Interpolatory Subdivision Schemes Generated by Splines

Valery A. Zheludev, Amir Z. Averbuch, and Marina Gruzd

Abstract. We present interpolatory subdivision schemes (ISS) based on interpolatory polynomial and discrete splines. These schemes converge on any initial data of power growth, and limit functions are regular. Although the masks of the ISS's are infinite, the computational cost of their implementation is competitive with the cost of schemes with finite masks.

§1. Introduction

A univariate stationary uniform ISS S_a consists in the following: A function f^j defined on the grid $\mathbf{G^j} = \{k/2^j\}_{k\in\mathbb{Z}}$: $f^j(k/2^j) = f_k^j$ is extended onto the grid $\mathbf{G^{j+1}}$ by filtering the array $\{f_k^j\}$:

$$f_k^{j+1} = \sum_{l\in\mathbb{Z}} a_{k-2l} f_l^j, \qquad (1)$$

where $a_0 = 1,\ a_{2k} = 0\ \forall k \neq 0$. This is one refinement step. The filter $\mathbf{a} = \{a_k\}$ is called the refinement mask of the ISS S_a. The z-transform of the mask (transfer function of the filter) $a(z) = \sum_{k\in\mathbb{Z}} z^k a_k$ is called the symbol of the ISS S_a. Eq. (1) is equivalent to the following relation in the $z-$domain: $f^{j+1}(z) = a(z)f^j(z^2)$.

The well-known interpolatory uniform subdivision scheme by Dubuc and Deslauriers (DD) [2] can be formulated in the following manner: The polynomial spline $Q_j^{2r}(x)$ of even order $2r$ (degree $2r-1$) of deficiency $2r-1$ is constructed, which interpolates the function f^j on the grid $\mathbf{G^j}$: $Q_j^{2r}(k/2^j) = f_k^j$. Then the samples f_k^{j+1} are defined as the values of the spline: $f_k^{j+1} = Q_j^{2r}(k/2^{j+1})$.

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For the spline of order $2r$ the mask $\mathbf{a}$ comprises $2r+1$ nonzero terms. In other words, the symbol of the subdivision scheme $a(z)$ is a Laurent polynomial.

Unlike DD, we construct a family of ISS's using regular splines of deficiency 1. Otherwise the construction procedure is similar to DD: We construct the polynomial spline of order p (degree $p-1$) $\sigma_j^p(x) \in C^{p-2}$ which interpolates the function f^j on the grid $\mathbf{G^j}$: $\sigma_j^p(k/2^j) = f_k^j$. Then the samples f_k^{j+1} are defined as the values of the spline: $f_k^{j+1} = \sigma_j^p(k/2^{j+1})$. Note that nodes of splines of even orders coincide with the grid points, and for odd orders are located at midpoints between the grid points.

A seeming drawback of using interpolatory splines is that it requires a convolution of the array $\mathbf{f^j}$ with the infinite mask. But actually it does not hamper the efficiency of implementation. On the other hand, the ISS's produced via such an approach enjoy remarkable approximation and regularity properties.

§2. Refinement Masks Derived from Splines

B-splines. Denote by $M^p(x)$ the central equidistant $B-$spline of order p on the grid $\{k\}_{-\infty}^{\infty}$. It is supported on the interval $(-p/2, p/2)$ and is positive within its support and symmetric around zero. The B-spline of order p is the piecewise polynomial of degree $p-1$ belonging to C^{p-2}. Nodes of $M^p(x)$ are located at points $\{k+p/2\}$.

We introduce two sequences $\mathbf{u}^p = \{M^p(k)\}$ and $\mathbf{w}^p = \{M^p\,(k+1/2)\}$, $k \in \mathbb{Z}$. Due to the compact support of B-splines, these sequences are finite. The $z-$transforms of the sequences $\mathbf{u}^p$ and $\mathbf{w}^p$ are the so called Euler-Frobenius polynomials [6].

