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Periodic Splines and Wavelets

VALERY A. ZHELUDEV

ABSTRACT. We present new computational techniques named as Spline Harmonic Analysis (SHA) and its applications to wavelet transforms. SHA is a version of Harmonic Analysis (HA) based on periodic splines of defect 1 with equidistant nodes. The discrete Fourier Transform (DFT) is a special case of SHA. The continuous Fourier Analysis is the limit case of SHA as the degree of splines involved tends to infinity. Thus SHA bridges the gap between the discrete and the continuous versions of the Fourier Analysis. SHA can be regarded as a computational version of the harmonic analysis of continuous periodic functions from discrete noisy data. We demonstrate this on the basis of periodic splines wavelet transforms of periodic functions. The SHA approach to wavelets yields a tool for constructing a diversity of spline wavelet bases, and a fast implementation of a decomposition of a function into a fitting wavelet representation. Via this approach we are able to construct *wavelet packets* (WP) bases for refined frequency resolution of signals.

1. Introduction

We present new computational techniques named as Spline Harmonic Analysis (SHA) and its applications to wavelet transforms. The starting point for this work was motivated by the papers [2], [3] by C.K.Chui and J.Z.Wang on spline wavelets. Once we have at our disposal a finite set of functionals of a function under consideration and any information on regularity properties of this function, usually, the best we can do is to approximate this function by an appropriate spline. This spline can be a polynomial but need not be. Then, as a matter of fact, processing the function is processing the spline.

Harmonic Analysis, in particular the Fourier Series is a powerful tool for solving many problems, especially the associated with the operators of convolution and of differentiation. But it should be emphasized that this apparatus is not quite suited for an analysis of a function of finite order of smoothness from discrete noisy data because of at least two reasons: 1) Basic functions of the Fourier Analysis — the exponentials are of infinite order of smoothness; 2) The practical computation of Fourier coefficients poses a lot of problems. The alternate version

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of HA — the Discrete Fourier Transform (DFT) is HA of discrete data without taking into account regularity properties of the function under study. Moreover, it is poorly suited to the operations of differentiation and of continuous convolution.

Therefore it is desirable to have a version of HA which operates in spline spaces of appropriate smoothness, moreover the "Fourier coefficients" in their HA should be computable immediately from discrete data. Such analysis promises remarkable benefits for dealing with splines. SHA to be presented is a version of HA based on periodic splines of defect 1 with equidistant nodes. DFT is a special case of SHA. The continuous Fourier Analysis is the limiting case of SHA as the degree of splines involved tends to infinity. Thus SHA bridges the gap between the discrete and the continuous versions of the Fourier Analysis. SHA can be regarded as a computational versions of the harmonic analysis of continuous periodic functions from discrete noisy data.

SHA techniques had been employed advantageously for solving some problems of spline functions [**13**] as well as for regularization of ill-posed problems arising in the numerical solution of some differential and integral equations connected with the convolution operator ([**14**]-[**17**]). However these topics are beyond of the scope; of this paper.

Here we discuss a recent application of SHA techniques, for the basis of periodic splines wavelet transforms of periodic functions. SHA approach to wavelets yields a tool for the constructing a diversity of spline wavelet bases, and a fast implementation of a decomposition of a function into a fitting wavelet representation ;and its reconstruction. It is worth noting that this family of bases contains periodizations of compactly supported spline-wavelets by Chui–Wang [**2**] and the Battle–Lemarie type orthonormal wavelets (cf. [**7**], [**1**]). Via this approach we are able to construct *wavelet packets* (WP) bases for refined frequency resolution of signals.

We consider first the basic notions of SHA and define the *ortsplines* (OS). Then, after an outline of the spline wavelet analysis of periodic functions, we introduce the notion of *father wavelet* (FW) and suggest a diversity of *father wavelet* bases. Next there introduce a *two-scale relation* and *ortwavelets* (OW). Then we discuss *mother wavelets* (MW) and a diversity of *mother wavelet* bases. In section 7 employing the techniques developed we construct WP. Further we describe the decomposition of a spline into a wavelet representation by means of *ortsplines* and *ortwavelets* and its reconstruction.

Abbreviations:

DFT — Discrete Fourier Transform,
FW — father wavelet,
HA — Harmonic Analysis,
MW — mother wavelet,
OS — ortsplines,

OW — ortwavelet,
SHA — Spline Harmonic Analysis,
WP — Wavelet packet.

2. Concept of Spline Harmonic Analysis

We introduce some notions. Throughout $N = 2^j$ and $\sum_k^j$ stands for $\sum_{k=0}^{2^j-1}$. Denote $\omega = \exp(2\pi i/N)$. The inner product of functions involved is

$$< f,g >= 1/N \int_0^1 f(y)\overline{g}(y)dy, \quad \|f\|^2 =< f,f > .$$

Direct and inverse DFT of a vector $a = \{a_k\}_0^{N-1}$ are

$$T_n^j(a) = 1/N \sum_k^j \omega^{-nk} a_k, \; a_k = \sum_n^j \omega^{nk} T_n^j(a), \tag{2.1}$$

Throughout ${}_p\mathfrak{D}^j$ will denote spaces of 1-periodic splines of degree $p-1$ and of defect 1 with their nodes in the points $\{k/2^j\}$, $j = 0,1\ldots$, $k = 0,\ldots 2^j - 1$. The central figure of most spline schemes is the B-spline. The function ${}_pB^j(x) = N^p\nabla_j^p(x_+^{p-1}/(p-1)!)$ where $x_+ = 0.5(x+|x|)$, is the B-spline of degree $p-1$ with nodes at the points $\{k/2^j\}$. The symbol denotes the descending difference with the step $1/2^j$. Note that the support of B-spline supp ${}_p\mathfrak{B}^j(x) = (0, p/2^j)$. The symbol ${}_pM^j(x)$ will denote the 1-periodic B-spline of degree $p-1$:

$${}_pM^j(x) = \sum_{l=-\infty}^{\infty} {}_p\mathfrak{B}^j(x+l) = \sum_{n=-\infty}^{\infty} e^{-\pi inp/N}\left(\frac{\sin(\pi n/N)}{\pi n/N}\right) e^{2\pi inx}. \tag{2.2}$$

Shifts of the B-spline ${}_pM^j(x)$ form a basis of the space ${}_p\mathfrak{D}^j$. Any spline ${}_pS^j \in {}_p\mathfrak{D}^j$ can be represented as follows:

$${}_pS^j(x) = \frac{1}{N}\sum_k^j q_k\, {}_pM^j(x - k/N). \tag{2.3}$$

This representation is the most suitable one for computing values of the spline.

