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PERIODIC SPLINES AND THE FAST FOURIER TRANSFORM†

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Periodic smoothing splines of arbitrary degrees and deficiency 1 on a uniform grid are studied. The method proposed for constructing such splines, based on the discrete Fourier transform, obviates the need to solve systems of equations for the coefficients and yields explicit expressions. This made it possible to investigate the approximating and smoothing properties of splines and their derivatives. To illustrate the applications, formulae are derived, by means of which it is possible to reduce the data arrays to be processed and the computational load involved in using fast Fourier transforms.

INTRODUCTION

TO CONSTRUCT smoothing and interpolating splines one usually has to solve systems of equations with band matrices, in which the number of diagonals increases with the degree of the spline. Thus the construction of high-degree splines, particularly in many dimensions, involves cumbersome computations. It is due to the lack of an explicit representation that relatively little progress is being made in the investigation of the approximating and smoothing properties of smoothing splines and their derivatives.

However, the situation becomes simpler for periodic splines of deficiency 1 on a uniform grid. It was pointed out in [1] that, since the matrix of the system to be solved in computing the coefficients of periodic interpolating splines is the circulant, there are explicit expressions for the coefficients in terms of the discrete Fourier transform (DFT) (see also [2, 3]). Similar formulae have been developed by others. Explicit formulae for calculating cubic periodic interpolating splines — based, however, on different ideas — were given in [4]. Formulae for the characteristic functions of cardinal interpolating splines have been given in [5]. In the same paper, and also in [6–8], these formulae were studied in detail, which proved rather useful for investigating periodic splines. Formulae for calculating cardinal interpolating splines were also given in [5–8]. Incidentally, such schemes were published even before, long before Schoenberg first introduced the concept of splines [9].

The technique proposed in this paper is based on a special operational calculus in the space of periodic splines. It enables one, in particular, to obtain explicit formulae for the coefficients of smoothing splines of arbitrary degrees. The use of DFTs makes it possible to apply fast Fourier transform (FFT) algorithms to reduce the volume of computations. This makes it possible to determine the optimal value of the smoothing parameter without having to calculate spline coefficients explicitly at each iteration, as is necessary in the usual algorithms. The prospects for the use of this technique are in fact considerably greater (see [10]).

The availability of an explicit representation for splines makes it possible to investigate a number

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of their properties. Asymptotic formulas, in terms of powers of the step size, are obtained for the remainder term in approximations of functions and their derivatives by smoothing splines. Sharp estimates are derived for the variances of these splines when the grid data are random variables.

If the components of the vector $\mathbf{z}=\{z_k\}_0^{2N-1}$ are the grid values (possible with errors) of a sufficiently smooth function and the FFT of $\mathbf{z}$ is to be computed, the use of splines of even degrees yields a marked reduction of the data blocks to be processed and the volume of computations. Error bounds will be derived for this technique.

Though the results outlined below pertain to one-dimensional splines, they can all be generalized directly to several dimensions.

1. PRELIMINARIES

1. *Notation; some known facts*

Let $w=e^{2\pi i/N}$ and let N be a natural number; $v_n=2\sin(\pi n/N)$, $V_n=[\sin(\pi n/N)]\times(\pi n/N)^{-1}$. The DFT of a vector $\mathbf{a}=\{a_k\}_0^{N-1}$ is

$$T_n(\mathbf{a})=\frac{1}{N}\sum_k w^{-nk}a_k.$$

Throughout this paper the symbol Σ_k will stand for $\Sigma_{k=0}^{N-1}$. The norm of the vector $\mathbf{a}$ is

$$\|\mathbf{a}\|=\left[\frac{1}{N}\sum_k a_k^2\right]^{1/2}.$$

The following properties of the DFT are known:

$$a_k=\sum_n w^{nk}T_n(\mathbf{a}), \tag{1.1}$$

$$\frac{1}{N}\sum_k a_k b_k=\sum_n T_n(\mathbf{a})\overline{T_n(\mathbf{b})}\Rightarrow\|\mathbf{a}\|^2=\sum_n|\|T_n(\mathbf{a})\||^2. \tag{1.2}$$

We introduce two grids over the x axis: $\Xi=\{x_k=k/N\}$, $\Xi^p=\{x_k^p=(k+p/2)/N\}$. Let $\mathfrak{S}^p$ denote the space of 1-periodic splines of degree $p-1$ and deficiency 1 with nodes at the points x_k^p. The symbol $M^p(x)$ will denote the central 1-periodic B-spline of degree $p-1$ (see [11]):

$$M^p(x)=\sum_{n=-\infty}^{\infty} V_n{}^p e^{2\pi i n x}. \tag{1.3}$$

The support of the B-spline is

$$\operatorname{supp} M^p(x)=\bigcup_{k=-\infty}^{\infty}\Omega_k{}^p,\qquad \Omega_k{}^p=(k+p/(2N),k-p/(2N)).$$

In the interval Ω_k^p we have

$$M^p(x)=N^p\partial^p(x_+^{p-1}/(p-1)!),\qquad x_+=0.5(x+|x|).$$

The symbol ∂, as usual, denotes the central difference with step size $1/N$. We set

$$m_n^p(x) = \frac{1}{N} \sum_k w^{-nk} M^p(x+x_k), \tag{1.4}$$

$$\mathbf{M}^p = \{M^p(x_k)\}_0^{N-1}, \quad u_n^p = T_n(\mathbf{M}^p) = \frac{1}{N} \sum_k w^{-nk} M^p(x_k). \tag{1.5}$$