Proposition 1 ([6])**.** *On the circle $z = e^{i\omega}$ the Laurent polynomials $u^p(z)$ are strictly positive. Their roots are all simple and negative. Each root ζ can be paired with a dual root θ such that $\zeta\,\theta = 1$. Thus, if $p = 2r-1$ or $p = 2r$, then $u^p(z)$ can be represented as:*

$$u^p(z) = \prod_{n=1}^{r-1} \frac{1}{\gamma_n}(1+\gamma_n z)(1+\gamma_n z^{-1}),\ 0 < |\gamma_n| \le g < 1. \tag{2}$$

We also introduce the rational functions which will be employed in the sequel: $U_i^p(z) = z\mathbf{w}^p(z^2)/\mathbf{u}^p(z^2)$.

Proposition 2. *If $p = 2r$, then*

$$1 + U_i^{2r}(z) = \frac{(1+z)^{2r} u^{2r}(z)}{2^{2r-1} z^r u^{2r}(z^2)}. \tag{3}$$

If $p = 2r-1$, then the following factorization formulas hold:

$$1 - U_i^{2r-1}(z) = (z - 2 + z^{-1})^r \vartheta_r(z), \quad \vartheta_r(z) = \frac{q_{r-2}(z - 2 + z^{-1})}{u^{2r-1}(z^2)}, \quad (4)$$

where q_{r-2} is a certain polynomial of degree $r-2$. The rational functions $\vartheta_r(z)$ can be written in the form

$$\vartheta_r(z) = A_r + \sum_{n=1}^{\infty} \epsilon_n^r (z - 2 + z^{-1})^n, \qquad A_r = \frac{(4^r - 1)}{r(2r-2)!}|b_{2r}|, \quad (5)$$

where b_s is the Bernoulli number of order s. The series in (5) converges for all z such that $|z| = 1$.

Interpolatory splines. Shifts of B-splines form a basis in the space $\mathbf{S}_j^p$ of splines of order p on the grid $\mathbf{G^j} = \{k/2^j\}$. Namely, any spline $\sigma_j^p \in \mathbf{S}_j^p$ has the representation $\sigma_j^p(x) = \sum_l q(l)\, M^p(2^j x - l)$. Let $\mathbf{q} = \{q(l)\}$, and let $q(z)$ be the z-transform of $\mathbf{q}$. We also introduce the sequences $\mathbf{s}_e^p = \{s_e^p(k) = \sigma^p(k/2^j)\}$, $\mathbf{s}_o^p = \{s_o^p(k) = \sigma^p((2k+1)/2^{j+1})\}$ and $\mathbf{s}^p = \{s^p(k) = \sigma^p(k/2^{j+1})\}$ of values of the spline on the grid points, on the midpoints, and on the whole set $\{k/2^{j+1}\}$. We have

$$\begin{aligned} s_e^p(k) &= \sum_l q(l)\, M^p(k - l), \quad s_o^p(k) = \sum_l q(l)\, M^p\left(k - l + \frac{1}{2}\right), \\ s_e^p(z) &= q(z) u^p(z), \quad s_o^p(z) = q(z) w^p(z), \quad s^p(z) = s_e^p(z^2) + z s_o^p(z^2) \end{aligned} \quad (6)$$

From these formulas, we can derive an expression for the coefficients of a spline σ_i^p which interpolates a given sequence $\mathbf{e} = \{e(k)\} \in l_1$ at $\mathbf{G^j}$:

$$\sigma_i^p(k/2^j) = e(k) \Longleftrightarrow q(z) = \frac{e(z)}{u^p(z)} \Longleftrightarrow q(l) = \sum_{n=-\infty}^{\infty} \lambda^p(l - n) e(n).$$