To start construction SHA we carry out a simple transform. Denoting $q = \{q_k^j\}_0^{N-1}$ and, exploiting Eq. (2.1), we write the spline as

$${}_pS^j(x) = \frac{1}{N}\sum_k^j {}_pM^j(x-k/N)\sum_r^j \omega^{rk} T_r^j(q) = \sum_r^j \xi_r^j\, {}_pm_r^j(x), \tag{2.4}$$

$$
\begin{aligned}
{}_pm_r^j(x) &= \frac{1}{N}\sum_k^j {}_pM^j(x-k/N)\omega^{rk} \\
{}_pM^j(x-k/N) &= \sum_r^j {}_pm_r^j(x)\omega^{-rk} \\
\xi_r^j &= T_r^j(q), \quad q_k^j = \sum_r^j \xi_r^j \omega^{rk}
\end{aligned}
\tag{2.5}
$$

Eqs. (2.5) imply that the splines ${}_pm_r^j(x), r = 0, 1, \ldots, 2^j - 1$ form a basis of the space ${}_p\mathfrak{D}^j$, and $\{\xi_r^j\}$ – are coordinates of a spline ${}_pS^j(x)$ with respect to this basis. These basic splines possess a variety of peculiar properties which can be readily checked by reference to relations established in [**2**]. These properties relate the splines $\{{}_pm_r^j(x)\}$ with the exponential functions $\mu_r(x) = \exp(2\pi i r x)$.

First define a sequence which will be needed in what follows:

$$
\begin{aligned}
{}_pu_r^j &= \sin(\pi r/N)^p \sum_{l=-\infty}^{\infty} (-1)^{pl}(\pi(r+lN)/N)^{-p} \\
&= \frac{1}{N}\sum_k^j \omega^{rk}\, {}_pM^j(p/2N - k/N)\, {}_pm_r^j(p/2N).
\end{aligned}
$$

This is a strictly positive N-periodic sequence. The sequences ${}_pu_r^j$ were studied in [**11**], [**12**]. These can be computed immediately. The the following two relations hold

$$
\begin{aligned}
&\langle {}_pm_r^j, {}_pm_n^j \rangle = \delta_n^r {}_{2p}u_r^j. \\
&\sum_k^j {}_pm_r^j(2p/N + k/N)\, {}_pm_n^j(2p/N + k/N) = \delta_n^r (u_n^p)^2.
\end{aligned}
\tag{2.6}
$$

Here δ_n^r is the Kroneker delta. Clearly,

$$
\| {}_pm_r^j \|^2 = {}_{2p}u_r^j. \tag{2.7}
$$

Eq. (2.6) implies, in particular, that the splines ${}_pm_r^j(x)$ form an orthogonal basis of the space ${}_p\mathfrak{D}^j$. Therefore we call these splines as ortsplines (OS).

It should be pointed out that OS have been suggested in [**6**] where the orthogonality of OS is established as well as the following property:

$$
{}_pm_r^j(x)/({}_{2p}u_r^j)^{1/2} \to \mu_r(x), r = -N/2+1, \ldots, N/2-1, \text{ as } p \to \infty. \tag{2.8}
$$

The techniques of SHA have been developed independently in [**13**]-[**17**] under the name Spline Operational Calculus and employed for solving some ill-posed problems.

In addition to Eq. (2.8) it can be easily verified that an orthogonal projection of the function $e^{2\pi inx}, n = r \bmod N$, onto the space ${}_p\mathfrak{D}^j$ is the spline

$$ {}_pm_r^j(x)e^{\pi inp/N}\left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^p / {}_{2p}u_r^j. $$

The spline ${}_p\hat{m}_r^j(x) = {}_pm_r^j(x + p/2N)/{}_pu_r^j$ is an interpolating one for the function $\mu_r(x)$ in the sense that ${}_p\hat{m}_r^j(k/N) = \mu_r(k/N)$. Splines which interpolate the functions $\mu_r(x)$ were suggested by Golomb **[5]** as linear combinations of the Bernoulli polynomials. Because of uniqueness of interpolating splines, the splines ${}_p\hat{m}_r^j(x)$ coincide with the splines by Golomb. There are estimations in **[5]** which lead to the relation ${}_p\hat{m}_r^j(x) \to \mu_r(x), r = -N/2, \dots, N/2$, as $p \to \infty$. And of course, ${}_p\hat{m}_r^j(x) \to \mu_r(x)$ provided $N \to \infty$, as an interpolating spline.

Besides the properties mentioned above we mention three remarkable properties of OS.

Property 1. ${}_pm_r^j(x + k/N) = {}_pm_r^j(x)\omega^{rk}$.

Property 2. ${}_pm_r^j(x)^{(s)} = N^s(1 - \omega^{-r})^s\, {}_{p-s}m_r^j(x) \in {}_{p-s}\mathfrak{D}^j$. Note that the spline ${}_{p-s}m_r^j(x)$ is a replica of the spline ${}_pm_r^j(x)$ in the space ${}_{p-s}\mathfrak{D}^j$.

Property 3. Given a spline ${}_lS^j(x) \in {}_l\mathfrak{D}^j$ as

$$ {}_lS^j(x)\sum_k^j \eta_k^j\, {}_lm_k^j(x), $$

the convolution is ${}_pm_r^j \times {}_lS^j(x) = {}_{l+p}m_r^j(x)\eta_r^j \in p + l\mathfrak{D}^j$.