Obviously,

$$u_n^p = m_n^p(x_0). \tag{1.6}$$

The functions u_n^p were investigated in [5–7]. Recurrence relations and explicitly formulae are available (see [6]) for the first values of p. The following representation holds for arbitrary p (see [6]):

$$u_n^p = \sum_{k=0}^{l} \gamma_k^p v_n^{2k}, \quad l = [(p+1)/2], \quad \gamma_0^p = 1. \tag{1.7}$$

It is important for our purposes that

$$0 < \varkappa_{p-1} = u_{N/2}^p \leqslant u_n^p \leqslant u_0^p = 1, \quad \varkappa_p = \mathrm{K}_p [2/\pi]^p, \tag{1.8}$$

where K_p is Favard's constant (see [7]).

If $g(x)$ is a 1-periodic continuous function and $\mathbf{g} = \{g(x_k)\}_0^{N-1}$, there is a connection between the Fourier coefficients of g:

$$c_n(g) = \int_0^1 e^{-2\pi i n y} g(y)\, dy,$$

and the DFT of the vector $\mathbf{g}$:

$$T_n(g) = \sum_{l=-\infty}^{\infty} c_{n+lN}(g).$$

This formula, together with (1.4), (1.5), imply the relation

$$u_n^p = \sum_{l=-\infty}^{\infty} V_{n+lN}^p. \tag{1.9}$$

2. *Auxiliary relations*

Any spline in $\mathfrak{S}^p$ may be written as

$$S^p(x) = \frac{1}{N} \sum_k q_k M^p(x-x_k). \tag{1.10}$$

It is evident from this formula that the spline $S^p(x)$ is uniquely defined by its order p and its coefficient vector $\mathbf{q} = \{q_k\}_0^{N-1}$. Let us put $\{T_n(\mathbf{q})\}_0^{N-1} = Q(S^p)$ and call this vector the transform of $S^p(x)$. The correspondence $S^p(x) \leftrightarrow Q(S^p)$ is one-to-one: given any vector Q it is easy to recover S^p by using (1.1) and (1.10).

Put $\mathbf{S}^p = \{S^p(x_k)\}_0^{N-1}$. By (1.10),

$$S^p(x_k) = \frac{1}{N} \sum_k q_l M^p(x_k - x_l) = \frac{1}{N} \sum_k q_l M^p(x_{k-l}).$$

This is a discrete convolution and, by formula (1.3),

$$T_n(\mathbf{S}^p)=T_n(\mathbf{q})\,u_n^p. \tag{1.11}$$

Formulas (1.10) and (1.4) now yield expressions for the Fourier coefficients:

$$c_n(S^p)=\int_0^1 e^{-2\pi i n y}S^p(y)\,dy= \tag{1.12}$$

$$=\frac{1}{N}\sum_k q_k\int_0^1 e^{-2\pi i n y}M^p(y-x_k)\,dy=T_n(\mathbf{q})\,V_n^p.$$

For the derivatives

$$c_n((S^p)^{(s)})=(2\pi i n)^s T_n(\mathbf{q})\,V_n^p=(iNv_n)^s T_n(\mathbf{q})\,V_n^{p-s}. \tag{1.13}$$

The following theorem is an obvious result of (1.12) and (1.13).

Theorem 1. If $S^p\in\mathfrak{S}^p$, then the derivative $(S^p)^{(s)}\in\mathfrak{S}^{p-s}$ may be written in the form

$$S^p(x)^{(s)}=\frac{1}{N}\sum_k q_k^s M^{p-s}(x-x_k).$$

Moreover, if we put $\mathbf{S}^{sp}=\{(S^p(x_k))^{(s)}\}_0^{N-1}$, $\mathbf{q}^s=\{q_k\}_0^{N-1}$, then

$$T_n(\mathbf{q}^s)=T_n(\mathbf{q})\,(iNv_n)^s,\qquad T_n(\mathbf{S}^{sp})=T_n(\mathbf{q})\,(iNv_n)^s u_n^{p-s} \tag{1.14}$$

Let $S^l\in\mathfrak{S}^l$ be a spline such that

$$S^l(x)=\frac{1}{N}\sum_k r_k M^l(x-x_k).$$

There are valid analogues of the Parseval equality. By this equality,

$$\int_0^1 S^p(x)S^l(x)\,dx=\sum_{n=-\infty}^{\infty}\bar{c}_n(S^l)\,c_n(S^p)=\sum_{n=-\infty}^{\infty}T_n(\mathbf{q})\overline{T_n(\mathbf{r})}\,V_n^{p+l}=$$

$$=\sum_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}\sum_{m=-\infty}^{\infty}V_m^{p+l}=\sum_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}\,u_m^{p+l}$$

by (1.12) and (1.9). Hence, in particular, using (1.14), we obtain

$$\int_0^1 [S^p(x)^{(s)}]^2\,dx=\sum_n |T_n(\mathbf{q})|^2 u_n^{2(p-s)}(Nv_n)^{2s}. \tag{1.15}$$

The discrete "Parseval equalities" now follow from (1.2) and (1.11):

$$\frac{1}{N}\sum_k S^p(x_k)S^l(x_k)=\sum_n T_n(\mathbf{q})\overline{T_n(\mathbf{r})}\,u_n^p u_n^l.$$

Hence, in particular, it follows that

$$\frac{1}{N}\sum_k S(x_k)^2=\sum_n |T(\mathbf{q})|^2(u_n^p)^2.$$

These formulae will be used to construct and study smoothing splines.