Here $\lambda^p = \{\lambda^p(k)\}$ is the sequence defined via its $z-$transform: $\lambda^p(z) = \sum_{k=-\infty}^{\infty} z^{-k} \lambda^p(k) = 1/u^p(z)$. It follows immediately from (2) that the coefficients $\{\lambda^p(k)\}$ decay faster than any power of k as $|k| \to \infty$. For the values of the spline in the midpoints we have $s_o^p(z^2) = w^p(z^2)/u^p(z^2)) = z^{-1} U_i^p(z) e(z^2)$. Substituting this relation in (6), we express one refinement step as follows: $f^{j+1}(z) = a_i^p(z) f^j(z^2)$, $\quad a_i^p(z) = 1 + U_i^p(z)$. The function $a_i^p(z)$ is the symbol of the ISS which is based on the interpolatory spline of order p. This is a rational function of z, and the corresponding mask $\mathbf{a_i^P} = \{a_k\}$ is infinite. Since the subdivision schemes are interpolatory, it is sufficient to derive only values in the midpoints. This can be achieved by filtering the array $\mathbf{f^j}$ with the filter whose transfer function is $z^{-1} U_i^p(z)$.

Remark. Similar construction can be carried out based on discrete rather than polynomial splines [5,1]. Without going into details we present filters derived from discrete splines of even orders. The discrete spline of order $2r$ provides the filter with the transfer function

$$U_d^{2r}(z) = \frac{\left(1+z^{-1}\right)^{2r} - (-1)^r \left(1-z^{-1}\right)^{2r}}{\left(1+z^{-1}\right)^{2r} + (-1)^r \left(1-z^{-1}\right)^{2r}}.$$

These filters are related to the Butterworth filters, which are used in signal processing. The symbol of the corresponding ISS is $a_d^{2r}(z) = 1 + U_d^{2r}(z)$.

Examples:

$$U_i^3(z) = \frac{4(z^{-1}+z)}{z^{-2}+6+z^2}, \quad U_i^4(z) = \frac{(z^{-3}+z^3)+23(z^{-1}+z)}{8(z^{-2}+4+z^2)},$$
$$U_i^5(z) = \frac{16((z^{-3}+z^3)+11(z^{-1}+z))}{z^{-4}+z^4+76(z^{-2}+z^2)+230},$$
$$U_d^6(z) = \frac{z^3+15z+15z^{-1}+z^{-3}}{2(3z^2+10+3z^{-2})}.$$

It is obvious that ISS based on a spline of order p (degree $p-1$) is exact on the polynomials of the same degree $p-1$. But for the interpolatory splines of odd orders (even degrees) the situation is more interesting. Proposition 2 implies the following super-convergence property of such splines:

Theorem 3 ([8]). *Let f be a function that does not grow faster than a power of x as $|x| \to \infty$. Let f be sampled on the grid $\{k\}$ and $\mathbf{f} = \{f(k)\}, k \in \mathbb{Z}$. Let the spline σ_i^{2r-1} of order $2r-1$ interpolates $\mathbf{f}$ on the grid $\{2k\}$. Then*

$$\sigma_i^{2r-1}(2k+1) = f(2k+1) - A_r \mathbf{D}^{2r} f(2k+1) - \sum_{n=1}^{\infty} \epsilon_n \mathbf{D}^{2(r+n)} f(2k+1),$$

where A_r is given in (5), $\mathbf{D}^{2m}$ denotes the operator of central difference of order $2m$ and the series converges absolutely.

Corollary 4. *Let f be a polynomial of degree $2r-1$. Then $\sigma_i^{2r-1}(2k+1) \equiv f(2k+1)$ for every $k \in \mathbb{Z}$.*

Remark. Equation (3) implies that the ISS based on the interpolatory spline of even order $2r$ is exact on the splines of that order. That means that, provided $f(x) = \sigma^{2r}(x)$ is a spline of order $2r$ with nodes on the grid $\mathbf{G^j} = \{k/2^j\}_{k\in\mathbb{Z}}$ and $f_k^j = f(k/2^j)$, all subsequent steps of subdivision produce values of this spline: $f_k^{j+s} = f(k/2^{j+s}),\ k \in \mathbb{Z},\ s \in \mathbb{N}$.

