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Spline Harmonic Analysis and Wavelet Bases

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ABSTRACT. We present here a short description of a new computational technique, Spline Harmonic Analysis, based on periodic splines, and its application to constructing wavelet schemes. This technique provides a promising tool for solving numerous problems connected with differential and integral equations, digital signal processing, geometric design, and so on.

1. Spline Harmonic Analysis

We first introduce some notation. Throughout, $N = 2^j$ and $\sum_k^j$ stands for $\sum_{k=0}^{2^j-1}$. Denote $\omega = \exp(2\pi i/N)$. The Discrete Fourier Transform (DFT) of a vector $\mathbf{a} = \{a_k\}_0^{N-1}$ is

$$T_n^j(\mathbf{a}) = \frac{1}{N}\sum_k^j \omega^{-nk} a_k, \qquad a_k = \sum_n^j \omega^{nk} T_n^j(\mathbf{a}).$$

The function ${}_pB^j(x) = N^p\nabla_j^p(x_+^{p-1}/(p-1)!)$, where $x_+ = 0.5(x+|x|)$, is the B-spline of degree $p-1$ with knots at the points $\{k/2^j\}$. The symbol ∇_j denotes a backward difference with step 2^{-j}. The symbol ${}_pM^j(x)$ denotes the 1-periodic B-spline of degree $p-1$: ${}_pM^j(x) = \sum_{l=-\infty}^{\infty} {}_pB^j(x+l)$.

Throughout, ${}_p\mathfrak{B}^j$ will denote the space of 1-periodic splines of degree $p-1$ and of defect 1 with knots at the points $\{k/2^j\}$. Any spline ${}_pS^j \in {}_p\mathfrak{B}^j$ can be represented as follows: ${}_pS^j(x) = \frac{1}{N}\sum_k^j q_k^j {}_pM^j(x-k/N)$. Note that this representation allows one to compute the spline ${}_pS^j(x)$ immediately.

Denoting $\mathbf{q}^j = \{q_k^j\}_0^{N-1}$ and exploiting the relations of DFT, we write the

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spline as ${}_pS^j(x) = \frac{1}{N}\sum_k^j {}_pM^j(x-k/N)\sum_r \omega^{rk}T_r^j(\mathbf{q}^j) = \sum_r^j \xi_r^j {}_pm_r^j(x)$, where

$$\begin{aligned}{}_pm_r^j(x) &= \frac{1}{N}\sum_k^j {}_pM^j(x-k/N)\omega^{rk} \\ &= \frac{1}{N}\sum_k^j {}_pM^j(x+k/N)\omega^{-rk}, \qquad \xi_r^j = T_r^j(\mathbf{q}^j).\end{aligned}$$

Hence, we see that the splines ${}_pm_r^j(x)$, $r = 0, 1, \dots, 2^j - 1$, form a basis of the space ${}_p\mathfrak{B}^j$, and $\{\xi_r^j\}$ are the coordinates of the spline ${}_pS^j(x)$ with respect to this basis. We mention some properties of the splines ${}_pm_r^j$.

1. There holds $\langle {}_pm_r^j, {}_pm_k^j\rangle = \int_0^1 {}_pm_r^j(y)\,{}_p\overline{m}_k^j(y)\,dy = \delta_k^r\,{}_{2p}u_r^j$, where δ_k^r is the Kronecker delta, ${}_pu_r^j = \frac{1}{N}\sum_k^j \omega^{-rk}\,{}_pM^j(k/N + p/2N) > 0$. This property implies that the splines ${}_pm_r^j$ form an orthogonal basis of ${}_p\mathfrak{B}^j$, and we name these splines *ortsplines.*
2. As $p \to \infty$ one has

$$(1.1)\qquad {}_pm_r^j(x)/({}_{2p}u_r^j)^{1/2} \to \mu_r(x) = \exp(2\pi i r x), \qquad r = 0, \dots, N-1.$$

3. There holds ${}_pM^j(x + k/N) = \sum_r^j {}_pm_r^j(x)\omega^{rk}$.
4. One has ${}_pm_r^j(x)^{(s)} = N^s(1-\omega^{-r})^s\,{}_{p-s}m_r^j(x) \in {}_{p-s}\mathfrak{B}^j$.
5. If a spline ${}_lS^j(x) \in {}_l\mathfrak{B}^j$ is given as ${}_lS^j(x) = \sum_k^j \eta_k^j\,{}_lm_k^j(x)$, then the convolution is ${}_pm_r^j * {}_lS^j(x) = {}_{l+p}m_r^j(x)\eta_r^j \in {}_{p+l}\mathfrak{B}^j$.