2. THE CONSTRUCTION OF SMOOTHING SPLINES

Suppose we are given a vector $\mathbf{z}=\{z_k\}_0^{N-1}$.

Problem 1. a. Find a spline $S^p(x)\in\mathfrak{S}^p$ that minimizes the functional

$$I(S^p)=N^{-2m}\int_0^1[S^p(x)^{(m)}]^2\,dx$$

provided that

$$E(S^p)=\frac{1}{N}\sum_k[S^p(x_k)-z_k]\leq\varepsilon^2.$$

b. Find a spline $S^p(x)\in\mathfrak{S}^p$ that minimizes the functional

$$J_\rho(S^p)=E(S^p)+\rho I(S^p).$$

c. Find a spline $S^p(x)\in\mathfrak{S}^p$ that satisfies the conditions

$$S^p(x_k)=z_k,\qquad k=0,1,\ldots,N-1.$$

Solution of problem 1. b. Write the unknown splines $S^p(x)$ in the standard form (1.10). By (1.15),

$$I(S^p)=\sum_n|T_n(\mathbf{q})|\,u_n^{2(p-m)}v_n^{2m},$$

and formulae (1.2), (1.11) yield

$$E(S^p)=\sum_n|T_n(\mathbf{q})u_n^p-T_n(\mathbf{z})|^2. \tag{2.1}$$

The DFT of the vector $\mathbf{a}$ may be written in the form $T_n(\mathbf{a})=C_n(\mathbf{a})-iS_n(\mathbf{a})$, where $C_n(\mathbf{a})$ and $S_n(\mathbf{a})$ are the cosine and sine DFTs, respectively. We now write the functional in the form

$$J_\rho(S^p)=E(S^p)+\rho I(S^p)=\sum_n\{\rho[C_n(\mathbf{q})^2+S_n(\mathbf{q})^2]u_n^{2(p-m)}v_n^{2m}+$$
$$+[C_n(\mathbf{q})u_n^p-C_n(\mathbf{z})]^2+[S_n(\mathbf{q})u_n^p-S_n(\mathbf{z})]^2\}.$$

It is easy to see that $J_\rho(S^p)$ will be a minimum if

$$C_n(\mathbf{q}(\rho))=\frac{C_n(\mathbf{z})u_n^p}{A_n(\rho)},\qquad S_n(\mathbf{q}(\rho))=\frac{S_n(\mathbf{z})u_n^p}{A_n(\rho)}, \tag{2.2a}$$

$$A_n(\rho)=\rho v_n^{2m}u_n^{2(p-m)}+(u_n^p)^2. \tag{2.2b}$$

This implies the following theorem.

Theorem 2. The following spline solves problem 1b:

$$S_\rho^p(\mathbf{z},x)=\frac{1}{N}\sum_k q_k(\rho)M^p(x-x_k),\qquad \mathbf{q}(\rho)=\{q_k(\rho)\}_0^{N-1}, \tag{2.3}$$

$$T_n(\mathbf{q}(\rho))=T_n(\mathbf{z})u_n^p/A_n(\rho), \tag{2.4}$$

where $A_n(\rho)$ is given by formula (2.2b).

Solution of problem 1a. Put $e(\rho)=E(S_\rho^p)$.

Lemma 1.

$$e(\rho)=\sum_{n=1}^{N-1}\frac{\rho^2|v_n^{2m}T_n(z)u_n^{2(p-m)}|^2}{A_n(\rho)^2} \tag{2.5}$$

The function $e(\rho)$ is strictly monotone increasing, and

$$e(0)=0,\quad \lim_{\rho\to\infty}e(\rho)=\tilde{e}=\|\mathbf{z}\|^2-\bar{\mathbf{z}}^2,\quad \bar{\mathbf{z}}=\frac{1}{N}\sum_k z_k, \tag{2.6}$$

moreover,

$$e(\rho)\leqslant\sum_n\frac{\rho^2|v_n^{2m}T_n(\mathbf{z})u_n^{2(p-m)}|^2}{(u_n^p)^2}\leqslant\sum_n\frac{\rho^2|v_n^{2m}T_n(\mathbf{z})|^2}{(\varkappa_{p-1})^2}\leqslant \tag{2.7}$$

$$\leqslant\frac{\rho^2}{(\varkappa_{p-1})^2}\|\delta^{2m}\mathbf{z}\|^2=e_1(\rho).$$

Proof. The truth of (2.5) follows in an obvious way from (2.1), (2.4). Relations (2.6) and (2.7) follow directly from (2.5). It is not difficult to calculate the derivative

$$e'(\rho)=\sum_n\frac{2\rho|v_n^{2m}T_n(\mathbf{z})u_n^{2(p-m)}u_n^p|^2}{A_n(\rho)^3}>0,\quad \rho>0.$$

This proves the lemma.

Theorem 3. Problem 1a has a unique solution for any value of ε such that

$$\varepsilon^2\leqslant\tilde{e}. \tag{2.8}$$

This solution is the spline $S_P^p(\mathbf{z},x)$ constructed using formulae (2.2)–(2.4), where P is determined from the equation $e(P)=\varepsilon^2$.