§3. Convergence and Regularity of Spline Subdivision Schemes

To investigate the convergence and regularity of our subdivision schemes, we apply a slightly modified technique developed by N. Dyn, J. Gregory and D. Levin [4, 3]. The difference is that, unlike these authors, we study subdivision schemes with infinite but exponentially decaying masks. Therefore, in the sequel we assume that the original sequence $\{f_k^0\}$ is of power growth, that is there exist positive constants A, M such that $|f_k^0| \le M k^A$, $k \in \mathbb{Z}$. Under this assumption the analysis of [3] can be carried out in our case with obvious alterations.

Definition 5. *A subdivision scheme S_a is convergent if for any initial data $\{f_k^0\}$, $k \in \mathbb{Z}$, of power growth, the sequence of polygonal lines interpolating the data $\{f_k^j\}$, $k \in \mathbb{Z}$, generated by S_a at the corresponding refinement level, converges uniformly to a continuous function $f^\infty(x)$.*

Theorem 6 ([3])**.** *The scheme S_a is convergent if and only if its symbol factorizes as: $a(z) = (1+z)q(z)$, and for the scheme S_q with the mask $\{q_k\}$, $k \in \mathbb{Z}$, there exists $L \in \mathbb{N}$ such that*

$$\|(S_q)^L\| = \max_i \left\{ \sum_k |q^{[L]}_{i+2^L k}| : 0 \le i \le 2^L - 1 \right\} < 1. \tag{6}$$

Here $q_k^{[L]}$ are the coefficients of the scheme $(S_q)^L$, which is the L times iterated scheme S_q.

Theorem 7 ([3])**.** *Let the symbol of the scheme S_a be factorized as: $a(z) = (1+z)^m 2^{-m} b_m(z)$. If S_b is convergent, then for any initial data $\mathbf{f^0}$ of power growth, the limit function $f^\infty \in C^m_{loc}(\mathbb{R})$ is of power growth, and $\frac{d^m}{dx^m}\left(f^\infty(x)\right) = \left(\triangle^m f^0\right)^\infty(x)$.*

Here $\triangle$ is the difference operator: $\triangle x_k = x_k - x_{k-1}$. The above propositions yield a practical algorithm for establishing the convergence of an ISS and analyzing its regularity. The key operation is evaluation of sums of coefficients of type (6). For subdivision schemes with finite masks, these sums can be calculated directly. But for infinite masks different methods of evaluation of the coefficients are required.

Evaluation of coefficients of subdivision masks via the discrete Fourier transform. For simplicity, we suppose that the number L in (6) is equal to 1. The cases with $L > 1$ are treated similarly. We assume that $N = 2^j$, $j \in \mathbb{N}$, $\sum_k^j$ stands for $\sum_{k=-N/2}^{N/2-1}$. The discrete Fourier transform (DFT) of an array $\mathbf{x}^j = \{x_k^j\}_{k=-N/2}^{N/2-1}$ and its inverse (IDFT) are $\widehat{x}^j(n) = \sum_k^j e^{-2\pi i k n/N} x_k^j$ and $x_k^j = \frac{1}{N}\sum_n^j e^{2\pi i k n/N}\widehat{x}^j(n)$. As before, $y(z)$ denotes the $z-$transform of a signal $\{y(k)\} \in l_1$. We assume that $z = e^{-i\omega}$.

Let a rational function $a(z) = Q(z)/P(z)$ be a symbol of a subdivision scheme whose mask $\mathbf{a} = \{a(k)\}_{k=-\infty}^{\infty}$. Here $P(z)$ and $Q(z)$ are Laurent polynomials. We assume the following properties of the symbol $a(z)$:

1) The polynomials $P(z)$ and $Q(z)$ are symmetric about inversion: $P(z^{-1}) = P(z)$, $Q(z^{-1}) = Q(z)$ and thus they are real on the unit circle $|z| = 1$.

2) Roots of the denominator $P(z)$ are real, simple and do not lie on the unit circle $|z| = 1$.

