible to define new specialized classes of local splines. Our approach is based on some new properties of B-splines constructed on a uniform mesh, established by the author, in particular, on the stability of discrete moments of B-splines.

As it was mentioneded in [19], $b^m(x)$ is the probability density of the sum of m independent random variables distributed uniformly on $[-h/2, h/2]$. It is given in [19] the generating function of moments

$$M_s^m = \frac{h^{-s}}{s!}\int_{-\infty}^{\infty} x^s b^m(x)dx$$

of this distribution. In the presented paper a recursive formula for evaluating the moments is given as well as explicit expressions for $s = 1, ..., 8$ and any m. The function

$$\mu_s^m(x) = \frac{h^{-s+1}}{s!}\sum_{k=-\infty}^{\infty}(x-hk)^s b^m(x-hk)$$

we call the discrete moment of the B-spline $b^m(x)$.

It was pointed out in [5] that $\mu_0^m(x) \equiv 1, \mu_1^m(x) \equiv 0$. It is established in presented paper that $\mu_s^m(x)$ by $s = 2, m-1$ are independent of x and coincides with the corresponding moments M_s^m. Moreover the structure of moments $\mu_s^m(x)$ by $s = m, ..., m+3$ is established. These results, which are of some concern in the theory of probability, are exploited essentially in the paper for the study of local splines.

The aforementioned properties of B-splines lead to asymptotic formulas in powers of h for the elementary splines. The latter splines which are being constructed by means of the formula

$$S_0^m(f,x) = \sum_k f(x_k)b^m(x-x_k),$$

are named the Bernstein-Schoenberg splines (BSS). First they were studied in [20] where it was pointed out that these splines reproduce exactly polynomials of the first degree. Moreover, the following asymptotic relation was established in [20] provided function $f \in C^2$:

$$S_0^m(f,x) = f(x) - \frac{mh^2}{24} f''(x) + o(mh^2) \quad \text{as} \quad mh^2 \to 0.$$

Although BSS were studied intensely (see for example [16], [7]), no one of researchers, to the author's knowledge, tried to establish the forthcoming terms of the asymptotic expansion of these splines and of their derivatives on a uniform mesh. However, it turned out that "deep" asymptotic formulas permit us to construct explicitly the splines ensuring an approximation of much more accuracy.

Exploiting the property of stability of discrete moments, we have established asymptotic expansions for BSS and their derivatives $S^{m+s}(f,x)^{(s)}$ in powers of h up to h^{m+1}. It was turned out that the terms which contain $h^k, k < m$, vanish when k is odd. If k is even, $k = 2l$ then they have numerators of the kind $c_{2l} f^{(2l+s)}(x)$ where c_{2l} are independent of x. Due to this fact we succeeded, via combining BSS, in obtaining formulas for constructing local splines of degree $m-1$, which reproduce exactly derivatives $P_{n+s}^{(s)}$ of polynomials of degree $n+s, n \le m-1$,. Among others, we establish formulas for constructing local splines which reproduce exactly derivatives $P_{m-1+s}^{(s)}$ of polynomials of most possible degree $m-1+s$ using least possible number of the samples $\{f(x_r)\}$(so called splines of minimal span -SMS). For SMS we have established an asymptotic expansion of the remainder term in powers of h up to h^{m+1}. In some cases which are important for practice, we succeeded in obtaining an explicit form of the remainder term. The explicit form derived makes it possible to establish exact estimations of the error of the approximation. The formulas obtained, which are of independent concern, permit the expansion in two directions.

On the one hand, the splines of arbitrary degree $m-1$ are constructed which quasiinterpolate functions and their derivatives of any order s in the nodes $\{x_k\}$. There is established an asymptotic expansion of the remainder term of these splines in powers of h. As an immediate consequence of this result we obtain the asymptotic expansions for cardinal interpolating splines. It should be pointed out especially that we succeeded in constructing splines of arbitrary degree which, using as initial data the *samples of a function f* in the mesh points, almost coincide with the splines *interpolating derivatives* of this function. In the cases $m = 4, s = 0, 1, 2$, we have established exact estimations of the error of approximation by these splines.

On the other hand, the opportunity appears to gain insight into the smoothing properties of local splines and to exploit these purposely for recovering functions and their derivatives from discrete noised data.

Many authors pointed out the smoothing properties of local splines (see, for example [19], [25], [26],[8]) but, to our knowledge, no one investigated this problem systematically. It is defined in the paper a new kind of local splines - the local smoothing splines with regularizing parameter (LPS). The approximating and filtering properties of these splines are studied as well as filtering properties of SMS, which are the special case of LPS. The computational experiments demonstrate the great efficiency of the LPS for recovering functions and their derivatives from discrete noised data.

To avoid overloading the text we shall omit the proofs of some propositions. The necessary references will be cited.

1. SOME PROPERTIES OF B-SPLINES

We introduce some notation. As usually, C^s is the space of functions f such that $f^{(s)}$ are continuous, L^s_∞ is the space of functions f such that $f^{(s-1)}$ are absolutely continuous, $f^{(s)}$ are locally bounded. If $f \in C$- then $\omega(f,\beta)$ will denote the modulus of continuity of f. Suppose $x_+ = \frac{1}{2}(x+|x|), f_k = f(hk)$,

$$x_k^m = \begin{cases} hk & \text{if } m \text{ is even} \\ h(k-1/2) & \text{if } m \text{ is odd.} \end{cases}$$

Hence, if $x \in [x_n^m, x_{n+1}^m]$ then $x = x_n^m + ht, t \in [0,1]$. We denote by δ_h^s the central difference:

$$\delta_h^s g(x) = h\sum_{k=0}^{s}(-1)^k\binom{s}{k}g(x-h(k-\frac{s}{2})).$$

Denote $b^m(x) = \delta_h^m(\frac{x_+^{m-1}}{(m-1)!})$. This is the central B- spline of degree $m-1$ with nodes in the points x_k^m. Point out that it is a spline of the defect 1 and it possesses the following properties:

$$b^m(x) \geq 0, \quad \operatorname{supp} b^m(x) = (-\frac{mh}{2}, \frac{mh}{2}), \quad b^m(-x) = b^m(x). \tag{1.1}$$

We define the moments

$$M_s^m = \frac{h^{-s}}{s!}\int_{-hm/2}^{hm/2} x^s b^m(x)dx, \quad M_0^0 = 1, \quad M_s^0 = 0\,\forall s \neq 0.$$

It is known (see [19]) that $M_1^m = 0, \quad M_2^m = \frac{m}{24}$. At the same paper [19] the Fourier transform is written:

$$\Phi(b^m,\omega) = \left(\frac{2\sin(h\omega/2)}{h\omega}\right)^m. \tag{1.2}$$

As a consequence of this can be obtained the generating function for the moments - compare with [17] :

$$\left(\frac{2\sin(h\omega/2)}{h\omega}\right)^m = \sum_{s=0}^{\infty} M_s^m v^s. \tag{1.3}$$

Multiplying the series, we come to the identity

$$M_r^{p+q} = \sum_{l=0}^{r} M_{r-l}^p M_l^q. \tag{1.4}$$

Let us write down some initial moments:

$$M_4^m = \frac{m(5m-2)}{5760}, M_6^m = \frac{m(35m^2-42m+16)}{2903040},$$
$$M_8^m = \frac{m(175m^3-420m^2+404m-144)}{1383459000}.$$

All of the odd moments are zero.

Point out the following property of the B- splines which can be verified from the definition:

$$b^m(x)^{(s)} = \delta_h^s b^{m-s}(x)$$

. We now introduce the discrete moments

$$\mu_s^m(t) = \frac{h}{s!}\sum_{r=0}^{m-1}(r+t-m/2)^s b^m(h(r+t-m/2)). \tag{1.5}$$

The relations are known ([5]):

$$\mu_0^m(t) \equiv 1, \mu_1^m(t) \equiv 0.$$

The functions $\mu_s^m(t)$ are of important concern in studying the approximating properties of local splines. Therefore we consider these functions with $s > 1$ in details.

Lemma 1.1. *([29]).The following relations hold:*

$$\int_0^1 \mu_s^m(t)dt = M_s^m, \tag{1.6}$$

$$\mu_s^m(t)' = \mu_{s-1}^m(t) - \sum_{r=0}^{s-1}\frac{[1-(-1)^{s-r}]}{2^{s-r}(s-r)!}\mu_r^{m-1}(t). \tag{1.7}$$

Point out the following property of the moments $\mu_s^m(t)$. If to keep in sight the symmetry of $b^m(x)$ then the relation

$$\mu_s^m(0) = \frac{h}{s!}\sum_{r=1}^{m-1}(r-m/2)^s b^m(h(r-m/2)),$$

entails that $\mu_s^m(0) = 0 = \mu_s^m(1)$ provided s is odd.

The following proposition is basic for a lot of the forthcoming results. We denote by $B_q(t)$ the Bernoulli polynomial of degree q-

Theorem 1.1. *If $s \le m-1$ then the moments $\mu_s^m(t)$ are independent of $t \in [0,1]$ and the following formulas hold*

$$\mu_s^m(t) \equiv M_s^m, s = 0, ..., m-1, \tag{1.8}$$

If $s = m, m+1$ then

$$\mu_m^m(t) = (-1)^{m+1}\frac{B_m(t)}{m!} + M_m^m, \tag{1.9}$$

$$\mu_{m+1}^m(t) = (-1)^{m+1}\frac{mB_{m+1}(t)}{(m+1)!} + M_{m+1}^m, \tag{1.10}$$

Proof. First we prove the formula (1.8). Make use of the induction. If $s = 0$ then $\mu_0^m(t) \equiv 1 \forall m > 0$. Suppose now that $\mu_p^l(t) \equiv \mu_p^l =$ const by all $p < s, l \geq s$. Let $m > s$. Then

$$\mu_s^m(t)' = \mu_{s-1}^m + \sum_{r=0}^{s-1} \xi_r \mu_r^{m-1} = a = \text{const.}$$

Hence $\mu_s^m(t) = at + c$. But, owing to symmetry of $b^m(x)$, we have $\mu_s^m(0) = \mu_s^m(1)$. Therefore $a = 0$ and $\mu_p^m(t) \equiv \mu_p^m =$ const. The relation (1.8) follows now immediately from the formula (1.6).