Proof. Let $\mathfrak{U}_\varepsilon$ denote the set of splines in $\mathfrak{S}^p$ that satisfy condition (2.8). If $S_0\equiv\bar{\mathbf{z}}$, then $E(S_0)=\tilde{e}$ and therefore $S_0\notin\mathfrak{U}_\varepsilon$. Suppose that $\inf I(S)$ in the space $\mathfrak{S}^p$ is achieved by a spline σ and $E(\sigma)=h^2<\varepsilon^2$. Obviously, there exists a neighbourhood $C(\sigma)$ in the n-dimensional space $\mathfrak{S}^p$ such that for any $S\in C(\sigma)$

$$\frac{1}{N}\sum_k[S(x_k)-\sigma(x_k)]^2<(\varepsilon^2-h^2)/2.$$

Hence it easily follows that $S\in\mathfrak{U}_\varepsilon$. But since $S_0\notin\mathfrak{U}_\varepsilon$, it follows that there is a spline S_1 in this neighbourhood for which $I(S_1)<I(\sigma)$. Therefore $\inf I(S)$ in $\mathfrak{S}^p$ cannot be achieved for a spline with $E(S)<\varepsilon^2$, so $E(\sigma)=\varepsilon^2$. But by condition (2.8) the equation $e(\mathrm{P})=\varepsilon^2$ has a unique solution and for the corresponding spline we have $E(S_\mathrm{P}^p)=\varepsilon^2$. Consequently, $S_\mathrm{P}^p(\bar{\mathbf{z}},x)$ is a solution of problem 1a.

Remarks. 1. If $\varepsilon\geqslant\|\mathbf{z}\|$, the solution of problem 1a is not unique. Any constant a such that $N^{-1}\Sigma_k(a-z_k)^2\leqslant\varepsilon^2$ is a solution. In particular, one can take $a=\bar{\mathbf{z}}$.

2. Put $\tau=1/\rho$, $b(\tau)=e(1/\tau)$. It is readily verified that $b(\tau)$ is monotone decreasing and convex from below. We can therefore use Newton's method to determine $\mathrm{T}=1/\mathrm{P}$ from the equation $b(\mathrm{T})=\varepsilon^2$. It is easy to see that the number $\mathrm{P}_1=(\varepsilon\varkappa_{p-1})^2\|\delta^{2m}\mathbf{z}\|^{-2}$ determined by solving the equation $e_1(\rho)=\varepsilon^2$ is less than P; accordingly $(\mathrm{T}_1=1/\mathrm{P}_1)>(\mathrm{T}=1/\mathrm{P})$. We may therefore find T by the method of chords, taking $\tau=0$, $\tau=\mathrm{T}_1$ as the initial points.

Solution of problem 1c. Put $\rho=0$ in formulae (2.2)–(2.4). Then we get the spline

$$S_0{}^p(\mathbf{z},x)=\frac{1}{N}\sum_k q_k(0)M^p(x-x_k),\qquad T_n(\mathbf{q}(0))=T_n(\mathbf{z})/u_n{}^p. \tag{2.9}$$

This spline solves problem 1c for any vector $\mathbf{z}$. The solution is unique, and $S_0^p(\mathbf{z},x)$ is an interpolating spline for the vector $\bar{\mathbf{z}}$.

Remark 3. Note that $S_0^p(\mathbf{z},x)$ is easily constructed directly. Thus, the relation $S_0^p(x_k)=z_k$, $k=0,1,\ldots,N-1$, is equivalent to $T_n(\mathbf{S}_0^p)=T_n(\mathbf{z})$, $n=0,1,\ldots,N-1$, $\mathbf{S}_0^p=\{S_0^p(x_k)\}_0^{N-1}$. But by formula (1.11), $T_n(\mathbf{S}_0^p)=T_n(\mathbf{q}(0))u_n^p$, and this implies (2.9).

Splines of the type $S_\rho{}^p(\mathbf{z},x)$ are known as smoothing splines. They made their first appearance in [12], where it was established that if $p=2m$ the spline $S_\rho{}^{2m}(z,x)$ is a solution of the following extremal problem (see also [11, 13]).

Problem 2a. Find a function $f\in\hat{\mathrm{W}}_2{}^m$ that minimizes the functional

$$I(f)=\int_0^1[f(x)^{(m)}]^2\,dx$$

on condition that

$$E(f)=\frac{1}{N}\sum_k[f(x_k)-z_k]^2\leqslant\varepsilon^2.$$

b. Find a function $f\in\hat{\mathrm{W}}_2{}^m$ that minimizes the functional $J_\rho(f)$.

Here $\hat{\mathrm{W}}_2{}^m$ is the space of periodic functions f such that $f^{(m-1)}$ are absolutely continuous and $f^{(m)}$ are square summable over [0, 1]. The norm in this space is

$$\|f\|_m=\left\{\int_0^1 f(x)^2\,dx+\int_0^1[f^{(m)}(x)]^2\,dx\right\}^{1/2}.$$

Note that when $p=2m$

$$T_n(\mathbf{q}(\rho))=T_n(\mathbf{z})/u_n{}^{2m}(\rho),\qquad u_n{}^{2m}(\rho)=\rho(v_n)^{2m}+u_n{}^{2m}. \tag{2.10}$$

A natural tool for the practical construction of splines is provided by fast Fourier transform algorithms. As the degree of the spline increases the computational load remains almost unchanged. Note, moreover, that our explicit representation has enabled us to investigate the approximating and smoothing properties of splines.