The Properties 1–3 implies that the splines ${}_pm_r^j(x)$ are eigenvectors of the operator of translation at 2^{-j} and generalized eigenvectors of the operators of convolution with any fixed spline and of differentiation. Therefore the expansion of a spline ${}_pS^j(x)$ with respect to the OS basis $\{{}_pm_r^j(x)\}$, can be treated as a version of HA of a spline ${}_pS^j(x)$ and, if the spline ${}_pS^j(x)$ approximates a function f, it can be looked upon as an approximation of HA of f. We name it as SHA. DFT is a special case of SHA in the space ${}_1\mathfrak{D}^j$, whereas, referring to Eq. (2.8) we can assert that the conventional Fourier Analysis is the limiting case of SHA in the spaces ${}_p\mathfrak{D}^j$ as $p \to \infty$. So, loosely speaking, SHA bridges the gap between the continuous Fourier Analysis and the discrete one. The coordinates of the spline ${}_pS^j(x) : \xi_r^j = \langle {}_pm_r^j(x), {}_pS^j(x)\rangle / {}_{2p}u_r^j$ we may regard as a spectra of the spline ${}_pS^j(x)$.

The natural fields of application of SHA methods, as in the case of classical Fourier Analysis, are one – and multidimensional problems in which various forms of convolution appear, including differential equations with constant coefficients. We enumerate some of these fields.

a) Problems of spline approximation, namely: interpolation, quasiinterpolation, smoothing, projection a function onto a spline space, error evaluation.

b) Numerical solution of differential and integral equations concerned with the convolution operator, including regularization of ill-posed problems arising, such as convolution integral equation of the first kind, inverse problem for the heat equation, Cauchy problem for the Lapalace equation etc.

We will not discuss here these topics ([**13**]-[**17**]) but turn to a recent and very promising application, namely to

c) Wavelet Analysis of periodic functions.

3. Outline of the Wavelet Analysis

We first formulate the concept of multiresolution analysis (MRA) of the space $L^2(T)$ of square integrable 1-periodic functions ([**8**], [**10**]).

DEFINITION. An MRA of the space $L^2(T)$ is a sequence of imbedded closed spaces V^j ($j \geq 0$) such that

(1) $V^0 \subset V^1 \subset \cdots \subset V^j \subset \cdots \subset L^2(T)$.

(2) $\bigcup_{j\geq 0} V^j$ is dense in $L^2(T)$.

(3) V^0 is {constant functions}, $f(x) \in V^j \Rightarrow f(2x) \in V^{j+1}$, $f(x) \in V^{j+1} \Rightarrow f(x/2) + f(x/2 + 1/2) \in V^j$.

(4) $\dim V^j = 2^j$ and for any value of j there exists a *scaling function* ϕ_o^j such that its shifts $\phi_k^j(x) = \phi_0^j(x - k/2^j), k = 0, 1, \ldots, 2^j - 1$ form a basis of the space V^j.

Because of Property (1), the space V^j and be represented as $V^j = V^{j-1} \oplus W^{j-1}$, where W^{j-1} is an orthogonal complementation of the subspace V^{j-1} in the space V^j. It is called the *wavelet space*. Property (4) implies $\dim W^{j-1} = 2^{j-1}$. Properties (1), (2) entail the representation $L^2(T) = V^0 \oplus \bigcup_{j\geq 0} W^j$.

As it is easily seen, the spaces ${}_p\mathfrak{D}^j$ generate MRA of the space $L^2(T)$ with B-spline ${}_pM^j(x)$ as a scaling function. The space ${}_p\mathfrak{D}^{j-1}$ of splines based on the sparse grid is subspace of ${}_p\mathfrak{D}^j$. Define the space of wavelets ${}_p\mathfrak{D}^{j-1}$ as the orthogonal complement of ${}_p\mathfrak{D}^{j-1}$ in ${}_p\mathfrak{D}^j$.

The outline of a practical wavelet analysis of a periodic function is as follows.

1) Projection (approximation) of a function onto a spline space ${}_p\mathfrak{D}^j \subset L^2(T)$ such that $f \to {}_pS^j(f, x) \in {}_p\mathfrak{D}^j$.

2) Decomposition of the spline

$$ {}_pS^j(f,x) = {}_pS^{j-1}(f,x) \otimes {}_pW^{j-1}. $$

where ${}_pS^{j-1} in\, {}_p\mathfrak{D}^{j-1}$, ${}_pW^{j-1} \in {}_p\mathfrak{D}^{j-1}$.

Then the spline ${}_pS^{j-1}$ is decomposed in a similar way:

$$ {}_pS^{j-1}(f,x) = {}_pS^{j-2}(f,x) \otimes {}_pW^{j-2}. $$

Continuing this procedure in accordance with the pyramidal diagram

(3.1)

$$ \begin{array}{ccccccccc} {}_pS^j(x) & \rightarrow & {}_pS^{j-1}(x) & \rightarrow & {}_pS^{j-2}(x) & \rightarrow & \cdots & \rightarrow & {}_pS^{j-m}(x) \\ & \searrow & {}_pW^{j-1}(x) & \searrow & {}_pW^{j-2}(x) & \searrow & \cdots & \searrow & {}_pW^{j-m}(x) \end{array} $$

where $m \leq j$, we obtain the representation:

$$(3.2) \qquad {}_pS^j(f,x) = {}_pS^{j-m} \otimes {}_pW^{j-1} \otimes {}_pW^{j-2} \otimes \dots {}_pW^{j-m}.$$

If there is a need for a refined frequency resolution one can decompose a spline ${}_pW^s(f,x)$ into "low frequency" and "high frequency" parts

$${}_pW^s(f,x) = {}_pW^s_l(f,x) \otimes {}_pW^s_h(f,x)$$

and continue this process with ${}_pW^s_l(f,x)$ or ${}_pW^s_h(f,x)$ or both of these together. Then there appears the so called *wavelet packet* (WP).

3) Multichannel processing the spline represented in a wavelet (or WP) basis. The appropriate choice of a wavelet basis is of prime importance for successful processing.