Proceed now to proving. The following properties of Bernoulli polynomials are known:

$$\frac{B_m{}'(t)}{m!} = \frac{B_{m-1}(t)}{(m-1)!}, \quad \int_0^1 B_m(t)dt = 0.$$

The formulas (1.8), (1.7) imply:

$$\begin{aligned}\mu_m^m(t)' =& M_{m-1}^m - \sum_{r=0}^{m-2} \frac{[1-(-1)^{m-r}]}{2^{m-r}(m-r)!} M_r^{m-1} - \mu_{m-1}^{m-1}(t) \\ =& c_m - \mu_{m-1}^{m-1}(t).\end{aligned}$$

Integrating the latter relation and employing (1.6) and the fact that $\mu_s^m(0) = 0 = \mu_s^m(1)$ provided s is odd, we have $c_{2n+1} = M_{2n}^{2n}$.

It is easily seen that $c_{2n} = 0$. Let us consider first an odd $m = 2n+1$ and prove (1.9) by means of the induction. If $m = 1$ then $\mu_1^1(t) = t - 1/2 = B_1(t)$. Suppose that the formula (1.9) is true if $m = 2n-1$. Then

$$\mu_{2n+1}^{2n+1}(t)'' = \mu_{2n-1}^{2n-1}(t) = \frac{B_{2n-1}(t)}{(2n-1)!}.$$

Therefore

$$\mu_{2n+1}^{2n+1}(t) = \frac{B_{2n+1}(t)}{(2n+1)!} + at + b.$$

But

$$\mu_{2n+1}^{2n+1}(0) = \mu_{2n+1}^{2n+1}(1) = 0.$$

The same is true for $B_{2n+1}(t)$. Hence

$$\mu_{2n+1}^{2n+1}(t) = \frac{B_{2n+1}(t)}{(2n+1)!}.$$

Differentiating the latter equality we arrive at

$$c_{2n+1} - \mu_{2n}^{2n}(t) = \frac{B_{2n}(t)}{(2n)!}.$$

But, as it was marked above, $c_{2n+1} = M_{2n}^{2n}$. This fact concludes the proof of (1.9). Let us prove now the formula (1.10). The formulas (1.8), (1.7) imply

$$\mu_{m+1}^{m}(t)' = \mu_{m}^{m}(t) - \sum_{r=0}^{m-2} \frac{[1-(-1)^{m+1-r}]}{2^{m-r}(m+1-r)!} M_r^{m-1} - \mu_m^{m-1}(t)$$
$$= c_m + \mu_m^m(t) - \mu_m^{m-1}(t).$$

It is easily seen that $c_{2n+1} = 0$. We prove (1.10) by means of the induction. It can be verified immediately that $\mu_3^2(t) = -\frac{2}{3}B_3(t)$. Suppose that (1.10) is true with $m = 2n-1$. Then

$$\mu_{2n+1}^{2n}(t)' = \mu_{2n}^{2n}(t) - \mu_{2n}^{2n-1}(t) + c_{2n} = -\frac{2nB_{2n}(t)}{(2n)!} + d_{2n},$$

$$\mu_{2n+1}^{2n}(t)'' = -\frac{2nB_{2n-1}(t)}{(2n-1)!} = -\frac{2nB''_{2n+1}(t)}{(2n+1)!}.$$

But $\mu_{2n+1}^{2n}(0) = B_{2n+1}(0) = B_{2n+1}(1) = \mu_{2n+1}^{2n}(1) = 0.$

Therefore $\mu_{2n+1}^{2n}(t) = -\dfrac{2nB_{2n+1}(t)}{(2n+1)!},$

$$\mu_{2n+2}^{2n+1}(t)' = \mu_{2n+1}^{2n+1}(t) - \mu_{2n+1}^{2n}(t) = \frac{(2n+1)B_{2n+1}(t)}{(2n+1)!}.$$

Hence,

$$\mu_{2n+2}^{2n+1}(t) = (2n+1)B_{2n+2}(t)/(2n+2)! + c.$$

It is easily seen that $c = M_{2n+2}^{2n+1}$.

Throghout the paper $\sum_r^m$ stands for $\sum_{r=n-\mu}^{n+\nu}$. Here $\mu = [m/2] = \nu$ if m is odd and $\mu = m/2 - 1, \nu = m/2$ if m is even. Denote

$$\lambda_{rs}^q(t) = h^{s-r+1} \sum_k^m \frac{1}{r!} \delta_h^s (hk - x)^r b^{q-s}(x - hk). \tag{1.11}$$

Theorem 1.2. ([29]).*The following relations hold: a)Is $m > p$ then*

$$\lambda_{p+s,s}^{m+s}(t) \equiv \lambda_{p+s,s}^{m+s} = (-1)^p \sum_{l=0}^{p} M_{p-l}^s M_l^m = (-1)^p M_p^{s+p} \tag{1.12}$$

b)If $p = m, m+1$ then

$$\lambda_{m+s,s}^{m+s}(t) = -\frac{B_m(t)}{m!} + M_m^{s+m}. \tag{1.13}$$

$$\lambda_{m+s+1,s}^{m+s}(t) = \frac{mB_{m+1}(t)}{(m+1)!} + M_{m+1}^{s+m}. \tag{1.14}$$

2. ASYMPTOTIC FORMULAS FOR BERNSTEIN-SCHOENBERG SPLINES

Let $f(x)$ be any continuous function and $f_k = f(x_k)$. Each local spline of defect 1 and of degree $m-1$, constructed on a uniform mesh with the step h from the data $\{f_k\}$, can be written in the following form if $x \in [x_n^m, x_{n+1}^m]$:

$$S^m(f,x) = h\sum_k^m F_k^m b^m(x-hk),$$

where F_k^m are finite linear combination of the samples $\{f_k\}$. A choice of the functionals F_k^m determines the properties of spline. The derivative of the spline of the degree $m+s-1$

$$S^{m+s}(f,x)^{(s)} = h\sum_k^{m+s} F_k^{m+s}\delta_h^s\, b^m(x-hk) \tag{2.1}$$

is a spline of the degree $m-1$. Note, that if $s = 2l$ is an even number then the formula (2.1) can be written in the following form:

$$S^{m+2l}(f,x)^{(2l)} = h\sum_k^m \delta_h^{2l}(F_k^{m+2l})b^m(x-hk). \tag{2.2}$$

Suppose that to compute the value of a spline in a point x one needs the values of function approximated in the points $\{hk\}_{k=l}^r$. Then we call the set $\{hk\}_{k=l}^r$ the span of the spline.

We consider at this section the BSS approximating $f^{(s)}$, namely

$$S_0^{m+s}(f,x)^{(s)} = h\sum_k^{m+s} f_k\delta_h^s b^m(x-hk). \tag{2.3}$$

In the case $s = 0$ these splines were studied by Schoenberg in [20]. In particular, it was established there that if $f \in C^2$, $m > 2$ then

$$S_0^m(f,x) = f(x) + \frac{h^2 m}{24}f''(x) + o(h^2).$$

We will obtain further formulas of such kind for the splines $S_0^{m+s}(f,x)^{(s)}$. These asymptotic formulas which are of independent concern, serve us as a basis for constructing and studying splines approximating functions and their derivatives with great accuracy. To establish these formulas we exploit essentially results of Section 1.

Let $f \in C^{s+p}$. The Taylor formula entails

$$f_k = \sum_{l=0}^{s+p}\frac{f^{(l)}(x)}{l!}(x_k-x)^l + R_{p+s}(x_k),$$

$$R_{p+s}(x_k) = \frac{1}{(s+p-1)!}\int_x^{x_k}(x_k-y)^{s+p-1}g_x^{s+p}(y)dy,$$

where $g_x^{s+p}(y) = f^{(s+p)}(y) - f^{(s+p)}(x).$

Hence

$$\begin{aligned} S_0^{m+s}(f,x)^{(s)} &= h\sum_k^{m+s} f_k\delta_h^s b^m(x-hk) \\ &= h\sum_k^{m+s}\left\{\sum_{l=0}^{s+p}\frac{f^{(l)}(x)}{l!}(x_k-x)^l + R_{p+s}(x_k)\right\}\delta_h^s b^{m+s}(x-hk) \\ &= h\sum_k^{m}\sum_{l=0}^{s+p}\frac{f^{(l)}(x)}{l!}\delta_h^s(x_k-x)^l b^m(x-hk) + F_p^{sm}(x) \\ &= f^{(s)}(x) + \sum_{l=1}^{p}\frac{f^{(s+l)}(x)}{(s+l)!}\delta_h^s(x_k-x)^{l+s} + F_p^{sm}(x), \end{aligned}$$

where $F_p^{sm}(x) = h\sum_k^{m+s} R_{p+s}(x_k)\delta_h^s b^m(x-hk)$.

In accordance with (1.11) we have

$$S_0^{m+s}(f,x)^{(s)} = f^{(s)}(x) + \sum_{l=1}^{p} h^l\lambda_{l+s,s}^{m+s}(t) f^{(s+l)}(x) + F_p^{sm}(x). \tag{2.4}$$

Point out that $F_p^{sm}(x) = o(h^{p+s})$. Keeping in sight (1.12), we can write

$$\begin{aligned} S_0^{m+s}(f,x)^{(s)} &= f^{(s)}(x) + \sum_{l=1}^{[(m-1)/2]} \varepsilon_{2l} h^{2l} M_{2l}^{m+s} f^{(s+2l)}(x) \\ &+ \sum_{l=m}^{p}\varepsilon_l h^l\lambda_{l+s,s}^{m+s}(t) f^{(s+l)}(x) + F_p^{sm}(x). \end{aligned}$$

Here $\varepsilon_l = \begin{cases} 1 & \text{if } \; l \le p \\ 0 & \text{if } \; l > p. \end{cases}$

Note that if $m > p$ then there holds

$$S_0^{m+s}(f,x)^{(s)} = f^{(s)}(x) + \sum_{l=1}^{[(m-1)/2]} \varepsilon_{2l} h^{2l} M_{2l}^{m+s} f^{(s+2l)}(x) + F_p^{sm}(x).$$

We point out especially the case $p = m+1$:

$$\begin{aligned} S_0^{m+s}(f,x)^{(s)} &= f^{(s)}(x) + \sum_{l=1}^{[(m-1)/2]} M_{2l}^{m+s} h^{2l} f^{(s+2l)}(x) \\ &+ h^m\lambda_{m+s,s}^{m+s}(t) f^{(s+m)}(x) + h^m\lambda_{m+s+1,s}^{m+s}(t) f^{(s+m+1)}(x) + F_{m+1}^{sm}(x). \end{aligned}$$

The formulas (1.13) and (1.14) now enable us to formulate the following result:

Theorem 2.1. *Let $x \in [x_n^m, x_{n+1}^m]$, $t = (x - x_n^m)/h, t \in [o, 1]$. Then the following equality holds provided $m > 2, f \in C^{s+m+1}$:*

$$S_0^{m+s}(f,x)^{(s)} = f(x)^{(s)} + \sum_{l=1}^{[(m-1)/2]} M_{2l}^{m+s} h^{2l} f^{(s+2l)}(x)$$
$$+ h^m f(x)^{(s+m)} \Big[\frac{B_m(t)}{m!} + M_m^{m+s}\Big] \tag{2.5}$$
$$+ h^{m+1} f(x)^{(s+m+1)} \Big[\frac{mB_{m+1}(t)}{(m+1)!} + M_{m+1}^{m+s} + + F_{m+1}^{sm}(x),$$
$$F_{m+1}^{sm}(x) = h^{m+1} O(\omega(f^{(s+m+1)}, h)).$$

For the linear splines ($m = 2$) there holds:

$$S_0^{2+s}(f,x)^{(s)} = f(x)^{(s)} + \frac{h^2}{2} f(x)^{(s+2)}(\theta + \frac{s}{12}) + \tag{2.6}$$
$$+ \frac{1}{3}\theta(t - 1/2)h^3 f(x)^{(s+2)} + F_3^{s2}(x), \qquad \theta = t(1-t).$$

Remark 2.1. We see from (2.6) , (2.5) that the splines $S_)^{m+s}(f,x)^{(s)}$ reproduce exactly the derivatives $P_{s+1}^{(s)}(x)$ of polynomials of the degree $s + 1$ and if $f \in C^{s+p}, p > 1$, then

$$S_0^{m+s}(f,x)^{(s)} == f(x)^{(s)} + O(h^2).$$

3. SPLINES OF THE GREAT APPROXIMATION ACCURACY

Inspecting (2.5) , we see that a part of addends in asymptotic expansion of BSS have constant coefficients with derivatives. To increase the order of approximation we eliminate these addends by means of combining appropriate BSS.