To end this section, we will present formulae for the derivatives of splines. By Theorem 1, the sth derivative of a spline $S_\rho^{p+s}\in\mathfrak{S}^{p+s}$ has the form

$$S_\rho{}^{p+s}(x)=\frac{1}{N}\sum_k q_k{}^{p+s}(\rho)M^{p+s}(x-x_k);$$

this is a spline in the space $\mathfrak{S}^p$,

$$S_\rho^{p+s}(x)^{(s)}=S_\rho{}^{ps}(x)=\frac{1}{N}\sum_k q_k{}^{ps}(\rho)M^p(x-x_k). \tag{2.11}$$

If we put $\mathbf{q}^{p+s}(\rho)=\{q_k^{p+s}(\rho)\}_0^{N-1}$, $\mathbf{q}^{ps}(\rho)=\{q_k^{ps}(\rho)\}_0^{N-1}$, then

$$T_n(\mathbf{q}^{ps})=T_n(\mathbf{q}^{p+s})\,(iNv_n)^s=(iNv_n)^s\,\frac{T_n(\mathbf{z})u_n^{p+s}}{A_n(\rho)}, \tag{2.12}$$

$$A_n(\rho)=\rho v_n{}^{2m}u_n^{2(p+s-m)}+(u_n^{p+s})^2.$$

Formulae (2.10) and (2.11) furnish an explicit expression for splines of any degree to approximate the derivatives of arbitrary order. If $\rho=0$ these are the derivatives of an interpolating spline. If the available data involve errors, it is natural to use derivatives of smoothing splines in the approximation, with the smoothing parameter determined as in the algorithm of problem 1a.

3. APPROXIMATING PROPERTIES OF SPLINES CONSTRUCTED FROM EXACT DATA

Let f be a continuous function. Suppose we have at our disposal a collection of data

$$z_k=\{f_k+e_k\}_{k=0}^{N-1}, \qquad f_k=f(x_k),$$

e_k are random errors, $x_k=k/N$. Put $\mathbf{f}=\{f_k\}_{k=0}^{N-1}$, $\mathbf{e}=\{e_k\}_{k=0}^{N-1}$. Then we can write

$$S_\rho^{ps}(\mathbf{z},x)=S_\rho^{ps}(\mathbf{f},x)+S_\rho^{ps}(\mathbf{e},x).$$

In this section we will consider the first term.

Put $b_k(t)=B_k(t)/k!$, where $B_k(t)$ is the Bernoulli polynomial. There are asymptotic formulae for interpolating splines.

Proposition 1 (see [14–16] for the case $s=0$). Let $f\in\mathbb{C}^{p+s+1}$. Then, if $x\in[x_k^{p+s},x_{k+1}^{p+s}]$, $t=N(x-x_k^{p+s})$, $h=1/N$, the following asymptotic formulae hold:

$$\begin{aligned}&S_0^{ps}(\mathbf{f},x)=f^{(s)}(x)+h^pP_0^{ps}(t)f(x)^{(p+s)}+\\&+h^{p+1}P_1^{ps}(t)f(x)^{(p+s+1)}+o(h^{p+1}f(x)^{(p+s+1)}),\\&P_0^{2m0}(t)=-[b_{2m}(t)-b_{2m}(0)],\qquad P_0^{2m-1,0}(t)=-b_{2m-1}(t),\end{aligned}$$

$$P_1^{2m0}(t)=2mb_{2m+1}(t),\qquad P_1^{2m-1,0}(t)=(2m-1)[b_{2m}(t)-b_{2m}(1/2)].$$

if $s>0$, then $P_0^{ls}(t)=B_l(t)/l!$, $P_1^{2m-1,1}(t)=(2m-1)b_{2m}(t)+b_{2m}(0)$. In all other cases, $P_1^{ls}(t)=lb_{l+1}(t)$.

Lemma 2 (see [16]). Let $f\in\mathbb{C}^{s+1}$. Then for any $p>1$

$$\frac{1}{N}\sum_k\partial^sf(x_k)M^p(x-x_k)=h^s[f^{(s)}+o(h)]. \tag{3.1}$$

We now consider smoothing splines. To avoid complications, we will confine our attention to the cases $p+s=2m$ and $p+s=2m-1$.

Theorem 4. Let $f\in\mathbb{C}^{p+s+1}$. Then if $x\in[x_k^{p+s},x_{k+1}^{p+s}]$, $t=N(x-x_k^{p+s})$, $h=1/N$, $p+s=2m$, $p+s=2m-1$, $\rho>0$, the following asymptotic formulae hold:

$$\begin{aligned}&S_\rho^{ps}(\mathbf{f},x)=f^{(s)}(x)+h^pW_0^{ps}(t)f(x)^{(p+s)}+\\&+h^{p+1}W_1^{ps}(t)f(x)^{(p+s+1)}+\sigma(f,x),\\&\|\sigma(\mathbf{f},x)\|_{L_2}=o(h^{p+1}f(x)^{(p+s+1)}),\end{aligned} \tag{3.2}$$