4) The reconstruction of the spline processed from its wavelet representation in accordance with the reciprocal pyramidal diagram

$$(3.3) \qquad \begin{array}{ccccccccc} {}_pS^{j-m}(x) & \to & \cdots & {}_pS^{j-2}(x) & \to & {}_pS^{j-1}(x) & \to & {}_pS^j(x) \\ {}_pW^{j-m}(x) & \nearrow & \cdots & {}_pW^{j-2}(x) & \nearrow & {}_pW^{j-1}(x) & \nearrow & \end{array}$$

We will discuss in this paper the following problems:

a) Bases in the space ${}_p\mathfrak{D}^v$, ${}_p\mathfrak{B}^v$, ${}_p\mathfrak{D}^v_l$, ${}_p\mathfrak{B}^v_l$ (the latter two are WP spaces).

b) Corresponding procedures of decomposition and reconstruction of splines.

c) Projection of a function onto a spline space.

4. Father Wavelets

We present in this section a family of bases of the space ${}_p\mathfrak{D}^v$. In what follows all splines involved are of degree p and usually we omit the index ${}_p\cdot$.

DEFINITION. We call a spline ${}^s\phi^j(x) \in {}_p\mathfrak{D}^j$ as *father wavelet* (FW) if its shifts ${}^s\phi^j(x - k/2^j), k = 0, 1, \dots, 2^j - 1$ form a basis of the space ${}_p\mathfrak{D}^j$. Two FW are regarded as *dual* ones if

$$\langle {}^s\phi^j(\cdot - k/2^j), {}^\sigma\phi^j(\cdot - l/2^j)\rangle = \delta^l_k.$$

We give necessary and sufficient conditions for a spline to be FW and for two FW to dual.

THEOREM 4.1. *A spline*

$$(4.1) \qquad {}^s\phi^j(x) = 2^{-j/2} \sum_r^j {}^s\rho^j_r m^j_r(x)$$

is an FW if and only if ${}^s\rho^j_r \neq 0, \forall r$. *Two FW are dual if and only if*

$$(4.2) \qquad {}^s\rho^j_r \, {}^\sigma\overline{\rho}^j_r \, {}_{2p}u^j_r = 1.$$

The following assertion relates the coordinates of a spline with respect to a FW basis with those in the OS one.

THEOREM 4.2. *Let*

$$^{s}\phi^{j}(x) = 2^{-j/2}\sum_{r}^{j} {}^{s}\rho_{r}^{j} m_{r}^{j}(x)$$

be an FW and suppose that a spline $S^j(x)$ is expanded with respect to two bases

$$\S^{j}(x) = \sum_{r}^{j} {}^{s}q_{k}^{j}\, {}^{s}\phi^{j}(x - k/N) = \sum_{r}^{j} \eta_{r}^{j} m_{r}^{j}(x).$$

Then

$$(4.3) \qquad {}^{s}q_{k}^{j} = 2^{-j/2}\sum_{r}^{j} \omega^{rk}\eta_{r}^{s}/{}^{s}\rho_{r}^{j}, \quad \eta_{r}^{j} = {}^{s}\rho_{r}^{j}\, 2^{-j/2}\sum_{k}^{j} {}^{s}q_{k}^{j}\omega^{-rk}.$$

Remark 1. If FW ${}^{\sigma}\phi^{j}$ is dual to FW ${}^{s}\phi^{j}$, then

$${}^{s}q_{k}^{j} = \langle S^{j}, {}^{\sigma}\phi^{j}(\cdot - l/2^{j})\rangle.$$

Remark 2. Eq. (4.2) implies that to make the change from an FW basis to the OS one or vice versa, one has to carry on a DFT. Of course, one should use a *fast Fourier transform* (FFT) algorithm for this purpose.

We give some examples of FW.

Examples

1) B-spline. Suppose ${}^{1}\rho_{r}^{j} \equiv 1$, then we can derive immediately form Eq. (2.5) that ${}^{1}\phi^{j}(x) = 2^{-j/2}M^{j}(x)$.
2) FW dual to ${}^{1}\phi^{j}(x)$. Suppose ${}^{2}\rho_{r}^{j} = 1/{}_{2p}u_{r}^{j}$. Then, in accordance with Eq. (4.2), the FW ${}^{2}\phi^{j}(x)$ is dual to ${}^{1}\phi^{j}(x)$.
   Note that if $S^{j}(x) = \sum_{k}^{j} {}^{2}q_{k}^{j}\, {}^{2}\phi^{j}(x - k/N)$, then

$${}^{2}q_{k}^{j} = 2^{-j/2}\int_{0}^{p/N} S^{j}(x - k/N)M^{j}(x)\,dx.$$

   Provided that $S^{j}(x) = S^{j}(f,x)$ is an orthogonal projection of a function f onto the spline space ${}_{p}\mathfrak{D}^{j}$ we have

$${}^{2}q_{k}^{j} = 2^{-j/2}\int_{0}^{p/N} f(x - k/N)M^{j}(x)\,dx.$$

3) Setting ${}^{3}\rho_{r}^{j} = ({}_{2p}u_{r}^{j})^{-1/2}$ we obtain the self-dual FW ${}^{3}\phi^{j}(x)$ those shifts from an orthonormal basis of ${}_{p}\mathfrak{D}^{j}$ (see [**1**]).
4) Interpolating FW. If we set ${}^{4}\rho_{r}^{j} = 1/{}_{p}u_{r}^{j}$ then the FW

$${}^{1}\phi^{j}(x) = 2^{-j/2}{}_{p}L^{j}(x),$$

   where ${}_{p}L^{j}(x)$ is the so called fundamental spline, namely

$${}_{p}L^{j}(k + p/2) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = 1, \dots, N-1. \end{cases}$$

Therefore the spline

$$S^j(x) = \sum_k^j z_{kp} L^j(x - k/N)$$

interpolates a vector $\{z_k\}_k^j$. To be specific, $S^j(k/N + p/2N) = z_k, \forall k$. In what follows u_r^j will stand for ${}_{2p}u_r^j$.

5. Two-scale Relations and Ortwavelets

The so called *two-scale relations* link basic vectors of the space V^{j-1} with those of the space V^j. These relations are fundamental for any wavelet construction. We establish these relations for ortsplines where these relations have an exceptional simplicity.