The Properties 4 and 5 imply that the splines ${}_pm_r^j(x)$ are generalized eigenvectors of operators of convolution with any fixed spline and of differentiation. Therefore, the expansion of a spline ${}_pS^j(x)$ with respect to the orthogonal basis $\{{}_pm_r^j(x)\}$ can be treated as a kind of harmonic analysis of a spline ${}_pS^j(x)$ and, if the spline ${}_pS^j(x)$ approximates a function f, it can be looked upon as an approximate harmonic analysis of f. We name this *Spline Harmonic Analysis* (SHA). DFT is a special case of SHA in the space ${}_1\mathfrak{B}^j$, whereas the common Fourier analysis is the limit case of SHA in the spaces ${}_p\mathfrak{B}^j$ as $p \to \infty$. So, loosely speaking, SHA bridges the gap between the continuous and the discrete Fourier analysis. The natural fields of application of SHA are one- and multidimensional problems in which various forms of convolution appear, including differential equations with constant coefficients. The remarkable effectiveness of SHA methods can be demonstrated by solving some ill-posed problems. We will not discuss this here (see, however, [4]) but turn to a recent and very promising application, namely, to the wavelet transforms of periodic splines.

2. Wavelets based on periodic splines

The spaces ${}_p\mathfrak{B}^j$, $j=0, 1, \ldots$, generate a multiresolution analysis (MRA) of the space $L^2(T)$ of 1-periodic square integrable functions [3]. Using the basis of ortsplines $\{{}_pm_r^j(x)\}$ yields a very feasible approach to constructing wavelets and to establishing wavelet relations. The outline of our technique is as follows:

1. The so-called two-scale relations are basic for wavelet construction. We establish these relations for ortsplines; the corresponding relations for B-splines follow from Theorem 1 below. Analogous relations may be established for other spline bases.

2. The two-scale relations for ortsplines enable us to construct a family of orthogonal bases of the space of wavelets ${}_p\boldsymbol{\mathfrak{W}}^{j-1}$, which is the orthogonal complement of ${}_p\mathfrak{B}^{j-1}$ in ${}_p\mathfrak{B}^j$ (Theorem 2). These bases involve the so-called ortwavelets, which generate a family of periodic wavelets in a conventional sense (Theorem 3). The shifts of each of these wavelets form a basis of the space ${}_p\boldsymbol{\mathfrak{W}}^{j-1}$. To change from the basis of ortwavelets to the basis of shifts of a wavelet, one can use the Fast Fourier Transform (FFT).

3. By means of ortwavelets we construct, along with a wavelet basis, the dual basis in the sense of Chui [2] (Theorem 4). The availability of the family of wavelet bases and their duals makes it possible to choose an appropriate basis for the problem to be solved.

4. We emphasize that the choice of a basis determines the procedures of decomposition of a spline into a wavelet representation and the inverse operation of reconstructing a spline from its wavelet representation. Once we use the SHA technique, these procedures become extremely simple. We cite a version of the decomposition procedure (Theorem 5).

We suggest the following scheme of wavelet processing of a function.

a) The function is projected onto a spline space ${}_p\mathfrak{B}^j$.

b) The spline obtained is represented via an ortspline basis by means of DFT.

c) The decomposition of the spline into an ortwavelet representation is carried out.

d) By means of DFT we change to the wavelet representation, whereupon we implement the processing of the spline.

e) The next step is the reciprocal one of changing to the ortwavelet representation, which is followed by the reconstruction of the spline via the ortspline basis of ${}_p\mathfrak{B}^j$.

f) The final step is changing by means of DFT to the B-spline basis of ${}_p\mathfrak{B}^j$ and computing the spline thus processed.

This technique may be extended immediately to the multidimensional case. To be specific, we cite here a few formulas illustrating our approach. For more details see [5].