where, if $s>0$,

$$\begin{aligned}&W_0^{2m-s,s}(t)g(x)=[P_0^{2m-s,s}(t)+(-1)^{m+1}\rho\delta^s]g(x),\\&W_0^{2m-s-1,s}(t)g(x)=P_0^{2m-s-1,s}(t)g(x),\\&W_1^{2m-s,s}(t)g(x)=P_1^{2m-s,s}(t)g(x),\\&W_1^{2m-s-1,s}(t)g(x)=[P_1^{2m-s-1,s}(t)+(-1)^{m+1}\rho\delta^s]g(x),\end{aligned}$$

and if $s=0$,

$$W_0^{2m0}(t)g(x)=[P_0^{2m0}(t)+(-1)^{m+1}\rho]g(x),$$
$$W_0^{2m-1,0}(t)g(x)=P_0^{2m-1,0}(t)g(x), \quad W_1^{2m0}(t)g(x)=P_1^{2m0}(t)g(x),$$
$$W_1^{2m-1,0}(t)g(x)=[P_1^{2m-1,0}(t)+(-1)^{m+1}\rho]g(x).$$

Proof. We will first consider the case when $p+s=2m$. Here formulae (3.1) and (3.2) give

$$S_\rho^{p+s}(\mathbf{f},x)^{(s)}=S_\rho^{ps}(\mathbf{f},x)=\frac{1}{N}\sum_k q_k^{ps}(\rho)M^p(x-x_k), \tag{3.2'}$$
$$T_n(\mathbf{q}^{2m-s,s}(\rho))=T_n(\mathbf{q}^{2m}(\rho))(iNv_n)^s=$$
$$=(iNv_n)^sT_n(\mathbf{f})/u_n^{2m}(\rho),$$
$$u_n^{2m}(\rho)=\rho(v_n)^{2m}+u_n^{2m}=\rho(v_n)^{2m}+u_n^{2m}.$$

Hence

$$T_n(\mathbf{q}^{2m-s,s}(\rho))=(iNv_n)^sT_n(\mathbf{f})/u_n^{2m}+\rho N^sT_n(\mathbf{Q}(\rho_1)),$$
$$T_n(\mathbf{Q}(\rho_1))=-(i)^sv_n^{2m+s}T_n(\mathbf{f})/[u_n^{2m}(\rho_1)]^2, \quad \rho_1\in[0,\rho].$$

It follows from (1.7) and (1.8) that

$$u_n^p=1+U_n^p, \quad U_n^p=\sum_{k=1}^{l}\gamma_k^pv_n^{2k}, \quad l=[(p-1)/2],$$

and moreover $\varkappa_{p-1}\leq U_n^p\leq 0$. Therefore,

$$u_n^{2m}(\rho_1)=1+v_n^2A_n(\rho_1,\ v_n^2)>0,$$

where $A_n(\rho_1,y)$ is a certain polynomial in y, $n=0, 1, \ldots, N-1$, and we can write

$$[u_n^{2m}(\rho_1)]^{-2}=1+v_n^2g_n(v_n^2), \qquad |g_n(v_n^2)|\leq G,$$
$$n=0,\ 1,\ldots,N-1.$$

Hence we obtain

$$T_n(\mathbf{Q}(\rho_1))=T_n(\mathbf{Q}^0(\rho_1))+T_n(\mathbf{Q}^1(\rho_1)),$$
$$T_n(\mathbf{Q}^0(\rho_1))=-(i)^sv_n^{2m+s}T_n(\mathbf{f})=(-1)^{m+1}T_n(\mathbf{f}^{2m+s}),$$
$$T_n(\mathbf{Q}^1(\rho_1))=-(i)^sv_n^{2(m+1)+s}g_n(v_n^2)T_n(\mathbf{f})=(-1)^mT_n(f^{2m+2+s})g_n(v_n^2).$$

We have used the notation $\mathbf{f}^q=\{\partial^qf(x_k)\}_{k=0}^{N-1}$ and the fact that $T_n(\mathbf{f}^q)=(iv_n)^qT_n(\mathbf{f})$. Let $\mathbf{Q}^i(\rho_1)=\{Q_k^i(\rho_1)\}_{k=0}^{N-1}$, $i=0, 1$. Put

$$S^i(x)=\frac{1}{N}\sum_k Q_k^i(\rho_i)M^{2m-s}(x-x_k).$$

Then by formula (3.2′) we obtain, using Lemma 2,

$$S_\rho^{2m-s,s}(\mathbf{f},x)=S_0^{2m-s,s}(\mathbf{f},x)+\rho N^s[S^0(x)+S^1(x)],\tag{3.3}$$
$$S^0(x)=(-1)^{m+1}N^{-1}\sum_k \partial^{2m+s}f(x_k)M^{2m-s}(x-x_k)=$$
$$=(-1)^{m+1}N^{-2m}\partial^s f^{(2m)}(x)+o(N^{-2m-1}\max|\partial^s f^{(2m+1)}(x)|).$$

Consider the term $S^1(x)$. It follows from formula (1.15) that

$$\int_0^1 [S^1(x)]^2\,dx=\sum_n |T_n(\mathbf{Q}^1(\rho_1))|^2 u_n^{4m-2s}\leqslant\tag{3.4}$$
$$\leqslant G^2\sum_n |T_n(\mathbf{f}^{2m+2+s})|^2 u_n^{4m-2s}.$$

Let

$$S^i(x)=\frac{1}{N}\sum_k Q_k^i(\rho_i)M^{2m-s}(x-x_k).$$

We then see by (3.4) that

$$\|S^1(x)\|_{L_2}\leqslant G\|S(x)\|_{L_2}\leqslant G\max_x|\partial^{2m+2+s}f(x)|\leqslant\tag{3.5}$$
$$\leqslant GN^{-2m-1}\max_x|\partial^{1+s}f^{(2m+1)}(x)|=o(N^{-2m-1}\max_x|\partial^s f^{(2m+1)}(x)|).$$

The assertion of the theorem for $p+s=2m$ now follows from (3.3), proposition 1 and (3.5).