THEOREM 5.1. *There hold the two-scale relations for* $r = 0, 1, \ldots, 2^{j-1} - 1$*:*

$$m_r^{j-1}(x) = b_r^j m_r^j(x) + b_{r+N/2}^j m_{r+N/2}^j(x), \quad b_r^j = 2^{-p}(1 + \omega^{-r})^p. \tag{5.1}$$

The following identity follows immediately from the latter relations:

$$u_r^{j-1} = {}_{2p}m_r^{j-1}(2p/N) =$$
$$4^{-p}\omega^{rp}\left[(1 + \omega^{-r})^{2p} u_r^j + (-1)^p (1 - \omega^{-r})^{2p} u_{r+N/2}^j\right].$$

The two-scale relations enable us to construct an orthogonal basis of the wavelet space ${}_p\mathfrak{B}^{j-1}$.

THEOREM 5.2. *There exists an orthogonal basis*

$$\{w_r^{j-1}(x)\}_0^{j-1} \text{ of } {}_p\mathfrak{B}^{j-1} \subset {}_p\mathfrak{D}^j$$

$$w_r^{j-1}(x) = a_r^j m_r^j(x) + a_{r+N/2}^j m_{r+N/2}^j(x), \tag{5.2}$$

$$a_r^j = \omega^r a_{r+N/2}^j u_{r+N/2}^j = 2^{-p}\omega^r(1 - \omega^r) u_{r+N/2}^j,$$

moreover

$$\langle w_r^{j-1}, w_r^{j-1} \rangle = v_r^{j-1}.$$

where $v_r^{j-1} = u_r^j u_{r+N/2}^j u_r^{j-1}$ *is a* 2^{j-1}*-periodic sequence.*

We name the splines $w_r^{j-1}(x)$ as *ortwavelets* (OW).

6. Mother Wavelets

We present here a family of bases of the space ${}_p\mathfrak{B}^{j-1}$. The contents of this section are related to Section 4 where we introduced the FW.

DEFINITION. We name a spline ${}^s\psi^{j-1}(x) \in {}_p\mathfrak{B}^{j-1}$ as *mother wavelet* (MW) if its shifts ${}^s\psi^{j-1}(x - k/2^{j-1})$, $k = 0, 1, \dots, 2^{j-1} - 1$ form a basis of the space ${}_p\mathfrak{B}^{j-1}$. Two MW are regarded as *dual* ones if

$$\langle {}^s\psi^{j-1}(\cdot - k/2^j), {}^\sigma\psi^{j-1}(\cdot - l/2^j)\rangle = \delta_k^l.$$

We give conditions for a spline to be MW and for two MW to be dual.

THEOREM 6.1. *A spline*

$$ {}^s\psi^{j-1}(x) = 2^{-j/2} \sum_r^{j-1} {}^s\tau_r^j W_r^{j-1}(x) \tag{6.1}$$

is an MW if and only if ${}^s\tau_r^j \neq 0, \forall r$. *Two MW are dual if and only if*

$$ {}^s\tau_r^{j-1\,\sigma}\tau_r^{j-1}\,{}_{2p}v_r^{j-1} = 1. \tag{6.2}$$

The following assertion relates the coordinates of a spline with respect to a MW basis with those in the OW one.

THEOREM 6.2. *Let*

$${}^s\psi^{j-1}(x) = 2^{-j/2} \sum_r^{j-1} {}^s\tau_r^j W_r^{j-1}(x)$$

be a MW and spline $W^{j-1}(x) \in {}_p\mathfrak{B}^{j-1}$ *is expanded with respect to two bases*

$$W^{j-1}(x) \sum_k^{j-1} {}^sp_k^{j-1}\, {}^s\psi^{j-1}(x - 2k/N) = \sum_r^{j-1} \eta_r^{j-1}\, w_r^{j-1}(x).$$

Then

$$\begin{aligned} {}^sp_k^{j-1} &= 2^{(1-j)/2} \sum_r^{j-1} \omega^{2rk} \eta_r^{j-1} / {}^s\tau_r^{j-1}, \\ \eta_r^{j-1} &= {}^s\tau_r^{j-1} 2^{(1-j)/2} \sum_k^{j-1} {}^sp_k^{j-1} \omega^{-2rk}. \end{aligned} \tag{6.3}$$

Remark 3. If MW ${}^\sigma\psi^{j-1}$ is dual to MW ${}^s\psi^{j-1}$, then

$${}^sp_k^{j-1} = \langle S^j, {}^\sigma\psi^{j-1}(\cdot - l/2^{j-1})\rangle.$$

Remark 4. Eqs. (6.2) imply that to make the change from an MW basis to the OW one or the reverse change, one has to perform a DFT.

We present some examples of MW.