Let $f \in C^{s+p}, 2 \le 2r \le p, 2r < m$. In accordance with (2.4)

$$S_0^{m+s+2r}(f,x)^{(s+2r)} \doteq \tag{3.1}$$

$$= \sum_{l=0}^{p-2r} h^l \lambda_{l+s+2r,s+2r}^{m+s+2r}(t) f^{(s+l+2r)}(x) - F_{p-2r}^{s+2r,m}(x).$$

Point out that $S_0^{m+s+2r}(f,x)^{(s+2r)} = h\sum_k^{m+s+2r} f_k \delta_h^{s+2r} b^m(x - hk)$ is a spline of the same degree $m - 1$ as $S_0^{m+s}(f,x)^{(s)}$. Suppose that an integer q is such that the inequalities hold: $2 \le 2q \le p, 2q < m$. Let us make the following linear combination of BSS of degree $m - 1$:

$$S_q^{m+s}(f,x)^{(s)} = \sum_{r=0}^{q} \beta_{2r}^{m+s} h^{2r} S_0^{m+s+2r}(f,x)^{(s+2r)} \doteq \tag{3.2}$$

$$= h\sum_k^{m+s} f_{qk}^{ms} \delta_h^s b^m(x - hk), f_{qk}^{ms} = f_k + \sum_{r=1}^{q} \beta_{2r}^{m+s} h^{2r} \delta_h^{2r} f_k, \beta_0^{m+s} = 1.$$

This is a spline of degree $m-1$. In accordance with (3.1) we have

$$S_q^{m+s}(f,x)^{(s)} = \sum_{r=0}^{q} \beta_{2r}^{m+s} \times \tag{3.3}$$

$$\times \sum_{l=0}^{p-2r} [h^{2r+l} \lambda_{l+s+2r,s+2r}^{m+s+2r}(t) f^{(s+l+2r)}(x) + h^{2r} F_{p-2r}^{s+2r,m}(x)]$$
$$= \sum_{r=0}^{q} \beta_{2r}^{m+s} \sum_{l=2r}^{p} [h^l \lambda_{l+s,s+2r}^{m+s+2r}(t) f^{(s+l)}(x) + h^{2r} F_{p-2r}^{s+2r,m}(x)]$$
$$= \sum_{l=0}^{q} h^{2l} f^{(s+2l)}(x) \sum_{r=0}^{l} \beta_{2r}^{m+s} M_{2(l-r)}^{m+s+2r}$$
$$+ \sum_{l=2q+1}^{p} h^l f^{(s+l)}(x) \sum_{r=0}^{q} \beta_{2r}^{m+s} \lambda_{l+s,s+2r}^{m+s+2r}(t) + F_{pq}^{s,m}(x),$$

where $$F_{pq}^{s,m}(x) = \sum_{r=0}^{q} \beta_{2r}^{m+s} h^{2r} F_{p-2r}^{s+2r,m}(x). \tag{3.4}$$

Let us choose now the coefficients β_{2r}^{m+s} in a way to eliminate the first group of addends in (3.3):

$$\sum_{r=0}^{l} \beta_{2r}^{m+s} M_{2(l-r)}^{m+s+2r} = 0, l = 1, 2, ..., q.$$

This relations produce the following recursive formula:

$$\beta_0^{m+s} = 1, \beta_{2n}^{m+s} = -\sum_{l=0}^{n} \beta_{2l}^{m+s} M_{2(n-l)}^{m+2l+s}, \beta_{2n+1}^{m+s} = 0. \tag{3.5}$$

To find β_{2r}^{m+s} one can use the generating function which was established in [19]:

$$\left(\frac{2\arcsin(v/2)}{v}\right)^m = \sum_{r=0}^{\infty} (-1)^r \beta_{2r}^m v^{2r}.$$

Substituting the derived values of β into (3.3), we obtain

$$S_q^{m+s}(f,x)^{(s)} = f^{(s)}(x)$$
$$+ + \sum_{l=2q+1}^{p} h^l f^{(s+l)}(x) \sum_{r=0}^{q} \beta_{2r}^{m+s} \lambda_{l+s,s+2r}^{m+s+2r}(t) + F_{pq}^{s,m}(x).$$

We distinguish two cases.

a) If $m > p$ then

$$S_q^{m+s}(f,x)^{(s)} = f^{(s)}(x) + \sum_{l=q+1}^{[p/2]} h^{2l} f^{(s+2l)}(x)\gamma_{2l,q}^{m+s} + F_{pq}^{s,m}(x),$$

$$\gamma_{n,q}^{m+s} = \sum_{r=0}^{q} \beta_{2r}^{m+s} M_{n-2r}^{m+s+2r}. \qquad (3.6)$$

As it easily seen from (3.5) that

$$\gamma_{2q+2,2q}^{m+s} = -\beta_{2q+2}^{m+s}. \qquad (3.7)$$

Point out that if $p \geq 2(q+1)$ then

$$S_q^{m+s}(f,x)^{(s)} = f^{(s)}(x) + O(h^{2(q+1)}),$$

and if $p < 2(q+1)$ then

$$S_q^{m+s}(f,x)^{(s)} = f^{(s)}(x) + o(h^p).$$

b)Let $p = m+1$. Then

$$\begin{aligned} S_q^{m+s}(f,x)^{(s)} &= f^{(s)}(x) \\ &+ \sum_{l=q+1}^{[m/2]+1} h^{2l} f^{(s+2l)}(x)\gamma_{2l,q}^{m+s} + h^m d_{mq}^{ms}(t) f(x)^{(s+m)} \\ &+ h^{m+1} d_{m+1,q}^{ms}(t) f(x)^{(s+m+1)} - F_{m+1,q}^{s,m}(x). \end{aligned} \qquad (3.8)$$

Here $\quad d_{mq}^{ms}(t) = \lambda_{m+s,s}^{m+s}(t) + \sum_{r=1}^{q} \beta_{2r}^{m+s} M_{m-2r}^{m+s+2r}$

$$== -\frac{B_m(t)}{m!} + \sum_{r=0}^{q} \beta_{2r}^{m+s} M_{m-2r}^{m+s+2r} = -\frac{B_m(t)}{overm!} + \gamma_{m,q}^{m+s}$$

in accordance with (3.6) and (1.13). Similarly we have

$$d_{m+1,q}^{ms}(t) = \frac{mB_{m+1}(t)}{(m+1)!} + \gamma_{m+1,q}^{m+s}.$$

Once m is given then the order of approximation can reach its climax if we choose $q = [(m-1)/2]$. The considerations carried out and (3.7) enable us to formulate the following important assertion.

Theorem 3.1.. *Let $f \in C^{s+m+1}, q = [(m-1)/2]$. If the coefficients β have been chosen in accordance with (3.5) then for any $x \in [x_n^m, x_{n+1}^m]$, $t = (x - x_n^m)/h$ the following relations are true:*

$$\begin{aligned} S_q^{m+s}(f,x)^{(s)} &= f(x)^{(s)} + h^m d_m^{ms}(t) f(x)^{(s+m)} + \\ &+ h^{m+1} d_{m+1}^{ms}(t) f(x)^{(s+m+1)} + F_{m+1,q}^{s,m}(x), \\ d_m^{ms}(t) &= -\frac{B_m(t)}{m!} - \beta_m^{m+s}, d_{m+1}^{ms}(t) \\ &= \frac{mB_{m+1}(t)}{(m+1)!} - \beta_{m+1}^{m+s}. \end{aligned}$$

Remark 3.1. One can conclude from the latter relation that the splines $S_q^{m+s}(f,x)^{(s)}$ with $q = [(m-1)/2]$ reproduce exactly the derivatives $P_{s+m-1}^{(s)}(x)$ of polynomials of degree $m+s+1$. This is the highest degree of a polynomial which can be reproduceed exactly by means of a spline of degree $m-1$. The span of this spline consists of $m+s+q$ points and is the least among splines which reproduce exactly the derivatives $P_{s+m-1}^{(s)}(x)$ of polynomials of degree $m+s+1$. We call the spline defined $S_q^{m+s}(f,x)^{(s)}$ with $q = [(m-1)/2]$ the Spline of Minimal Span (SMS).

Let us write now some initial values of β_{2r}^m:

$$\beta_2^m = \frac{-m}{24}, \beta_4^m = \frac{m(5m+22}{5760}, \quad \beta_6^m = -\frac{m(35m^2+462m+1528)}{2903040},$$
$$\beta_8^m = \frac{m(175m^3-462m^2+40724m-119856)}{138345900}.$$

4. EXPLICIT REPRESENTATION OF THE REMAINDER TERM OF APPROXIMATION

The asymptotic expansions of the remainder terms of approximation functions and their derivatives by local splines established in previous sections enable us sometimes to derive the explicit representations of these terms. Such representations turn establishing the exact estimates of errors of the approximation into a routine procedure. The explicit representations are obtained for BSS of arbitrary degree which reproduce exactly the derivatives $P_{s+1}^{(s)}(x)$ of polynomials of degree $s+1$, for SMS of the first – fifth degrees which approximate $f^{(s)}$ (s is arbitrary). One of Kornejchuk's results [12] enables us to establish the explicit representations of the remainder terms for SMS of arbitrary odd degrees approximating f. We study the splines mentioned by means of the integral representations of the remainder terms. Furthermore the so called Peano kernels appear.