To get the proof for $p+s=2m-1$, we write

$$T_n(\mathbf{q}^{2m-s-1,s}(\rho))=T_n(\mathbf{q}^{2m-1}(\rho))(iNv_n)^s=(iNv_n)^s T_n(\mathbf{f})/\hat{u}_n^{2m-1}(\rho),$$
$$\hat{u}_n^{2m-1}(\rho)=\rho(v_n)^{2m}\gamma_n+u_n^{2m-1},\qquad \gamma_n=u_n^{2(m-1)}/u_n^{2m-1}=1+v_n^2G_n(v_n^2),$$
$$|G_n(v_n^2)|\leqslant c,\qquad n=0,1,\dots,N-1.$$

The rest of the proof is more or less analogous to that of the previous case and is therefore omitted.

Remarks. 4. If ρ is bounded as $N\to\infty$ (this may be ensured by coordinating ρ with N as the error level decreases), then the smoothing splines with $p+s=2m, p+s=2m-1$ will approximate $f^{(s)}$ to the same order in terms of h as the corresponding interpolating splines.

5. For what follows we specially mention some asymptotic formulas for splines of even degrees with $s=0$:

$$S_\rho^{2m-1,0}(\mathbf{f},x)=f(x)-h^{2m-1}[b_{2m-1}(t)-b_{2m-1}(1/2)]f(x)^{(2m-1)}+$$
$$+h^{2m}\{(2m-1)[b_{2m}(t)-b_{2m}(1/2)]+(-1)^{m+1}\rho N^{-2m}\}f(x)^{(2m)}+o(h^{2m}f(x)^{(2m)}).$$

If $x=\tilde{x}_k=(k-\frac{1}{2})/N$, $t=0$, then

$$S_\rho^{2m-1,0}(\mathbf{f},\tilde{x}_k)=f(\tilde{x}_k)+h^{2m}\{(2m-1)[b_{2m}(0)-b_{2m}(1/2)]+$$
$$+(-1)^{m+1}\rho\}f(\tilde{x}_k)^{(2m)}+o(h^{2m}f(x)^{(2m)})\quad(\rho\ \text{ограничен}).$$

Thus, for splines of even degrees, the points $\tilde{x}_k$ are "quasi-interpolation" points, in the sense that at these points the first term of the asymptotic expansion vanishes. Of course, we are understanding $o(h^{2m}f(x)^{(2m)})$ in the sense of the norm $\|\cdot\|_{L_2}$.

4. SMOOTHING PROPERTIES OF THE PERIODIC SPLINES

Throughout this section, $\{z_k=f_k+e_k\}$, $f_k=f(x_k)$, e_k will be assumed to be uncorrelated, identically distributed random variables with expectation $\mathbf{E}(e_k)=0$ and variance $\mathbf{D}(e_k)=d$. We also put $a=\mathbf{E}(|e_k|)$. Then

$$S_\rho^{ps}(x)=\frac{1}{N}\sum_k q_k^{ps}(\rho)M^p(x-x_k)=S_\rho^{ps}(\mathbf{f},x)+S_\rho^{ps}(\mathbf{e},x)$$

is a random variable, $\mathbf{E}(S_\rho^{ps}(\mathbf{z},x))=S_\rho^{ps}(\mathbf{f},x)$, $\mathbf{D}(S_\rho^{ps}(\mathbf{z},x))=\mathbf{D}(S_\rho^{ps}(\mathbf{e},x))=d_\rho^{ps}(x)$, $\mathbf{E}(|S_\rho^{ps}(\mathbf{e},x)|)=a_\rho^{ps}(x)$.

Let

$$S_\rho^{ps}(\mathbf{y},x)=L_\rho^{ps}(x),\qquad \mathbf{y}=\{1,0,\dots,0\},\quad T_n(y)=1/N, \tag{4.1a}$$

$$L_\rho^{ps}(x)=\frac{1}{N}\sum_k l_k^{ps}(\rho)M^p(x-x_k),\qquad \mathbf{l}^{ps}(\rho)=\{l_k^{ps}(\rho)\}_0^{N-1}, \tag{4.1b}$$

$$T_n(\mathbf{l}^s(\rho))=(iNv_n)^s\frac{u_n^{p+s}}{NA_n^s(\rho)}, \tag{4.1c}$$

$$A_n^s(\rho)=\rho|v_n|^{2m}u_n^{2(p+s-m)}+(u_n^{p+s})^2. \tag{4.1d}$$

By (2.12), $T_n(\mathbf{q}^{ps}(\rho))=NT_n(\mathbf{z})T_n(\mathbf{l}^s(\rho))$. Hence

$$q_k^{ps}(\rho)=\sum_\nu l_\nu^{ps}(\rho)z_{k-\nu},$$

$$S_\rho^{ps}(\mathbf{z},x)=\frac{1}{N}\sum_k M^p(x-x_k)\sum_\nu l_\nu^{ps}(\rho)z_{k-\nu}=$$