THEOREM 1. *The following two-scale relations hold for* $r = 0, 1, \ldots, 2^{j-1} - 1$:

$$ {}_p m_r^{j-1}(x) = {}_p b_r^j \, {}_p m_r^j(x) + {}_p b_{r+N/2}^j \, {}_p m_{r+N/2}^j(x), \; {}_p b_r^j = 2^{-p}(1+\omega^{-r})^p $$
$$ \Rightarrow {}_p M^{j-1}(x) = 2^{-p} \sum_{l=0}^{p} \binom{p}{l} {}_p M^j(x - l/N). $$

THEOREM 2. *There exists a family of orthogonal bases*

$$ \{{}_p^\nu w_r^{j-1}(x)\}_0^{N/2-1} \quad of \; {}_p\mathfrak{W}^{j-1} \subset {}_p\mathfrak{V}^j, $$
$$ {}_p^\nu w_r^{j-1}(x) = {}_p^\nu a_r^j \, {}_p m_r^j(x) + {}_p^\nu a_{r+N/2}^j \, {}_p m_{r+N/2}^j(x), \qquad r = 0, 1, \ldots, 2^{j-1} - 1, $$
$$ {}_p^\nu a_r^j = 2^{-p} \omega^r \, {}_p^\nu \tau_r^{j-1} \, {}_p \bar{b}_{r+N/2}^j = 2^{-p} \omega^r (1 - \omega^r)^p \, {}_p^\nu \tau_r^{j-1} ({}_{2p} u_r^j)^{-1}, $$

where $\{{}_p^\nu \tau_r^{j-1}\}$ *is an arbitrary nonzero* 2^{j-1}*-periodic sequence.*

Choosing various sequences $\{{}_p^\nu \tau_r^{j-1}\}$, we obtain various bases $\{{}_p^\nu w_r^{j-1}(x)\}$ of the wavelet space ${}_p\mathfrak{W}^{j-1}$. The family of ortwavelets $\{{}_p^\nu w_r^{j-1}(x)\}$ generates the family of periodic wavelets in a conventional sense. To be precise, denote ${}_p^\nu A_k^{j-1} = \sum_r^j \omega^{rk} \, {}_p^\nu a_r^j$ and define the splines

$$ {}_p^\nu \psi^{j-1}(x) = \frac{1}{N} \sum_k^j {}_p^\nu A_k^{j-1} \, {}_p M^j(x - k/N) = \sum_r^j {}_p^\nu a_r^j \, {}_p m_r^j(x) \in {}_p\mathfrak{W}^{j-1}. $$

There hold the dual relations

$$ {}_p^\nu \psi^{j-1}(x + 2l/N) = \sum_r^{j-1} \omega^{2rl} \, {}_p^\nu w_r^{j-1}(x), $$
$$ {}_p^\nu w_r^{j-1}(x) = \frac{2}{N} \sum_l^{j-1} \omega^{-2rl} \, {}_p^\nu \psi^{j-1}(x + 2l/N). $$

THEOREM 3. *The splines* $\{{}_p^\nu \psi^{j-1}(x - 2l/N)\}_{l=0}^{N/2-1}$ *form a family of bases of the space* ${}_p\mathfrak{W}^{j-1}$; *any spline* ${}_p W^{j-1}(x) \in {}_p\mathfrak{W}^{j-1}$ *can be written as*

$$ {}_p W^{j-1}(x) = \sum_r^{j-1} {}_p^\nu \eta_r^{j-1} \, {}_p^\nu w_r^{j-1}(x) = \frac{2}{N} \sum_l^{j-1} {}_p^\nu t_l^{j-1} \, {}_p^\nu \psi^{j-1}(x - 2l/N). $$

Moreover, the following relations hold:

$$ {}_p^\nu t_l^{j-1} = \sum_r^{j-1} \omega^{2rl} \, {}_p^\nu \eta_r^{j-1} \Leftrightarrow {}_p^\nu \eta_r^{j-1} = \frac{2}{N} \sum_l^{j-1} \omega^{-2rl} \, {}_p^\nu t_l^{j-1}. $$

The theorem enables us to affirm that the splines ${}_p^\nu \psi^{j-1}(x)$ appear as wavelets in the sense of [3]. We emphasize that the shifts of any wavelet ${}_p^\nu \psi^{j-1}(x)$ form a basis of the wavelet space ${}_p\mathfrak{W}^{j-1}$. This family of wavelets contains the compactly supported (up to periodicity) spline wavelet (compare [3]) as well as the spline wavelet whose shifts form an orthonormal basis of ${}_p\mathfrak{B}^{j-1}$ (compare [1]). Together with a basis, this family contains its dual basis. Namely, the following proposition holds.