4.1. Peano kernels. Let $\mathbb{T}$ be a linear operator $C \to C$. Denote by $\mathbb{T}(f,x)$ the result of the application this operator to a function f- Suppose that operator $\mathbb{T}_s^m$ is such that there holds the following relation for any polynomial $P_{m+s}(x)$ of the degree $m+s$:

$$\mathbb{T}_s^m(P_{m+s},x) = P_{m+s}^{(s)}(x). \tag{4.1}$$

If $f \in L_\infty^{m+s+1}$ then provided $x \in [a,b]$

$$f(x) = P_{s+m}(x) + \frac{1}{(m+s)!}\int_a^b (x-u)_+^{m+s} f^{(m+s+1)}(u)du$$
$$\Leftrightarrow P_{s+m}^{(s)}(x) = f(x)^{(s)} - \frac{1}{(m)!}\int_a^b (x-u)_+^m f^{(m+s+1)}(u)\,du.$$

In accordance with (4.1) we have

$$\mathbb{T}_s^m(f,x) = P_{m+s}^{(s)}(x) + \frac{1}{(m+s)!}\int_a^b \mathbb{T}_s^m((x-u)_+^{m+s},x) f^{(m+s+1)}(u)du$$

$$= f(x)^{(s)} - \int_a^b K_s^m(x,u) f^{(m+s+1)}(u)du, \qquad (4.2)$$

$$K_s^m(x,u) = \frac{1}{m!}(x-u)_+^m - \frac{1}{(m+s)!}\mathbb{T}_s^m((x-u)_+^{m+s},x). \qquad (4.3)$$

The function $K_s^m(x,u)$ we call the Peano kernel of the operator $\mathbb{T}_s^m$. Thus we have

$$f(x)^{(s)} - S(f,x)^{(s)} = \int_a^b K_s^m(x,u) f^{(m+s+1)}(u)du. \qquad (4.4)$$

Lemma 4.1. *([30]).Let $\mathbb{T}_s^m$ be a linear operator and $\phi(x)$ be any continuous function. If the following representation is true for each function $f \in C^{s+m+1}$:*

$$\mathbb{T}_s^m(f,x) = f(x)^{(s)} + h^{m+1}\phi(x) f(x)^{(s+m+1)} + h^{m+1}O(\omega(f^{(s+m+1)},h)), \qquad (4.5)$$

then: **1.***The relation (4.1) holds.* **2.***Provided $K_s^m(x,u)$ is the Peano kernel of type (4.3), the following relation is true:*

$$\int_a^b K_s^m(x,u)du = -h^{m+1}\phi(x).$$

Corollary 4.1. *Let the representation (4.5) be true for each function $f \in C^{s+m+1}$. If the Peano kernel $K_s^m(x,u)$ of the operator $\mathbb{T}_s^m$ doesn't change the sign within its domain of definition then the following relation holds:*

$$f(x)^{(s)} - \mathbb{T}_s^m(f,x) = -h^{m+1}\phi(x) f(\xi)^{(s+m+1)}, \xi \in [a,b].$$

The Peano kernels of operators, connected with splines, which are studied in this section do not change the sign. We adduce the proof of this fact for SMS of second and fifth degrees only. The proof for other splines the reader can find in [30], [31].

Point out that when $u > x$, we have in accordance with (4.18):

$$K_s^m(x,u) = -\frac{1}{(m+s)!}\mathbb{T}_s^m((x-u)_+^{m+s},x) \qquad (4.6)$$

If m is odd then it can be easily verified that

$$K_s^m(x,u) = \frac{1}{m!}(u-x)_+^m) - \frac{1}{(m+s)!}(-1)^s\mathbb{T}_s^m((u-x)_+^{m+s},x), \qquad (4.7)$$

and, provided $x > u$,

$$K_s^m(x,u) = \frac{(-1)^{s+1}}{(m+s)!}\mathbb{T}_s^m((u-x)_+^{m+s},x). \qquad (4.8)$$

Assume that a local spline $S(f,x)^{(s)}$ reproduces exactly $P^{(s)}_{s+m-1}(x)$ and, given an x, the span of this spline is contained in $[a,b]$. Suppose that the following representation is true with any function $f \in C^{s+m+1}$:

$$S(f,x)^{(s)} = f(x)^{(s)} + h^m\psi(x)f(x)^{(s+m)} \tag{4.9}$$
$$+h^{m+1}\phi(x)f(x)^{(s+m+1)} + h^{m+1}O(\omega(f^{(s+m+1)};h)).$$

Here $\phi(x), \psi(x)$ are some continuous functions. Let us define the operator $\mathbb{T}^m_s$ on the space C^{s+m}:

$$\mathbb{T}^m_s(f,x) = S(f,x)^{(s)} - h^m\psi(x)f(x)^{(s+m)}. \tag{4.10}$$

There holds the following relation provided $u \geq x$

$$K^m_s(x,u) = -\frac{1}{(m+s)!}\mathbb{T}^m_s((x-u)^{m+s}_+, x) \tag{4.11}$$
$$= -\frac{1}{(m+s)!}S((x-u)^{m+s}_+, x)^{(s)} + h^m\psi(x)(x-u)^0_+$$
$$= -\frac{1}{(m+s)!}S((x-u)^{m+s}_+, x)^{(s)}.$$

\# If $u < x$ and m is odd then

$$K^m_s(x,u) = \frac{(-1)^{s+1}}{(m+s)!}\, S((u-x)^{m+s}_+, x)^{(s)}. \tag{4.12}$$

The operators $\mathbb{T}^m_s$ defined above are used to study SMS of even degrees. In the other cases we suppose that the splines $S(f,x)^{(s)}$ under consideration reproduce exactly $P^{(s)}_{s+m}(x)$. In those cases we define the operator as follows:

$$\mathbb{T}^m_s(f,x) = S(f,x)^{(s)} \tag{4.13}$$

The latter relation means that in (4.9) $\psi(x) \equiv 0$. The formulas (4.10) and (4.11) remain, of course.

Lemma 4.2. *([30]).Let $x \in [x^{p-s}_n, x^{p-s}_{n+1}], x = x^{p-s}_n + ht$. Denote*

$$x' = x^{p-s}_{n+1} - ht, u = x^{p-s}_n + hv, u' = x^{p-s}_{n+1} - hv, t, v \in [0,1].$$

Suppose that a spline $S^p(f,x)^{(s)}$ of the degree $p-s-1$ reproduces exactly $P^{(s)}_{s+m}(x)$, m is an odd number, $K^m_s(x,u)$ is the Peano kernel connected with this spline. Then: 1).$K^m_s(x+hN, u+hN) = K^m_s(x,u)$ for each integer N. 2).$K^m_s(x,u) = K^m_s(x',u')$.

Corollary 4.2. *Suppose the assumptions of Lemma 4.2 hold. If $K^m_s(x,u) \geq 0 (\leq 0)$ by $u \geq x$ then $K^m_s(x,u) \geq 0 (\leq 0)$ within the entire its domain of definition.*

Kornejchuk in the paper [12] presents without a proof the following assertion concerning the Peano kernels $K(x,u)$ connected with SMS of arbitrary odd degree $2q-1$ which reproduces exactly polynomials of the degree $2q-1$.

Proposition 4.1. *The Peano kernels $K(x,u)$ do not change their signs within their entire domain of definition.*

Proceed now to considering specific splines. It turns out that for all of splines studied, except SMS of even degrees, the first term of the asymptotic expansion determines the form of the remainder terms. For an SMS of even degrees the form of the remainder term is determined by two initial terms of the asymptotic expansion. The asymptotic formulas we use follow from the general formulas of Section 3.

4.2.Representation of the remainder term of approximation.. Throughout this section we shall use the notations: $\kappa = t - 1/2$, $\theta = t(1-t)$, $\omega(f^{(s+2)}, h)$ is the modulus of continuity.

a)BSS of degree $p-1$ approximating $f^{(s)}$:.

$$S_0^{p+s}(f,x)^{(s)} = h\sum_k^{p+s} f_k \delta_h^s b^p(x-hk). \tag{4.14}$$

Provided $p > 2, f \in C^{s+2}$, the following relation is true:

$$S_0^{p+s}(f,x)^{(s)} = f(x)^{(s)} + h^2\frac{p+s}{24}f(x)^{(s+2)} + h^2 O(\omega(f^{(s+2)}, h)). \tag{4.15}$$

For the linear splines ($p=2$):

$$S_0^{2+s}(f,x)^{(s)} = f(x)^{(s)} + \frac{h^2}{2}f(x)^{(s+2)}(\theta + \frac{s}{12}) + h^2 O(\omega(f^{(s+2)}, h)). \tag{4.16}$$

These splines reproduce exactly $P_{s+1}^{(s)}(x)$. If $u > x$ then

$$K_s^1(x,u) = \frac{-h}{(1+s)!}\sum_k^p b^p(x-hk)\delta_h^s(h(k+j)-u)_+^{1+s}.$$

But $\delta_h^s(z-u)_+^{1+s} = (\zeta - u)_+^1 \geq 0$. Since $b^p(x-hk) \geq 0 \forall t$ then $K_s^1(x,u) \leq 0$ provided $u > x$.

Corollary 4.1 and formulas (4.15), (4.16) enable us to formulate the following assertion.