It follows from 1 and 2 that $P(z)$ can be represented in the form (2). The above properties imply that the coefficients $a(k)$ of the filter are symmetric about zero and decay exponentially as $k \to \infty$:

$$|a(k)| \le Ag^k \Rightarrow \sum_{k=N}^{\infty} |a(k)| \le Bg^N, \ B = \frac{A}{1-g}. \tag{7}$$

We need to evaluate the sums $S_e(a) = \sum_{k=-\infty}^{\infty} |a(2k)|$, $S_o(a) = \sum_{k=-\infty}^{\infty} |a(2k+1)|$. We write $F(\omega) = a(e^{-i\omega}) = Q(e^{-i\omega})/P(e^{-i\omega}) = \sum_{k=-\infty}^{\infty} e^{-i\omega k}\, a(k)$, and calculate the function F in the discrete set of points

$$\widehat{a}(n) = F\left(\frac{2\pi n}{N}\right) = \sum_{k=-\infty}^{\infty} e^{\frac{-2\pi i k n}{N}}\, a(k) = \sum_{r=-N/2}^{N/2-1} e^{\frac{-2\pi i r n}{N}}\, \varphi(r),$$

$$\varphi(r) = \sum_{l=-\infty}^{\infty} a(r+lN) = a(r) + \psi(r), \quad \psi(r) = \sum_{l\in\mathbb{Z}\setminus 0} a(r+lN).$$

It follows from (7) that

$$|\psi(r)| \le 2Bg^{N/2} \Rightarrow |a(r)| = |\varphi(r)| + \alpha_N(r), \quad |\alpha_N(r)| \le 2Bg^{N/2}. \tag{8}$$

The samples $\varphi(k)$ are available via IDFT: $\varphi(k) = \frac{1}{N}\sum_n^j e^{2\pi i k n/N}\, \widehat{a}(n)$. Using, (8) we can evaluate the sums we are interested in as:

$$S_e(a) = \sum_{k=-N/4}^{N/4-1} |a(2k)| + 2\sum_{k=N/4}^{\infty} |a(2k)| = \sum_{r=-N/4}^{N/4-1} |\varphi(2k)| + \rho_N,$$

$$\rho_N = \sum_{r=-N/4}^{N/4-1} |\alpha_N(2k)| + 2\sum_{k=N/4}^{\infty} |a(2k)|, \quad |\rho_N| \le B(N+2)g^{N/2}.$$

Hence, it follows that doubling N, we can approximate the infinite series $S_e(a)$ by the finite sum $\sigma_e^N(a) = \sum_{k=-N/4}^{N/4-1} |\varphi(2k)|$, whose terms are available via DFT. An appropriate value of N can be found theoretically using estimations of the roots of the denominator $P(z)$. But practically, we can iterate calculations gradually, doubling N until the result of calculation $\sigma_e^{2N}(a)$ becomes identical to $\sigma_e^N(a)$ (up to machine precision). The same approach is valid for evaluation of the sum $S_o(a)$ and of the sums $\sum_k |q^{[L]}_{i+2^L k}|$ with any L.

§4. A Few Particular Cases

In this section we establish the convergence and evaluate smoothness of a few subdivision schemes based on splines.

Quadratic interpolatory spline.

The symbol of the scheme is

$$a_i^3(z) = 1 + U_i^3(z) = \frac{(1+z)^4}{z^4+6z^2+1} = (1+z)\,q(z),\ \ q(z) = \frac{(1+z)^3}{z^4+6z^2+1}.$$

To establish the convergence we have to prove that the scheme S_q with the rational symbol $q(z)$ and the infinite mask $\{q(k)\}$ is contractive. For this purpose we evaluate the norms $\|(S_q)^L\| = \max\left\{\sum_k |q^{[L]}_{i+2^L k}|\right\}$ using DFT as described in the previous section. Let us start with $L = 1$. Then

$$\widehat{q}^{[1]}(n) = q(e^{-2\pi i n/N}) = \frac{2e^{\frac{i\pi n}{N}}\cos^3\frac{i\pi n}{N}}{1+\cos^2\frac{2\pi n}{N}}.$$