Examples

1) B-wavelet. Suppose ${}^1\tau^j \equiv 1$. The determining feature of the wavelet ${}^1\psi^{j-1}(x)$ is the compactness (up to periodization) of its support. To be precise. supp ${}^1\psi^{j-1}(x) \subseteq ((-2p)/N, (2p-2)/N) (\bmod N)$. The wavelet $2^{(1-j)/2}\, {}^1\psi^{j-1}(x)$ appears as a periodization of the B-wavelet suggested by Chui and Wang in [**2**].
2) MW is dual to ${}^1\psi^{j-1}(x)$. Suppose ${}^2\tau^j = 1/v_r^{j-1}$. Then, in accordance with Eq. (6.1), the MW ${}^2\psi^{j-1}(x)$ is dual to ${}^1\psi^{j-1}(x)$.
Observe that if $S^j(x) = S^{j-1}(x) \oplus W^{j-1}(x)$ and
$$W^{j-1}(x) = \sum_k^{j-1} {}^2p_k^{j-1}\, {}^2\psi^{j-1}(x - 2k/N)$$
then
$${}^2p_k^{j-1} = \int_{-2p/N}^{(2p-2)/N} S^j(x - 2k/N)\, {}^1\psi^{j-1}(x)\, dx.$$
Provided that $S^j(x) = S^j(f,x)$ is an orthogonal projection of a function f onto the spline space ${}_p\mathfrak{D}^j$ we have
$${}^2p_k^{j-1} = \int_{-2p/N}^{(2p-2)/N} f^j(x - 2k/N)\, {}^1\psi^{j-1}(x)\, dx.$$
Loosely speaking, expanding a spline with respect to the basis $\{{}^2\psi^{j-1}(x - 2k/N)\}$ can be looked upon as a spatially local spectral analysis of the spline.
3) Setting ${}^3\tau^j = (v_r^{j-1})^{-1/2}$ we obtain the self-dual MW ${}^3\psi^{j-1}(x)$; those shifts from an orthonormal basis of ${}_p\mathfrak{D}^{j-1}$ This MW appears as a periodization of the Battle-Lemarié wavelet ([**1**], [**10**], [**7**]).
4) Cardinal MW. If we set ${}^4\tau^{j-1} = 1/v_r^{j-1}$ then obtain the MW
$${}^4\psi^{j-1}(x) = 2^{(1-j)/2}\,{}_{2p}L^j(x + 1/N)^{(p)},$$
where ${}_{2p}L^j(x)$ is the fundamental spline of degree $2p-1$ introduced in Section 4. $2^{(1-j)/2}\, {}^4\psi^{j-1}(x)$ is a periodization of the cardinal wavelet suggested by Chui and Wang in [**3**].

7. Wavelet Packets

To obtain a refined frequency resolution we use the so called *wavelet packets* (WP) ([**9**], [**4**]). We construct WP on the basis of OW.

We call the splines
$${}^l w_r^{j-2}(x) = b_r^j w_r^{j-1}(x) + b_{r+N/4}^{j-1}\, w_{r+N/4}^{j-1}(x) \in {}_p\mathfrak{B}^{j-1},$$
$r = 0, 1, \ldots, 2^{j-2} - 1$, *low-frequency OW* (LOW) and the splines
$${}^h w_r^{j-2}(x) = {}^1a_r^j w_r^{j-1}(x) + {}^1a_{r+N/4}^{j-1}\, w_{r+N/4}^{j-1}(x) \in {}_p\mathfrak{B}^{j-1},$$
$${}^1a_r^j = \omega^{2r}\, b_{r+N/4}^{j-1} \|w_{r+N/4}^{j-1}\|^2 = 2^{-p}\omega^{2r}(1 - \omega^{2r})^p\, v_{r+N/4}^{j-1},$$

. $r = 0, 1, \dots, 2^{j-2} - 1$, *high-frequency OW* (HOW).

THEOREM 7.1. *The following relations hold:*

$$\langle {}^l w_r^{j-2}, {}^l w_s^{j-2} \rangle = \delta_r^s \, {}^l V_r^{j-2}, \quad \langle {}^h w_r^{j-2}, {}^h w_s^{j-2} \rangle = \delta_r^s \, {}^h V_r^{j-2},$$

where

$$\begin{aligned} {}^l V_r^{j-2} &= \|b_r^{j-1}\|^2 \, v_r^{j-1} + \|b_{r+N/4}^{j-1}\|^2 \, v_{r+N/4}^{j-1} \\ &= (\cos(2\pi r/N))^{2p} \, v_r^{j-1} + (\sin(2\pi r/N))^2 p \, v_{r+N/4}^{j-1}, \\ {}^h V_r^{j-2} &= \|{}^1 a_r^{j-1}\|^2 \, v_r^{j-1} + \|{}^1 a_{r+N/4}^{j-1}\|^2 \, v_{r+N/4}^{j-1} \\ &= (\sin(2\pi r/N))^{2p} \, v_r^{j-1} (v_{r+N/4}^{j-1})^2 + (\sin(2\pi r/N))^2 p \, v_{r+N/4}^{j-1} (v_r^{j-1})^2, \end{aligned}$$

Moreover, $\langle {}^l w_r^{j-2}, {}^l w_s^{j-2} \rangle = 0 \forall r, s.\ 2^{-p} \omega^{2r} (1 - \omega^{2r})^p \, v_{r+N/4}^{j-1},$

COROLLARY 7.2. *The splines* $\{{}^l w_r^{j-2}(x), {}^l w_s^{j-2}(x)\}_r^{j-1}$ *form an orthogonal basis of the space* ${}_p\mathfrak{B}^{j-1}$.

These results enable us to decompose the wavelet space ${}_p\mathfrak{B}^{j-1}$ into an orthogonal sum of spaces. To be specific, we define the space ${}^l\mathfrak{B}^{j-2} \subset {}_p\mathfrak{B}^{j-1}$ as ${}^l\mathfrak{B}^{j-2} = \operatorname{span} \{{}^l w_r^{j-2}(x)\}_r^{j-2}$ and the space ${}^h\mathfrak{B}^{j-2} \subset {}_p\mathfrak{B}^{j-1}$ as ${}^h\mathfrak{B}^{j-2} = \operatorname{span} \{{}^h w_r^{j-2}(x)\}_r^{j-2}$. Then it is readily seen that

$${}_p\mathfrak{B}^{j-1} = {}^l\mathfrak{B}^{j-2} \oplus {}^h\mathfrak{B}^{j-2}.$$

It is reasonable to name the space ${}^l\mathfrak{B}^{j-2}$ as the *low-frequency wavelet space* and the space ${}^h\mathfrak{B}^{j-2}$ as the *high-frequency wavelet space.*

If needed we can decompose in a similar manner one (or both) of the spaces ${}^l\mathfrak{B}^{j-2}, {}^h\mathfrak{B}^{j-2}$ into orthogonal sums of spaces ${}^{ll}\mathfrak{B}^{j-3}, \oplus {}^{hl}\mathfrak{B}^{j-3}$ and ${}^{lh}\mathfrak{B}^{j-3}, \oplus {}^{hh}\mathfrak{B}^{j-3}$ respectively and iterate this process. So we have the diagram.