Theorem 4.1. *If $p > 2, f \in C^{s+2}$ then the following relation holds:*

$$S_0^{p+s}(f,x)^{(s)} = f(x)^{(s)} + h^2\frac{p+s}{24}f(\xi)^{(s+2)}.$$

For the linear splines ($p=2$):

$$S_0^{2+s}(f,x)^{(s)} = f(x)^{(s)} + \frac{h^2}{2}f(\xi)^{(s+2)}(\theta + \frac{s}{12}).$$

If $x \in [x_n^p, x_{n+1}^p]$, then $\xi \in [x_n^p + h(1 - \frac{p+s}{2}), x_{n+1}^p + h\frac{p+s}{2}]$.

b)Cubic SMS reproducing exactly $P_{s+3}^{(s)}(x)$:.

$$S_1^{4+s}(f,x)^{(s)} = h\sum_k^{4+s}(f_k - h^2\frac{4+s}{24}\delta_h^2 f_k)\delta_h^s b^4(x-hk).(4.17)$$

If $f \in C^{s+4}$ then the following relation is true:

$$S_1^{4+s}(f,x)^{(s)} = f(x)^{(s)} - \frac{h^4}{24}f(x)^{(s+4)}(\theta^2 + \frac{5s^2+62s+160}{240}) + +h^4 O(\omega(f^{(s+4)},h)).$$

Theorem 4.2. *Let $S_1^{4+s}(f,x)^{(s)}$ be an SMS of the third degree defined by (4.17) and s be arbitrary natural number. Then $\forall f \in C^{s+4}$ the following relation holds:*

$$f(x)^{(s)} - S_1^{4+s}(f,x)^{(s)} = \frac{h^4}{24}f(\xi)^{(s+4)}\left(\theta^2 + \frac{5s^2+62s+160}{240}\right)$$

. If $x \in [hn, h(n+1)]$ then $\xi \in \left[h(n-2-\frac{s}{2}), h(n+3+\frac{s}{2})\right]$.

c)Splines of the fifth degree which reproduce exactly $P_{s+3}^{(s)}(x)$.. These will be needed for the study of the fifth degree SMS .

$$S_1^{6+s}(f,x)^{(s)} = h\sum_k^{4+s}(f_k - h^2\frac{6+s}{24}\delta_h^2 f_k)\delta_h^s b^6(x-hk). \qquad (4.18)$$

Theorem 4.3. *Let $S_1^{6+s}(f,x)^{(s)}$ be a spline of the fifth degree defined by formula (4.18) and s be an arbitrary natural number. Then $\forall f \in C^{s+4}$ the following relation holds:*

$$f(x)^{(s)} - S_1^{6+s}(f,x)^{(s)} = \frac{h^4}{5760}f(\xi)^{(s+4)}(6+s)(5s+52).$$

If $x \in [hn, h(n+1)]$ then $\xi \in \left[h(n-3-\frac{s}{2}), h(n+4+\frac{s}{2})\right]$.

d)Fifth degree SMS which reproduce exactly $P_{s+5}^{(s)}(x)$:.

$$S_2^{6+s}(f,x)^{(s)} = h\sum_k(f_k - h^2\frac{6+s}{24}\delta_h^2 f_k \qquad (4.19)$$

$$+\frac{h^4}{5760}(6+s)(5s+52)\delta_h^4 f_k)\delta_h^s b^6(x-hk).$$

If $f \in C^{s+6}$ then the following relation is true:

$$S_2^{6+s}(f,x)^{(s)} = f(x)^{(s)} + \frac{h^6}{720}f(x)^{(s+6)}[\theta^2(\theta+1/2) + \frac{35s^3+1092s^2+10852s+33264}{4032}] + h^6 O(\omega(f^{(s+6)},h)). \qquad (4.20)$$

Lemma 4.2. *The Peano kernel $K_0^5(x,u) \leq 0$ within its entire domain of definition.*

This is a special case of Proposition 4.1.

Lemma 4.3. *Let $S_2^{6+s}(f,x)^{(s)}$ be SMS of fifth degree defined by (4.19) and s be an arbitrary natural number. Then its Peano kernel $K_s^5(x,u) \leq 0$ within its entire domain of definition.*

Proof. Take use of the induction. We assume that $K_r^5(x,u) \leq 0$ and prove that $K_{r+1}^5(x,u) \leq 0$. In accordance with (4.19)

$$\begin{aligned}
S_4^{6+r+1}(F,x)^{(r+1)} =& h \sum_k^{6+r+1} (F(x_k) - h^2 \frac{7+r}{24}\delta_h^2 F(x_k) \\
&+ \frac{h^4}{5760}(7+r)(5r+57)\delta_h^4 F(x_k)\delta_h^{r+1} b^6(x-hk)) \\
&= h \sum_k^{6+r} (\delta_h F(x_k) - h^2 \frac{6+r}{24}\delta_h^2 \delta_h F(x_k) \\
&+ \frac{h^4}{5760}(6+r)(5r+52)\delta_h^4 \delta_h F(x_k)\delta_h^r b^6(x-hk)) \\
&- \frac{h^3}{24}\sum_k^{6+r+1} (\delta_h F(x_k - h^2 \frac{6+r}{24}\delta_h^2 \delta_h^3 F(x_k)\delta_h^r(b^6(x-hk)) \\
&+ \frac{3h^5}{640}\sum_k \delta_h^5 F(x_k)\delta_h^r b^6(x-hk) \\
=& S_2^{6+r}(\delta_h F,x)^{(r)} - \frac{h^2}{24} S_1^{6+r}(\delta_h^3 F,x)^{(r)} + \frac{3h^4}{640} S_0^{6+r}(\delta_h^5 F,x)^{(r)}.
\end{aligned}$$

If $F \in L_\infty^{6+r}$ then the formulas (4.4) and Theorems 4.1, 4.3 enable us to write

$$\begin{aligned}
S_2^{6+r+1}(F,x)^{(r+1)} &= \delta_h F^{(r)}(x) \\
&- \delta_h \int_a^b K_r^5(x,y)F^{(6+r)}(y)dy - \frac{h^2}{24}\delta_h^3 F^{(r)}(x) \\
&+ \frac{h^6(6+r)(5r+52)}{138240}\delta_h^3 F^{(r+4)}(\xi) \\
&+ \frac{3h^4}{640}\delta_h^5 F^{(r)}(x) + \frac{h^6}{5120}\delta_h^5 F^{(r+2)}(\vartheta).
\end{aligned}$$

If $u > x$ then

$$K^5_{r+1}(x,u) = -\frac{1}{(6+r)!}S_2^{6+r+1}((x-u)_+^{6+r},x)^{(r+1)}$$
$$= -\frac{1}{120}\{N(x,u) + M(x,u)),$$
$$M(x,u) = -720\delta_h\int_a^b K^5_r(x,y)b^1(y-u)dy$$
$$+\frac{h^6(6+r)(5r+52)}{384}b^3(\xi) + +\frac{3h^4}{512}b^5(\vartheta),$$
$$N(x,u) = \delta_h(x-u)_+^6 - \frac{h^2}{24}\delta_h^3(x-u)_+^6 + \frac{3h^4}{640}\delta_h^5(x-u)_+^6.$$

Since, by supposition, $K^5_r(x,u) \le 0$, we have $M(x,u) > 0$. Let us consider the function $N(x,u)$. If $u \ge x+3h/2$ then $N(x,u) \ge 0$. Let $h/2 \le u-x \le 3h/2$. Denote $x-u = hz, z \in [-\frac{3}{2}, -\frac{1}{2}]$. Then

$$N(x,u) = -h^5\left(\frac{1}{24}(z+3/2)^6 - \frac{3}{640}[(z+5/2)^6 - 5(z+3/2)^6]\right) > 0.$$

Assume now that $0 \le u-x \le h/2, z \in [-1/2, 0]$. Then

$$N(x,u) = h^5\Bigg((z+1/2)^6 - \frac{1}{24}[(z+3/2)^6 - 3(z+1/2)^6]$$
$$+\frac{3}{640}[(z+5/2)^6 - 5(z+3/2)^6 + 10(z+1/2)^6]\Bigg)$$
$$= \frac{h^5}{1920}[2250(z+1/2)^6 - 125(z+3/2)^6 + 9(z+5/2)^6] > 0$$

. Thus $K^5_{r+1}(x,u) \le 0$ as $x < u$ and, consequently, this inequality is true within the entire domain of definition of the function $K^5_{r+1}(x,u)$.

As a consequence of this lemma and Corollary 4.1 we have

Theorem 4.4. *Let $S_2^{6+s}(f,x)^{(s)}$ be the SMS of the fifth degree defined by (4.19) and s be arbitrary natural number. Then $\forall f \in C^{s+6}$ the following relation holds:*

$$f(x)^{(s)} - S_2^{6+s}(f,x)^{(s)}$$
$$= \frac{h^6}{720}f(\xi)^{(s+6)}\left[\theta^2(\theta+\frac{1}{2}) + \frac{35s^3+1092s^2+10852s+33264}{4032}\right]$$

If $x \in [hn, h(n+1)]$ then $\xi \in [h(n-4-\frac{s}{2}), h(n+5+\frac{s}{2})]$.

e) Splines of odd degrees. Theorem 3.1 and proposition 4.1 lead us to establishing the explicit representation of the remainder term of approximation for SMS of arbitrary odd degrees approximating f

Theorem 4.5. *Let $S_q^{2m}(f,x), q=m-1$, be a SMS of the degree $2m-1$ which reproduces exactly polynomials of the same degree and m be arbitrary natural number. Then $\forall f \in C^{2m}$ the following relation holds:*

$$f(x) - S_q^{2m}(f,x) = h^{2m} f^{(2m)}(\xi)\left(\frac{B_{2m}(t)}{(2m)!} - \beta_{2m}^{2m}\right).$$

If $x \in [hn, h(n+1)]$ then $\xi \in [h(n-2m+2), h(n+2m-1)]$.

The aforesaid consideration enable us to come up with the following

Conjecture 4.1. *Let $S_q^{2m+s}(f,x)^{(s)}, q = m-1$, be SMS of the degree $2m-1$ which reproduces exactly $P_{2m-1+s}^{(s)}(x)$ and m, s be arbitrary natural numbers. Then $\forall f \in C^{2m+s}$ the following relation holds:*

$$f^{(s)}(x) - S_q^{2m+s}(f,x)^{(s)} = h^{2m} f^{(2m+s)}(\xi)\left(\frac{B_{2m}(t)}{(2m)!} - \beta_{2m+s}^{2m}\right).$$

If $x \in [hn, h(n+1)]$ then $\xi \in [h(n-2m+2-s/2), h(n+2m-1+s/2)]$.