The sums $\sum_{k=-\infty}^{\infty} |q^{[1]}_{i+2k}| \simeq \sum_{r=-N/4}^{N/4-1} |\varphi(2k+i)|,\ i = 0, 1,$ provided N is sufficiently large. The values $\varphi(k)$ are calculated via IDFT: $\varphi(k) = N^{-1}\sum_n^j e^{2\pi i k n/N}\,\widehat{q}^{[1]}(n)$. Direct calculation yields the estimate: $\|S_q^1\| \leq 0.7071$. Thus the scheme is convergent.

To establish the differentiability of the limit function f^∞ of the scheme S_a, we have to prove that the scheme S_{b_1} with the symbol $b_1(z) = 2(1+z)^{-2}\,a_i^3(z)$ is contractive. The norm of the operator S_{b_1} does not meet the requirement $\|S_{b_1}\| < 1$. But we succeeded in proving that $\|(S_{b_1})^2\| \leq 0.6667$. Hence the limit function $f^\infty \in C^1$.

But an even stronger assertion is true: the limit function $f^\infty \in C^2$. To establish it, we prove that the scheme S_{b_2} with the symbol $b_2(z) = 4(1+z)^{-3}\,a_i^3(z)$ is contractive. As in the previous case our calculations lead to the estimate: $\|(S_{b_2})^2\| \leq 0.6667$, which proves the assertion.

Interpolatory spline of fifth order (fourth degree)

The symbol of the scheme is

$$a_i^5(z) = 1 + U_i^5(z) = \frac{(1+z)^6(z^2+10z+1)}{z^8+76(z^6+z^2)+230z^4+1} = (1+z)\,q(z),$$

$$\widehat{q}(n) = q(e^{-2\pi i n/N}) = \frac{4e^{\pi i n/N}\cos^5\left(\frac{\pi n}{N}\right)\left(5+\cos\frac{2\pi n}{N}\right)}{5+18\cos^2\frac{2\pi n}{N}+\cos^4\frac{2\pi n}{N}}.$$

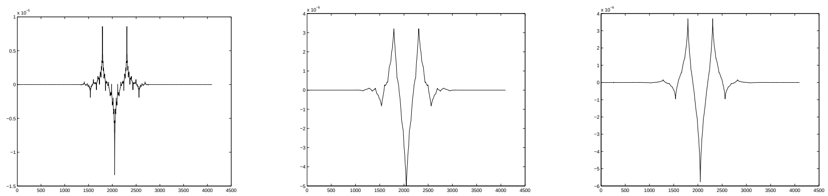

Fig. 1. Second derivatives of the BLFs: 4-point DD ISS (left), 6-point DD ISS (center), quadratic spline (right).

As in the previous case, we find that the scheme with the symbol $a_i^5(z)$ is convergent and, moreover, the limit function $f^\infty \in C^4$.

Discrete interpolatory spline of sixth order

The symbol of the scheme is

$$a_d^6(z) = 1 + U_d^6(z) = \frac{(1+z)^6}{2z(3z^4 + 10z^2 + 3)}. \tag{9}$$

This scheme is also convergent and the limit function $f^\infty \in C^4$.

We illustrate in Table 1 the process of evaluation of norms of operators $\|S_b^L\|$ for the schemes with the symbols a_i^5 and a_d^6 .

	L	$\|\|S_b^L\|\|$
C^0	1	0.8532
C^1	2	0.5965
C^2	2	0.8070
C^3	3	0.9962
C^4	5	0.8902

	L	$\|\|S_b^L\|\|$
C^0	1	0.8333
C^1	2	0.6000
C^2	2	0.8000
C^3	4	0.6830
C^4	11	0.9512

Table 1: Convergence of the schemes with the symbols a_i^5 (left) and a_d^6 (right).