$$\begin{array}{ccccccc} & & & {}_p\mathfrak{B}^{j-1} & & & \\ & & \swarrow & & \searrow & & \\ & {}^l\mathfrak{B}^{j-2} & & \oplus & & {}^h\mathfrak{B}^{j-2} & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ {}^{ll}\mathfrak{B}^{j-3} & \oplus & {}^{hl}\mathfrak{B}^{j-3} & \oplus & {}^{lh}\mathfrak{B}^{j-3} & \oplus & {}^{hh}\mathfrak{B}^{j-3} \\ \dots\ \dots\ \dots\ \dots & & \dots & \dots & \dots & & \dots\ \dots\ \dots\ \dots \end{array} \tag{7.1}$$

As in the precious sections we can find splines whose shifts form bases of the spaces ${}^l\mathfrak{B}^{j-2}$ and ${}^h\mathfrak{B}^{j-2}$. For example, we call a spline ${}^l_s\psi^{j-2}(x) \in {}^l\mathfrak{B}^{j-2}$ a *low-frequency MW* (LMW) if its shifts ${}^l_s\psi^{j-2}(x - k/2^{j-2}), k = 0, 1, \dots, 2^{j-2} - 1$ form a basis of the space. Tow LMW are regarded as dual ones if

$$\langle {}^l_s\psi^{j-2}(\cdot - k/2^j), {}^l_\sigma\psi^{j-2}(\cdot - l/2^j) \rangle = \delta_k^l.$$

THEOREM 7.3. *A spline*

$$(7.2) \qquad {}^{l}_{s}\psi^{j-2} = 2^{(2-j)/2} \sum_{r}^{j-2} {}^{l}_{s}\tau_r^{j-2} {}^{l}w_r^{j-2}(x)$$

is an LMW is and only if ${}^{l}_{s}\tau_r^{j-2} \neq 0, \forall r$. *Two MW are dual if and only if*

$${}^{l}_{s}\tau_r^{j-2} \, {}^{l}_{\sigma}\tau_r^{j-2} = {}^{l}V_r^{j-2}$$

Note that, setting ${}^{l}_{s}\tau_r^{j-2} \equiv 1$, we obtain the LMW of minimal support, so to say, B-LW.

Similar considerations can be conducted in the space ${}^{l}_{h}\psi^{j-2}$.

8. Decomposition of a Spline into Wavelet Representation

We discuss here the decomposition of a spline $S^j(f,x) \in {}_p\mathfrak{D}^j$ approximating a function f into the orthogonal sum

$$S^j(f,x) = {}_pS^{j-1}(f,x) \otimes {}_pW^{j-1}(f,x).$$

where ${}_pS^{j-1}(f,x) \in {}_p\mathfrak{D}^{j-1}, W^{j-1}(f,x) \in {}_p\mathfrak{B}^{j-1}$.

Assume that the spline is expanded with respect to the OS basis

$$(8.1) \qquad S^j(f,x) = \sum_{r}^{j} \xi_r^j \, m_r^j(x).$$

The coefficients $\xi_r^{j-1}, \eta_r^{j-1}$ are needed for the representation

$$S^j(f,x) = \sum_{r}^{j-1} \xi_r^{j-1} \, m_r^{j-1}(x) + \sum_{r}^{j-1} \eta_r^{j-1} \, w_r^{j-1}(x) +$$

To obtain these coefficients we should form inner products $\langle S^j, m_r^{j-1} \rangle$ and $\langle S^j, w_r^{j-1} \rangle$ and use Eq. (5.1), (5.2). We then obtain

THEOREM 8.1. *The following relations hold:*

$$\begin{aligned} \xi_r^{j-1} &= (u_r^{j-1})^{-1}(\xi_r^j \bar{b}_r^j u_r^j + \xi_{r+N/2}^j \bar{b}_{r+N/2}^j u_{r+N/2}^j), \\ \eta_r^{j-1} &= (u_r^{j-1})^{-1}(\xi_r^j \bar{a}_r^j u_r^j + \eta_{r+N/2}^j \bar{a}_{r+N/2}^j u_{r+N/2}^j), \end{aligned}$$

Then the procedure of decomposition is iterated in accordance with diagram (3.1). Formulas of Section 7 enable us if necessary to subject the splines $W^{j-1}(f,x)$ to a similar decomposition which leads to WP.

Remark 5. For analyzing and processing a signal we usually need the representation of the splines $W^{j-1}(f,x)$ in an MW basis $\{{}^s\psi^{j-1}(x-k/2^{j-1})\}^{j-1}$. Because of (6.3) and (6.4) we can maintain that the basis of shifts of MW ${}^2\psi^{j-1}(x)$ which is dual to B-wavelet ${}^1\psi^{j-1}(x)$ appears as the most feasible basis for spatially local analysis of a signal. For changing from the OW basis to an MW one we employ Eq. (6.2).

We now discuss the problem of obtaining the representation (8.1) which is the starting point for the decomposition.

1. If we construct $S^j(f,x)$ as a spline interpolating the function f, then $\xi_r^j = T_r^j(f)/_p u_r^j$, $f = \{f(k/N + p/2N)\}_k^j$.

2. The most natural choice of the spline $S^j(f,x)$ for wavelet algorithms is the original projection of the function f onto the spline space ${}_p\mathfrak{D}^j$. In this case $\xi_r^j = T_r^j(F)/_{2p}u_r^j$, $\mathbf{F} = \{F_k^j\}_k^j$,

$$F_k^j = \int_0^{p/N} f(x - k/N) M^j(x)\,dx.$$

Once we have at out disposal samples of the function f we can suggest the following approximate formula for computing the values F_k^j, which follows from some results established in **[18]**.

THEOREM 8.2. *If $f \in \mathbf{C}^p$ and $t \in [0,1]$ is any fixed value, then*

$$F_k^j = 2^{-j} \sum_{l=0}^{p} f((l-k+t)/N) M^j((l+t)/N) + O(N^{-p}).$$

Remark 6. In the case when a function f is of less smoothness than $\mathbf{C}^p$ one may use that fact that B-spline ${}_pB^j(x)$ (recall that the spline ${}_pM^j(x)$ is a periodization of ${}_pB^j(x)$) is the probability density of the sum of p random variables uniformly distributed on $[0, 1/N]$. Therefore F_k^j can be looked upon as mean values of the function $f(x - k/N)$ with respect to the distribution ${}_pB^j(x)$ and can be computed by means of the Monte Carlo Method.