Turn now to splines of even degrees.

f) SMS of the second degree which reproduce exactly $P_{s+2}^{(s)}(x)$:

$$S_1^{3+s}(f,x)^{(s)} = h\sum_k^{3+s}(f_k - h^2\frac{3+s}{24}\delta_h^2 f_k)\delta_h^s b^3(x-hk). \qquad (4.20)$$

If $f \in C^{s+4}$ then the following relation holds:

$$S_1^{3+s}(f,x)^{(s)} = f(x)^{(s)} + \frac{h^3}{6}f(x)^{(s+3)}\theta_K$$
$$+\frac{h^4}{24}f(x)^{(s+4)}(3\theta^2 - \frac{5s^2+52s+135}{240}) + h^4 O(\omega(f^{(s+4)}, h)).$$

To study the spline $S_1^{3+s}(f,x)^{(s)}$, as well as SMS of fourth degree, we use the operators $\mathbb{T}_s^m$ of such kind as in formula (4.10). Namely,

$$\mathbb{T}_s^3(f,x) = S_1^{3+s}(f,x)^{(s)} - \frac{h^3}{6}f(x)^{(s+3)}\theta_K.$$

Then, provided $u > x$, the Peano kernel is

$$K_0^3(x,u) = -\frac{1}{(3+s)!}S_1^{3+s}((x-u)_+^{3+s}, x)^{(s)}.$$

We consider first the case when $s=0$. It is easy to transform the spline to the following form:

$$S_1^3(f,x) = -\frac{1}{16}\sum_{i=-2}^{2} f_{n+i}P_i(t), t = h^{-1}(x - h(n-1/2)),$$
$$P_2(t) = t^2, P_1(t) = (1+t)^2 - 13t^2,$$
$$P_0(t) = (2+t)^2 - 13(1+t)^2 + 34t^2, P_{-i}(t) = P_i(1-t).$$

Provided $u > x \iff v > t$, the Peano kernel can be written as follows:

$$K_0^3(x,u) = \frac{-1}{6}S_1^3((x-u)_+^3, x) = \frac{h^3}{96}\sum_{i=0}^{2}(i+1/2-v)_+^3 P_i(t).$$

Lemma 4.4. *The Peano kernel of the operator* $\mathbb{T}_0^3$
– $K_0^3(x,u) \geq 0$ *within its entire domain of definition.*

Proof. It can be verified immediately that $P_0(t) < 0, P_2(t) \geq 0 \, \forall t \in [0,1]$. Denote by $\mathcal{D}$ the domain $\{x \in (h(n-0.5), h(n+0.5)]; u \in [x, h(n+2.5))\}$. Assume that in some point $H_0(x_0,u_0) \in \mathcal{D}$ the both following relations hold simultaneously:

$$K_0^3(x_0,u_0) = 0 \iff \sum_{i=0}^{2}(i+1/2-v_0)_+^3 P_i(t_0) = 0,$$

$$K_0^3(x_0,u_0)\prime = 0 \iff \sum_{i=1}^{4}(i+1/2-v_0)_+^2 P_i(t_0) = 0.$$

Substituting one relation into the other, we have:

$$(2.5-v_0)_+^2 P_2(t_0) = (0.5-v_0)_+^2 P_0(t_0),$$

that is impossible with $t \in [0,1], v < 2.5$. Therefore, if there exists a point $H_0 \in \mathcal{D}$ such that $K_0^3(x_0,u_0) = 0$ then
$K_0^3(x_0,u_0)0$, and the equation $K_0^3(x,u) = 0$ defines an implicit function $u(x)$ on the interval $(hn, h(n+1)]$. If the variable u with any given x passes through the value $u(x)$ then the function $K_0^3(x,u)$ must change its sign.

Let us consider first the subdomain $\mathcal{D}^1 = \{x \in (h(n-0.5), h(n+0.5)]; u \in [x, h(n+1.5)]\}$. It can be verified easily that if $u = x$ then

$$96h^{-3}K_0^3(u,u) = \sum_{i=0}^{2}(i+1/2-v)_+^3 P_i(v) > 0.$$

If $x = h(n+0.5), u \geq h(n+0.5)$ then

$$96h^{-3}K_0^3(h(n+0.5),u) = (2.5-v)_+^3 - 9(1.5-v)_+^3 \geq 0 \quad \text{as} \quad v \geq 1.$$

If $u \geq h(n+1.5)$ then $96h^{-3}K_0^3(x,u) = (2.5-v)_+^3 t^3 \geq 0$. Therefore neither in the domain $\mathcal{D}^1$, nor by $u \geq h(n+1.5)$, the function $K_0^3(x,u)$ can change its sign. Hence it follows the assertion of the lemma.

The proof of the following proposition is related to the proof of Lemma 4.3.

Lemma 4.5. ([30])*Let* $S_1^{3+s}(f,x)^{(s)}$ *be an SMS of the second degree defined by (4.20) and* s *be an arbitrary natural number. Then its Peano kernel* $K_s^3(x,u) \geq 0$ *within its entire domain of definition.*

Hence it follows

Theorem 4.5. *Let* $S_1^{3+s}(f,x)^{(s)}$ *be an SMS of the second degree defined by (4.20) and* s *be arbitrary natural number. Then* $\forall f \in C^{s+4}$ *the following relation holds:*

$$f(x)^{(s)} - S_1^{3+s}(f,x)^{(s)} = -\frac{h^3}{6} f(x)^{(s+3)} \theta_K$$

$$+\frac{h^4}{24}f(x)^{(s+4)}(3\theta^2 - \frac{5s^2+52s+135}{240}).$$

If $x \in [h(n-1/2), h(n+1/2)]$ *then* $\xi \in [h(n-2-\frac{s}{2}), h(n+2+\frac{s}{2})]$.

g) Fourth degree SMS *which reproduce exactly* $P^{(s)}_{s+5}(x)$:

$$S_2^{5+s}(f,x)^{(s)} = h\sum_k (f_k - h^2\frac{5+s}{24}\delta_h^2 f_k \tag{4.21}$$

$$+\frac{h^4}{5760}(5+s)(5s+47)\delta_h^4 f_k)\delta_h^s b^5(x-hk).$$

Theorem 4.6. *([30]) Let* $S_2^{5+s}(f,x)^{(s)}$ *be an SMS of the fourth degree defined by (4.21) and* s *be arbitrary natural number. Then* $\forall f \in C^{s+6}$ *the following relation holds:*

$$f(x)^{(s)} - S_2^{5+s}(f,x)^{(s)} = \frac{h^5}{120}f(x)^{(s+5)}\theta\kappa(\theta+\frac{1}{3})$$
$$+\frac{h^6}{720}f(\xi)^{(s+6)}(5\theta^2(\theta+1/2) - \frac{35s^3+987s^2+8773s+24045}{4032})$$

If $x \in [h(n-1/2), h(n+1/2)]$ *then* $\xi \in [h(n-4-\frac{s}{2}), h(n+4+\frac{s}{2})]$.

Theorem 3.1 enable us to propound the following

Conjecture 4.2. *Let* $S_q^{2m-1+s}(f,x)^{(s)}, q = m-1$, *be an SMS of the degree* $2m-2$ *which reproduces exactly* $P^{(s)}_{2m-2+s}(x)$ *and* m, s *be arbitrary natural numbers. Then* $\forall f \in C^{2m+s}$ *the following relation holds:*

$$f^{(s)}(x) - S_q^{2m-1+s}(f,x)^{(s)}$$
$$= h^{2m-1}f^{(2m-1+s)}(x)\frac{B_{2m-1}(t)}{(2m-1)!}$$
$$- h^{2m}f^{(2m+s)}(\xi)\left((2m-1)\frac{B_{2m}(t)}{(2m)!} - \beta_{2m}^{2m-1+s}\right).$$

5.QUASIINTERPOLATING SPLINES

Denote $y_k(t) = B_k(t) - B_k, z_k(t) = B_k(t) - \tilde{B}_k$, where $B_k(t)$ is the Bernoulli polynomial of a degree k, $B_k = B_k(0)$ is the Bernoulli number and $\tilde{B}_k = B_k(1/2)$. It is well known (see [6] e.g.) that

$$y_k(1) = y_k(0) = 0, \quad y_1(\frac{1}{2}) = \frac{1}{2}, \quad y_{2n+1}(1/2) = z_{2n+1}(0) = 0,$$
$$y_{2n}(1/2) = -(2-2^{1-2n})B_{2n}, \quad y_{2n+1}(t) = z_{2n+1}(t) = B_{2n+1}(t), n = 1,2,....$$

We consider first splines of odd degrees ($m = 2q$). In accordance with Theorem 3.1, an SMS can be written in the following form:

$$S_{q-1}^{2q+s}(f,x)^{(s)} = f(x)^{(s)} - h^{2q}\left(\frac{y_{2q}(t)}{(2q)!} + \kappa^s\right) f(x)^{(2q+s)} \tag{5.1}$$

$$+ 2qh^{2q+1}\frac{y_{2q+1}(t)}{(2q+1)!} f(x)^{(2q+s+1)} + o(h^{2q+1}),$$

$$\kappa_{2q}^s = \frac{B_{2q}}{(2q)!} + \beta_{2q}^{2q+s}.$$

Hence it is seen that the approximating properties of the SMS can be improved via eliminating the constants κ_{2q}^s in the formula (5.1). Let us construct the spline

$$S_Q^{2q,s}(f,x) = h\sum_k^{2q+s} f_{Qk}^{2q,s}\delta_h^s b^{2q}(x-hk), \tag{5.2}$$

$$f_{Qk}^{2q,s} = f_k + \sum_{i=1}^{q-1}\beta_{2i}^{m+s}h^{2r}\delta_h^{2r}f_k + h^{2q}\kappa_{2q}^s\delta_h^{2q}f_k. \tag{5.3}$$

The span of this spline contains $4q+s$ points, two points more then the span of the corresponding SMS. If $f \in C^{2q+s+1}$, $x = h(n+t)$ then the asymptotic expansion holds:

$$S_Q^{2q,s}(f,x) = f(x)^{(s)} - h^{2q}\frac{y_{2q}(t)}{(2q)!}f(x)^{(2q+s)} \tag{5.4}$$

$$+ 2qh^{2q+1}\frac{y_{2q+1}(t)}{(2q+1)!}f(x)^{(2q+s+1)} + o(h^{2q+1}).$$

For SMS of even degrees ($m = 2q-1$) the following expansion holds provided $f \in C^{2q+s}$, $x = h(n-1/2+t)$:

$$S_{q-1}^{2q-1+s}(f,x)^{(s)} = f^{(s)}(x) - h^{2q-1}\frac{z_{2q-1}(t)}{(2q-1)!}f(x)^{(2q-1+s)} \tag{5.5}$$

$$+ h^{2q}f^{(2q+s)}(x)\left((2q-1)\frac{z_{2q}(t)}{(2q)!} - \kappa_{2q-1}^s\right) + o(h^{2q}).$$

Considerations similar to aforesaid ones lead to constructing a spline which possess the great approximation accuracy:

$$S_Q^{2q-1,s}(f,x) = h\sum_k f_{Qk}^{2q-1,s}\delta_h^s b^{2q-1}(x-hk), \tag{5.6}$$

$$f_{Qk}^{2q-1,s} = f_k + \sum_{i=1}^{q-1}\beta_{2i}^{m+s}h^{2i}\delta_h^{2i}f_k + h^{2q}\kappa_{2q-1}^s\delta_h^{2q}f_k.$$

There holds the asymptotic expansion:

$$S_Q^{2q-1,s}(f,x) = f(x)^{(s)} - h^{2q-1}\frac{z_{2q-1}(t)}{(2q-1)!}f(x)^{(2q-1+s)} \tag{5.7}$$
$$+ (2q-1)h^{2q}\frac{z_{2q}(t)}{(2q)!}f(x)^{(2q+s)} + o(h^{2q}).$$

The formulas (5.4), (5.7) imply that whereas the approximation of $f(x)^{(s)}$ by meanf of the splines $S_Q^{m,s}(f,x)$, inside the interval $[x_n, x_{n+1}]$ is of the order $O(h^m)$,, the approximation in the points $x = x_n, (t = 0$ if $m = 2q, t = 1/2$ if $m = 2q-1)$ is of the order $o(h^{m+1})$. If f is a polynomial of the degree $m+s+1$, then the spline $S_Q^{m,s}(f,x)$ reduces to the interpolating one. These are the reasons to call the splines $S_Q^{m,s}(f,x)$ the Quasyinterpolating Splines (QIS). It should be pointed out that the cubic splines $S_Q^{4,0}(f,x)$ quasiinterpolating function (not derivatives) had been constructed via another approaches in [11], [52], [69]. Their approximating properties haven't been studied systematically.