Definition. *Let S be a convergent subdivision scheme. Then the function $\phi_S = S^\infty \delta$, where δ is the Kroneker delta, is called the* basic limit function *(BLF) of the scheme S.*

The limit function $f^\infty(x)$ generated by a convergent subdivision scheme S from initial data $\{f_k^0\}$ of power growth, as well as its derivatives, can be expressed via the BLF:

$$(f^\infty(x))^{(s)} = \sum_{k \in \mathbb{Z}} f_k^0 \phi_S^{(s)}(x-k).$$

In Figure 1 we display second derivatives of the BLFs of the 4-point DD ISS (left picture), of the 6-point DD ISS (central picture) and of the ISS based on quadratic splines (right picture).

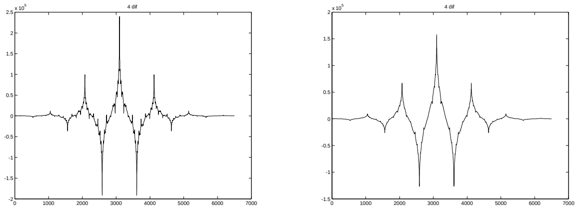

Fig. 2. Fourth derivatives of BLF of the fifth order splines ISS (left) and of BLF of the sixth order discrete splines ISS (right).

It is well known that the second derivative of the BLF of the 4-point DD ISS does not exist. The BLF of the 6-point DD ISS belongs to C^{α}, $\alpha < 2.830$. In the picture, the second derivative of the BLF of the quadratic spline scheme looks smoother than the second derivative of the BLF of the 6-point DD ISS. Thus, we conjecture that it belongs to C^{β} for some $\beta > \alpha$.

In Figure 2 we display fourth derivatives of the BLF's of the ISS based on polynomial splines of fifth order (left picture), and on discrete splines of sixth order (right picture).

We observe that the fourth derivative of the BLF of the sixth order discrete splines ISS is of near-fractal appearance. Nevertheless, it is proved that it is continuous.

§4. Remarks on implementation

Although masks of our spline subdivision schemes are infinite, the rational structure of symbols allows us to implement the refinement via recursive filtering, which is commonly used in signal processing. For example, in the quadratic spline ISS, the passage from f^j to f^{j+1} is conducted as follows: $f^{j+1}_{2k} = f^j(k)$, $f^{j+1}_{2k+1} = s^j(k)$, where values $s^j(k)$ are derived from f^j by a cascade of elementary recursive filters: $x(k) = 4\alpha(f^j(k) + f^j(k+1))$, $x_1(k) = x(k) - \alpha x_1(k-1)$, $s^j(k) = x_1(k) - \alpha s^j(k+1)$. The parameter $\alpha = 3 - 2\sqrt{2} \approx 0.172$. The cost of computing a value f^{j+1}_{2k+1} is 3 M(ultiplications) and 3 A(dditions). For comparison, the 4-point DD ISS requires 2M+3A, but the regularity of the limit function is inferior to the regularity of the limit function of the above scheme. The 6-point DD ISS based on quintic polynomials, which produces the limit function of approximately the same regularity as the spline ISS, requires 3M+5A. The scheme based on the discrete splines of sixth order requires 4M+5A. However, it produces limit functions belonging to C^4. Thus, we argue that ISS's based on splines are competitive with conventional ISS's based on polynomial interpolation.

Conclusion. We recall that, due to the super-convergence property, the limit functions of the ISS based on splines of even degrees are more regular than the splines themselves. Actually, these limit functions form a new class of functions which deserves a thorough investigation. Also, the approximation order of the ISS is higher than that of the splines.

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Valery A. Zheludev and Amir Z. Averbuch
School of Computer Science
Tel Aviv University
Tel Aviv 69978, Israel
zhel@math.tau.ac.il and amir@math.tau.ac.il

Marina Gruzd
Dept. of Applied Mathematics
Technion
32000, Haifa, Israel
gruzd@techunix.technion.ac.il