9. Reconstruction of a Spline from its Wavelet Representation

After a processing a spline $S^j(x) \in {}_p\mathfrak{D}^j$ in a wavelet basis it is required to reconstruct this into the standard form suited for computation.

$$S^j(x) = \frac{1}{N} \sum_k^j q_{kp}^j M^j(x - k/N)$$

First we should change from FW-MW bases to OS-OW ones in accordance with Eq. (4.3), (6.2) exploiting in the process the FFT techniques. So suppose we have two splines

$$\begin{aligned} S^{j-1}(x) &= \sum_r^{j-1} m_r^{j-1}(x)\,\xi_r^{j-1} \in {}_p\mathfrak{D}^{j-1}, \\ W^{j-1}(x) &= \sum_r^{j-1} w_r^{j-1}(x)\,\eta_r^{j-1} \in {}_p\mathfrak{W}^{j-1}. \end{aligned}$$

Let $S^j(x) = S^{j-1} \oplus W^{j-1}(x)$. We are able to prove the following assertion.

THEOREM 9.1. *The following relations hold:*

$$
\begin{aligned}
S^j(x) &= \frac{1}{N}\sum_k^j q_k^j M^j(x-k/N) = \sum_r^j \xi_r^j {}_p m_r^j(x), \\
\xi_r^j &= b_r^{j-1}\xi_r^{j-1} + {}_p a_r^{j-1}\eta_r^{j-1}, \quad q_k^j = \sum_r^j \omega^{kr}\xi_r^j.
\end{aligned}
$$

This means that, given the representation of a spline in the form (3.2), it is possible to reconstruct it into the conventional form (9.1) in line with the diagram (3.3).

The algorithm suggested allows a fast implementation.

Remark 7. To compute values and draw a spline

$$
W^{j-1}(x) = \sum_r^{j-1} w_r^{j-1}(x)\,\eta_r^{j-1} \in {}_p\mathfrak{W}^{j-1}
$$

one may carry on the suggested reconstruction procedure assuming $\xi_r^j \equiv 0$.

10. Concluding Remarks

1. The techniques suggested can be expanded immediately to the multidimensional case.

2. One can develop a *whole-axis* version of the SHA which is related in some sense to the Fourier Transform. By means of this SHA version we are able to construct the *whole-axis wavelet analysis.*

3. The relations of the wavelet analysis established in the paper by means of the SHA approach as well as some results of the theory of local splines [**18**] enable us to construct local wavelet algorithms processing a signal in *real-time conditions.*

4. Since OS are generalized eigenvectors of operators of convolution with any fixed spline and of differentiation, the approach suggested is natural for solving problems where these operators are involved.

All these topics will be discussed in subsequent papers of the author.

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References

1. Battle, G. *A block spin construction of ondelettes* Part I, Lemarie functions. Comm. Math. Phys. **110** (1987), 601–615.
2. Chui, C.K. and Wang, J.Z. *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc. **330** (1992), 903–915.
3. Chui, C.K. and Wang, J.Z. *Cardinal spline approach to wavelets*, Proc. Amer. Math. Soc. **113** (1991), 785–793.
4. Chui, C.K. and Chun, L. *Nonorthogonal wavelet packets*, SIAM J. Math. Anal. **24** (1993), 712–738.
5. Golomb, M. *Approximation by periodic spline interpolants on uniform mesh*, J. Approx. Theory **1** (1968), 26–65.
6. M. Kamada, K. Toraici and Riochi Mori *Periodic splines orthonormal bases*, J. Approx. Theory **55** (1988), 27–34.
7. Lemarié, P.G. *Ondelettes á localisation exponentielle*, J de Math. Pure et Appl. **67** (1988), 227–236.
8. Mallat, S. *Theory for multiresolution signal decomposition: the wavelet representation*, IEEE Trans. Pattern Anal. & Mash. Intel. **11** (1989), 674–693.
9. Meyer, Y. *Wavelets; Algorithms & Applications*, SIAM, Philadelphia, 1993.
10. V. Perrier et C. Basdevant *La décomposition en ondelettes périodiques, un outil pour l'analyse de champs inhomogènes*, Théorie et algorithmes. La recerche aérospatiale. **3** (1983), 53–67.
11. I. J. Schoenberg, *Contribution to the problem of approximation of equidistant data by analytic functions*, Quart. Appl. Math. **4** (1946), 45–99, 112–141.
12. Yu.N.Subbotin, *On the relation between finite differences and the corresponding derivatives*, Proc. Steklov Inst¿ Math. **78** (1965), 24–42.
13. V.A.Zheludev, *Periodic splines and the fast Fourier transform*, Comput. Maths Math. Phys. **32** (1992), 149–165.
14. V.A.Zheludev, *An operational calculus connected with periodic splines*, Soviet. Math. Dokl. **42** (1991), 162–167.
15. V.A.Zheludev, *Spline-operational calculus and inverse problem for heat equation*, Proc. Int. Conference on Approximation Theory, Kecskemét (hungary), 1990, editors J.Szabados, KTandoi. Colloq. Math. Soc. J. Bolyai, **58** (1991), 763–783.
16. V.A.Zheludev, *Spline-operational calculus and numerical solving convolution integral equations of the first kind*, Diff. Equations **28** (1992), 269–280.
17. V.A.Zheludev, *Spline-operational calculus and its application for numerical solving ill-posed Cauchy problem for the Laplace equation*, Proc. of the Jyväskylä – St. Petersburg Seminar on Partial Differential Equations and Numerical methods, Ber. Univ. Jyväskyä Math. Inst. **56** (1993), 133–154.
18. V.A.Zheludev, *Local spline approximation on a uniform grid*, Compt. Maths Math. Phys. **27** (1987) No. 5.

Department of Mathmatics, Military Institute for Constructive Engineering, Zakharjevskaja 22, 191194 St. Petersburg, Russia