Remark 5.1. Emphasize that the spline $S_Q^{m,s}(f,x)$, being a local one and using the closest to the point x *samples of a function* f only, *approximates the derivative* $f(x)^{(s)}$ with almost the same accuracy as the spline interpolating samples of $f(x)^{(s)}$. Due to this property, quasiinterpolating splines $S_Q^{m,s}(f,x)$ perform recovering functions f and their derivatives $f^{(s)}$ from samples of functions f with distinguished accuracy. We shall cite further corresponding evaluations for the cubic splines.

Remark 5.2. Point out that, varying the constant κ_{2q}^s, in the formula (5.3) we can provide vanishing the first term of asymptotic expansion of splines of odd degree in two arbitrary points of each interval $[x_n, x_{n+1}]$ which are symmetric with respect to the centre of the interval. The appropriate choice of the parameter provides vanishing in the centre of the interval the two initial terms of asymptotic expansion. The related assertion holds for splines of even degrees.

Denote by $S^m(f,x)$ a spline of the degree $m-1$ which interpolates a function f in the points $\{x_k\}_{-\infty}^{\infty}$. Since the spline $S_Q^{m,0}(f,x)$ is interpolating one provided f is a polynomial of the degree $m+1$, then the formulas (5.4), (5.7) with $s=0$ yield the asymptotic expansion for cardinal interpolating splines $S^m(f,x)$.

It should be pointed out that asymptotic expansions for periodic interpolating splines and their derivatives have been established in [46],[47] via another approach.

Now we present evaluations of the errors which appear in process of approximating functions and their derivatives $f(x)^{(s)}, s = 0,1,2$ by means of cubic quasiinterpolating splines.

Theorem 5.1. *Assume* $f^{(s+4)} \in L_\infty, s = 0,1,2$. *Then the following inequalities hold:*

$$|S_Q^{4,s}(f,x) - f^{(s)}(x)| \le h^4 R_s \|f^{(4+s)}\|_\infty,$$

$$R_0 \cong 1.60881\cdot 10^{-2}, R_1 \cong 1.93092\cdot 10^{-2}, R_2 \cong 2.29846\cdot 10^{-2}.$$

If $f \in C^{6+s}, s = 0, 1, 2$ then the following relations hold in the nodes of the spline ($x = x_k^s$):

$$S_Q^{4,s}(f, x_k^s) - f^{(s)}(x_k^s) = -h^6 \rho_s f^{(6+s)}(\xi),$$

$$\rho_0 = \frac{1}{216}, \rho_1 = \frac{7031}{967680}, \rho_2 = \frac{1}{189}, \xi \in [x_{n-2+s}, x_{n+2+s}].$$

The constants occurring in the relations are unimproveable on the given classes of functions.

For comparison, we adduce the corresponding evaluations of errors for interpolating splines as well as for SMS.

If $S(f, x)$ is a periodic interpolating spline of third degree constructed by values $\{f(x_k)\}$ then the following inequality holds provided $f^{(4)} \in L_\infty$ (see [25], e.g.):

$$|S(f, x) - f(x)| \leq h^4 \Omega \|f^{(4+s)}\|, \Omega = \frac{5}{384} \cong 1.3 \cdot 10^{-2}.$$

The constant Ω occurring in the inequality is unimproveable on the classes of functions considered. For SMS the following unimproveable evaluations of errors are being derived from Theorem 4.2.

If $f^{(s+4)} \in L_\infty, s = 0, 1, 2$ then the following inequalities hold:

$$|S_r^{4+s}(f, x)^{(s)} - f^{(s)}(x)| \leq h^4 T_s \|f^{(4+s)}\|,$$

$$T_0 = \frac{35}{1152} \cong 3.04 \cdot 10^{-2}, T_1 = \frac{121}{2880} \cong 4.2 \cdot 10^{-2},$$

$T_2 = \frac{319}{5760} \cong 5.54 \cdot 10^{-2}$.

The estimations presented confirm our contention advanced in Remark 5.1. These show that quasiinterpolating splines demonstrate remarkable advantages before SMS when approximating functions and, especially their derivatives and that quasiinterpolating splines are comparable in sense of approximating accuracy with the splines interpolating derivatives. The computational experiments carried out support also these conclusions.

6. LOCAL SMOOTHING SPLINES WITH REGULARIZING PARAMETER.

We shall present in this section a new kind of local splines - the Local smoothing Splines with regulariziing Parameter (LSP). These splines, provided the appropriate choice of the parameter value, provide remarkably efficient tool for recovering a function and, especially, its derivatives from noised samples of the function. The algorithm of their constructing is related to the algorithm of constructing the quasiinterpolating splines. To be specific, we add to a SMS the certain BSS furnished a parameter. This BSS influences the constant entries in the asymptotic expansion of the SMS. The difference is that for quasiinterpolating we eliminate these entries but for smoothing we enhance these.

Henceforth we assume that $r=[(m-1)/2]$, $\vec{z}=\{z_k\}_{-\infty}^{\infty}$. Let

$$S_r^{m+s}(\vec{z},x)^{(s)}=\sum_{l=0}^{r}\beta_{2l}^{m+s}S_0^{m+s+2l}(\vec{z},x)^{(s+2l)}$$

be SMS of the degree $m-1$ which approximates $f^{(s)}$ provided $z_k=f(x_k), f\in C^s$. We define now the new local spline of the degree $m-1$:

$$S_{l\rho}^{m,s}(\vec{z},x)=S_r^{m+s}(\vec{z},x)^{(s)} tag 6.1$$
$$+(-1)^r\rho h^{2(r+1)}S_0^{m+s+2(r+1)}(\vec{z},x)^{(s+2(r+1))}.$$

Here ρ is a parameter, $S_{l\rho}^{m,s}(\vec{z},x)$ we call the LSP. Point out that if $\rho=0$ then the LSP reduces into the corresponding SMS. If we choose $\rho=(-1)^r\kappa_m^s<0$ (κ_m^s are defined in formulas (5.1), (5.5)), then we obtain the quasiinterpolating spline. If the parameter ρ turns positive then approximating properties of the spline deteriorate but the smoothing properties improve, as it will be shown in Section 7.

We can write

$$S_{l\rho}^{m,s}(\vec{z},x)=h\sum_{k}^{m+s}F_k(\rho)\delta_h^s b^m(x-hk); \tag{6.2}$$

$$F_k(\rho)=z_k+\sum_{l=1}^{r}\beta_{2l}^{m+s}h^{2l}\delta_h^{2l}z_k+(-1)^r\rho h^{2(r+1)}\delta_h^{2(r+1)}z_k.$$

First we discuss the approximating properties of LSP provided $z_k=f(x_k)$. The following assertions hold.

Theorem 6.1. *([32]). Let $z_k=f(x_k), f\in C^{s+m+1}$. If $x\in[x_n^m,x_{n+1}^m]$, $t=(x-x_n^m)/h$ then*

$$S_{l\rho}^{m,s}(\vec{z},x)=f(x)^{(s)}+h^m d_m^{ms}(t,\rho)f(x)^{(s+m)}$$
$$+h^{m+1}d_{m+1}^{ms}(t,\rho)f(x)^{(s+m+1)}+F_{m+1}^{s,m}(x,\rho).$$

If $m=2n$ then

$$d_m^{ms}(t,\rho)=d_m^{ms}(t)+(-1)^{n-1}\rho, d_{m+1}^{ms}(t,\rho)=d_{m+1}^{ms}(t),$$

if $m=2n-1$ then

$$d_m^{ms}(t,\rho)=d_m^{ms}(t), d_{m+1}^{s}(t,\rho)=d_{m+1}^{ms}(t)+(-1)^{n-1}\rho.$$

The functions $d_m^{ms}(t), d_{m+1}^{ms}(t)$ are defined in Theorem 3.1.

Remark 6.1. The formula (6.3) implies that the splines $S_{l\rho}^{m,s}(\vec{z},x)$ with arbitrary value of the parameter ρ reproduce exactly the derivatives $P_{s+m-1}^{(s)}(x)$ of polynomials of degree $m+s-1$ just as the corresponding SMS.

Theorem 6.2. *([32]). Let* $z_k = f(x_k), f \in C^{s+2q}, r = q, \rho \geq 0$. *1) If* $s = 0, q$ *is arbitrary natural number then*

$$S_{l\rho}^{2q,0}(\vec{z}, s) = f(x) + h^{2q} d_{2q}^{2q,0}(t,\rho) f(\xi)^{(2q)}.$$

2) *If* $q = 2, 3$, s *is arbitrary natural number then*

$$S_{l\rho}^{2q,s}(\vec{z}, s) = f(x)^{(s)} + h^{2q} d_{2q}^{2q,s}(t,\rho) f(\xi)^{(2q+s)},$$

$$S_{l\rho}^{2q-1,s}(\vec{z}, s) = f(x)^{(s)} - h^{2q-1} d_{2q-1}^{2q-1,s}(t,\rho) f(x)^{(2q-1+s)} + h^{2q} d_{2q-1}^{2q-1,s}(t,\rho) f(\xi)^{(2q+s)}.$$

If $x \in [x_n^m, x_{n+1}^m]$ *then*

$$\xi \in [x_n^m + h(-r - \frac{m+s}{2}), x_{n+1}^m + h(r + 1 + \frac{m+s}{2})], m = 2q, 2q-1.$$

If $x \in [h(n-1/2), h(n+1/2)]$ *then* $\xi \in [h(n - 2m + 2 - s/2), h(n + 2m - 2 + s/2)]$.

7. SMOOTHING PROPERTIES OF LOCAL SPLINES.

The most frequently used apparatus for recovering a function and its derivatives from noised samples of the function are the global smoothing splines which had appeared for the first time in the papers of Schoenberg [22] and Reinsch [18]. Nevertheless, at certain circumstances, in particular, when the data array is rather large, or when it is required to process the data in the real time scale, local splines offer remarkable advantages before the global smoothing splines constructed by the entire array. We discuss opportunities for recovering which provide the local smoothing splines with regularizing parameter (LSP) defined in Section 6.