THEOREM 4. *The relation* $\langle {}^\nu_p\psi^{j-1}(x-2l/N),\ {}^\mu_p\psi^{j-1}(x-2s/N)\rangle = \delta^l_s$ *is true if and only if* ${}^\nu_p\tau^{j-1}_r\,{}^\mu_p\bar{\tau}^{j-1}_r({}_{2p}v^j_r\,{}_{2p}u^j_r)^{-1}\,{}_{2p}u^{j-1}_r = 1/2N$.

We now establish formulae of decomposition.

THEOREM 5. *The following representation holds for* $r = 0, 1, \dots, 2^j - 1$:

$$
\begin{gathered}
{}_pm^j_r(x) = {}_ph^j_r\,{}_pm^{j-1}_r(x) + {}^\nu_pg^j_r\,{}^\nu_pw^{j-1}_r(x),\\
{}_ph^j_r = \omega^r(1+\omega^r)^p\,{}_{2p}u^j_r/2^p\,{}_{2p}u^{j-1}_r,\\
{}^\nu_pg^j_r = (1-\omega^{-r})^p\omega^{-r}\,{}_{2p}v^j_r\,{}_{2p}u^j_r/2^p\,{}_{2p}u^{j-1}_r\,{}^\nu_p\tau^{j-1}_r.
\end{gathered}
$$

Any spline ${}_pS^j(x) = \frac{1}{N}\sum_k^j q^j_k\,{}_pM^j(x-k/N) = \sum_r^j \xi^j_r\,{}_pm^j_r(x) \in {}_p\mathfrak{B}^j$, $\mathbf{q} = \{q_k\}_0^{N-1}$, $\xi^j_r = T^j_r(\mathbf{q})$, *can be written as the following orthogonal sum*:

$$
\begin{gathered}
{}_pS^j(x) = {}_pS^{j-1}(x) + {}^\nu_pW^{j-1}(x),\\
{}_pS^{j-1}(x) = \sum_r^{j-1} {}_pm^{j-1}_r(x)\xi^{j-1}_r = \frac{2}{N}\sum_k^{j-1} q^{j-1}_k\,{}_pM^{j-1}(x-2k/N) \in {}_p\mathfrak{B}^{j-1},\\
{}^\nu_pW^{j-1}(x) = \sum_r^{j-1} {}^\nu_pw^{j-1}_r(x)\,{}^\nu\eta^{j-1}_r = \frac{2}{N}\sum_k^{j-1} {}^\nu t^{j-1}_k\,{}^\nu_p\psi^{j-1}(x-2k/N) \in {}_p\mathfrak{B}^{j-1},\\
\xi^{j-1}_r = \xi^j_r\,{}_ph^j_r + \xi^j_{r+N/2}\,{}_ph^j_{r+N/2}, \qquad {}^\nu\eta^{j-1}_r = \xi^j_r\,{}^\nu_pg^j_r + \xi^j_{r+N/2}\,{}^\nu_pg^j_{r+N/2},\\
q^{j-1}_k = \sum_r^{j-1}\omega^{2kr}\xi^{j-1}_r, \qquad {}^\nu t^{j-1}_k = \sum_k^{j-1} {}^\nu\eta^{j-1}_r.
\end{gathered}
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REFERENCES

1. G. Battle, *A block spin construction of ondelettes. Part* I: *Lemarié functions*, Comm. Math. Phys. **110** (1987), 601–615.
2. C. K. Chui and J. Z. Wang, *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc. **330** (1992), 903–915.
3. Y. Meyer, *Ondelettes et opérateurs*. I, Hermann, Paris, 1990.
4. V. A. Zheludev, *Operational calculus connected with periodic splines*, Soviet. Math. Dokl. **42** (1991), 162–167.
5. ———, *Wavelets based on periodic splines*, Dokl. Akad. Nauk (1994) (to appear).

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