Throughout this section we assume that $\{z_k = f_k + e_k\}$, $f_k = f(x_k), x_k = kh, k = 0, ..., N-1$; e_k are uncorrelated equally distributed random errors with the mathematical expectation $\mathcal{E}(e)_k) = 0$ and the variance $\mathcal{D}(e_k) = d$. Denote $a \leftrightharpoons \mathcal{E}(|e_k|$. Let us construct LSP $S_{l\rho}^{p,s}(\vec{z}, x)$ with data $\vec{z} = \{z_k\}$. Then

$$S_{l\rho}^{p,s}(\vec{z}, x) = S_{l\rho}^{p,s}(\vec{f}, x) + S_{l\rho}^{p,s}(\vec{e}, x), \quad (\vec{e} = \{e_k\})$$

is a stochastic variable, moreover

$$\mathcal{E}(S_{l\rho}^{p,s}(\vec{z}, x)) = S_{l\rho}^{p,s}(\vec{f}, x),$$

$$\mathcal{D}(S_{l\rho}^{p,s}(\vec{z}, x)) = D(S_{l\rho}^{p,s}(\vec{e}, x)) = d_{l\rho}^{p,s}(x),$$

$$\mathcal{E}(|S_{l\rho}^{p,s}(\vec{e}, x)|) \doteq a_{l\rho}^{p,s}(x).$$

To study the smoothing properties of LSP we use techniques developed in [33] for smoothing periodic splines.

The smoothing properties of LSP we characterize by ratio the variance of the spline $\mathcal{D}(S_{l\rho}^{p,s}(\vec{z}, x)$ to the variance of initial data $\mathcal{D}(z_k) = \mathcal{D}(e_k) = d$. This ratio is, obviously, independent of the step of a mesh and of the interval where the point x lies. Therefore we can assume without loss of generality that $h = 1/N, N \in \mathbb{N}$. Suppose also that $x \in [x_\nu^{p+s}, x_\nu^{p+s}] \subseteq (0,1)$ and that the span of the spline $S_{l\rho}^{p,s}(\vec{z},x)$ with a given $x - -\{x_k\}_{k=\alpha}^{\beta} \in (0,1)$.

$$\text{Let} \quad Z_k = \begin{cases} z_k & \text{if} \quad k = \alpha,\dots,\beta, \\ 0 & \text{if} \quad k = 0,\dots,\alpha-1,\beta+1,\dots,N-1, \end{cases} \quad \vec{Z} = \{Z_k\}_{k=0}^{N-1}.$$

Now we construct a periodic spline (compare with (6.2)):

$$S_\rho(\vec{Z},x) = N^{s-1}\sum_{k=0}^{N-1} Q_k(\rho)\delta^s M^p(x-hk), \tag{7.1}$$

$$Q_k(\rho) = Z_k + \sum_{l=1}^{r}\beta_{2l}^{p+s}\delta^{2l}Z_k + (-1)^r\rho\delta^{2(r+1)}Z_k,$$

where $M^p(x)$ is the 1-periodically expanded B-spline $b^p(x)$. Since the functions b^p and M^p coincide on the interval $(-\frac{p}{2N},\frac{p}{2N})$, the splines $S_\rho(\vec{Z},x) = S)_{l\rho}^{p,s}(\vec{z},x)$ if $x \in [x_\nu^{p+s}, x_{\nu+1}^{p+s}]$ and, consequently,

$$d_{l\rho}^{p,s}(x) = \mathcal{D}(S_\rho(\vec{Z},x)).$$

The value of $\mathcal{D}(S_\rho(\vec{Z},x))$ we evaluate by means of the methods of [33]. To carry it out we write in accordance with (7.1)

$$S_\rho(\vec{Z},x) = \sum_{k=0}^{N-1} Z_k L_\rho(x-x_k),$$

$$L_\rho(x) = N^{s-1}\delta^s[M^p(x) + \sum_{l=1}^{r}\beta_{2l}^{p+s}\delta^{2l}M^p(x) + (-1)^r\rho\delta^{2(r+1)}M^p(x)].$$

$$\text{Therefore} \quad \mathcal{D}(S_\rho(\vec{Z},x)) = d\sum_{k=0}^{N-1} L_\rho(x-x_k)^2 = dN\sum_n |\Lambda_n(x)|^2,$$

$$\Lambda_n(x) = \frac{1}{N}\sum_{k=0}^{N-1} w^{-nk}L_\rho(x+x_k) =$$

$$= N^{s-1}(iv_n)^s[1 + \sum_{l=1}^{r}\beta_{2l}^{p+s}(-1)^l(Nv_n)^{2l} - \rho(Nv_n)^{2(r+1)}]m_n^p(x),$$

where

$$m_n^p(x) = \frac{1}{N}\sum_{k=0}^{N-1} w^{-nk}M^p(x+x_k), \quad v_n = 2\sin(\pi n/N).$$

Hence

$$\mathcal{D}(S)_\rho(\vec{Z},x)) = \frac{d}{N}N^{2s}\sum_{n=0}^{N-1}(v_n)^{2s}\{[1$$

$$+\sum_{l=1}^{r}\beta_{2l}^{p+s}(-1)^l(Nv_n)^{2l} - \rho(Nv_n)^{2(r+1)}]m_n^p(x)\}^2.$$

If $x = x_\nu$ then we have, setting

$$u_n^p = m_n^p(0),$$

$$\mathcal{D}(S_\rho(\vec{Z},x_\nu)) = \frac{d}{N}N^{2s}\sum_{n=0}^{N-1}(v_n)^{2s}\{[1$$

$$+\sum_{l=1}^{r}\beta_{2l}^{p+s}(-1)^l(Nv_n)^{2l} - \rho(Nv_n)^{2(r+1)}]u_n^p\}^2.$$

We define now the following polynomial of the degree $2(r+1)+s\}$

$$P_{l\rho}^{p,s}(y) = y^s\sum_{l=0}^{r}\beta_{2l}^{p+s}(-1)^l y^{2l} + (-1)^r\rho y^{2r+2}, r = [(p-1)/2]. \quad (7.2)$$

It can be verified immediately that $u_n^p = \sum_{l=0}^{r}\gamma_l^p v_n^{2l} = Q_{2r}(v_n)$ is a polynomial of degree $2r$ with respect to v_n. Denote now

$$W_{l\rho}^{p,s}(y) = [P_{l\rho}^{p,s}(y)Q_{2r}(y)]^2 = \sum_{l=s}^{4r+2+s}\alpha_l^p(\rho)y^{2l}. \quad (7.3)$$

This is a polynomial of the degree $8r+4+2s$. The following identity is well known:

$$\frac{1}{N}\sum_{n=0}^{N-1}\sin(\frac{\pi n}{N})^{2l} = \binom{2l}{l} = \frac{(2l)!}{(l!)^2}.$$

It follows from this identity that

$$\mathcal{D}(S_\rho(\vec{Z},x_\nu)) = \frac{d}{N}N^{2s}\sum_{n=0}^{N-1}W_{l\rho}^{p,s}(v_n) \quad (7.4)$$

$$= \sum_{l=s}^{4r+2+s}\alpha_l^p(\rho)\frac{(2l)!}{(l!)^2} = d\Delta^{p,s}(\rho).$$

We refer the reader to [33] for the proof of the following assertion.

Theorem 7.1. *If p is arbitrary even number then $d_\rho^{p,s}(x) \le d_\rho^{p,s}(x_\nu), s = 0,1,\ldots$ If $p = 2,\ldots,6$ then*

$$d_\rho^{p,s}((\nu+1/2)/N) \le d_\rho^{p,s}(x) \le d_\rho^{p,s}(x_\nu), s = 0,1,\ldots$$

Since $d_{l\rho}^{p,s}(x) = \mathcal{D}(S_\rho(\vec{Z},x))$, the relation (7.4) and Theorem 7.1 enable us to come up with the following proposition.

Theorem 7.2. *If: 1)p is arbitrary even number or 2)$p = 3, 5$ then the following unimprovable estimation is true*

$$d_{l\rho}^{p,s}(x) \leq h^{-2s} d\Delta_{l\rho}^{p,s}(\rho), s = 0, 1, ...$$

Conjecture 7.1. *The assertion of the theorem holds with arbitrary natural values p.*

As it follows from (7.2)-(7.4), the function $\Delta^{p,s}(\rho)$ is of such structure:

$$\Delta^{p,s}(\rho) = a^{p,s}\rho^2 - 2b^{p,s}\rho + c^{p,s}.$$

Therefore for each specific spline we can find the value $\rho = P$ which minimize the variance of the spline. The spline provides the greatest smoothing effect with ρ =P. It is obvious that

$$\text{P} = b^{p,s}/a^{p,s}, \Delta^{p,s}(\text{P}) = c^{p,s} - (b^{p,s})^2/a^{p,s} \leq c^{p,s}.$$

Point out, that $\Delta^{p,s}(0) = c^{p,s}$. In this case LSP reduces into SMS. We demonstrate now the smoothing characteristics of some LSP ($\Delta^{p,s}(\text{P})$) and SMS ($\Delta^{p,s}(0)$). For comparison we cite also the corresponding characteristics of BSS ($\Delta_0^{p,s}$).

p	s	$\Delta_0^{p,s}$	$\Delta^{p,s}(0)$	$\Delta^{p,s}(\text{P})$	P
2	0	1	1	1/3	1/3
2	1	2	2	0.2	0.3
2	2	6	6	0.285	0.29
3	0	0.58	0.83	0.438	0.135
3	1	0.81	1.69	0.346	0.134
3	2	2.11	5.71	0.586	0.138
4	0	0.5	0.72	0.415	0.162
4	1	0.563	1.25	0.3	0.155
4	2	1.285	3.84	0.48	0.156
5	0	0.532	0.75	0.49	0.062
5	1	0.43	1.39	0.44	0.063
5	2	0.802	4.52	0.801	0.067
6	0	0.396	0.69	0.467	0.078
6	1	0.309	1.14	0.395	0.077
6	2	0.551	3.44	0.687	0.08

Tab.7.1

It is manifested in the table that the effect of smoothing by means of LSP is of the same order as by means of BSS and exceeds remarkably this effect for SMS, especially when recovering derivatives. But the approximation of $f^{(s)}$ from the samples $f(x_k)$ by means of LSP is of the same order as by means of SMS and much more accurate than by means of BSS. It enable us to say that LSP combine the advantages of BSS and SMS. In [32] results of computational experiments are adduced, which demonstrate a remarkable efficiency of LSP for recovering functions and, especially, derivatives in one- and multidimensional cases.

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