$$=\sum_\nu z_\nu\frac{1}{N}\sum_k M^p(x-x_{k-\nu}l_\nu^{ps}(\rho))=\sum_k z_kL_\rho^{ps}(x-x_k),$$

$$S_\rho^{ps}(\mathbf{e},x)=\sum_k e_kL_\rho^{ps}(x-x_k).$$

Therefore, since the r.v's e_k are uncorrelated, we obtain

$$d_\rho^{ps}(x)=d\sum_n L_\rho^{ps}(x-x_n)^2. \tag{4.2}$$

Let us transform $L_\rho^{ps}(x-x_n)$, using relation (4.1b):

$$L_\rho^{ps}(x-x_n)=\frac{1}{N}\sum_k l_{k-n}^{ps}(\rho)M^p(x-x_k)=$$

$$=\frac{1}{N}\sum_h M^p(x-x_h)\sum_\nu (iNv_\nu)^s w^{(k-n)\nu}\frac{u_\nu^{p+s}}{NA_\nu^s(\rho)}=$$

$$=\sum_\nu (iNv_\nu)^s w^{-n\nu}\frac{u_\nu^{p+s}}{NA_\nu^s(\rho)}\frac{1}{N}\sum_h M^p(x+x_h)w^{-h\nu}=$$

$$=\sum_\nu w^{-n\nu}F_\nu(x),$$

$$F_\nu(x)=(iNv_\nu)^s\frac{u_\nu^{p+s}}{NA_\nu^s(\rho)}m_\nu^p(x).$$

The function $m_\nu^p(x)$ is defined by (1.4). Hence it follows, by (1.2) and (4.2), that

$$d_\rho^{ps}(x)=dN\sum_\nu |F_\nu(x)|^2= \tag{4.3}$$

$$=dN^{2s-1}\sum_n\left[v_n^s\frac{u_n^{p+s}m_n^p(x)}{A_n^s(\rho)}\right]^2$$

If $x=x_k$, we obtain, by (1.6),

$$d_\rho^{ps}(x_h)=dN^{2s-1}\sum_n\left[v_n^s\frac{u_n^{p+s}u_n^p}{A_n^s(\rho)}\right]^2. \tag{4.4}$$

If $s=0$,

$$d_\rho^{p0}(x_h)=\frac{d}{N}\sum_n\left[\frac{(u_n^p)^2}{A_n^0(\rho)}\right]^2,$$

in particular,

$$d_\rho^{2m0}(x_h)=\frac{d}{N}\sum_n\left[\frac{u_n^{2m}}{u_n^{2m}(\rho)}\right]^2,\qquad u_n^{2m}(\rho)=\rho|v_n|^{2m}+u_n^{2m}. \tag{4.5}$$

Theorem 5. a. If
(1) $p=3, 5$,
(2) p is any even number,
then $d_\rho^{ps}(x)\leqslant d_\rho^{ps}(x_k)$, $s=0, 1, \ldots$.
b. If $p=2, \ldots, 6$, then $d_\rho^{ps}(x)\geqslant d_\rho^{ps}(h(k+1/2))$.

Proof. a. (1) was proved in [17].
a. (2) By (1.3),

$$M^p(x+x_\nu)=\sum_{n=-\infty}^{\infty}V_n^p e^{2\pi inx}w^{n\nu}.$$

Hence

$$m_\lambda^p(x) = \sum_{n=-\infty}^{\infty} V^p e^{2\pi i n x} \frac{1}{N} \sum_{\nu} w^{(n-\lambda)\nu} = \tag{4.6}$$

$$= e^{2\pi i \lambda x} \sin^p(\pi\lambda/N) \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n N x}}{[\pi(\lambda+Nn)/N]^p},$$

$$|m_\lambda^p(x)|^2 = \sin^{2p}(\pi\lambda/N) \sum_{n,l=-\infty}^{\infty} \frac{e^{2\pi i (l-n) N x}}{[\pi(\lambda+Nn)/N]^p [\pi(\lambda+Nl)/N]^p} \leqslant$$

$$\leqslant \sin^{2p}(\pi\lambda/N) \sum_{n,l=-\infty}^{\infty} \frac{1}{[\pi(\lambda+Nn)/N]^p [\pi(\lambda+Nl)/N]^p} = |m_\lambda^p(x_h)|^2.$$

Part b was proved in [17]. We need only note that there the assertion for $p=6$ was proved on the assumption that a certain quadratic form $A_5(\mathbf{x})$, whose arguments are the coefficients of the spline, is non-negative. Later [18], however, we were able to prove that this is true for any values of the arguments.

Conjecture. Theorem 5 holds for any value of p.

5. THE RECOVERY OF FUNCTIONS AND DERIVATIVES

The accuracy with which $f^{(s)}(x)$ can be recovered depends on two factors: the accuracy with which $f^{(s)}(x)$ is approximated by $S_\rho^{ps}(\mathbf{f}, x)$ and the spread of values of $S_\rho^{ps}(\mathbf{e}, x)$ relative to zero. These quantities may differ greatly for different splines. The specific spline used should be chosen depending on the actual conditions of the problem. Namely, the choice of the spline is influenced by the value of $\mathbf{D}(e_k)=d$, the required accuracy and smoothness of the recovered function, the possibility of varying the step size, and so on. It should be noted that the simplicity with which splines can be constructed and the smoothing parameters selected offers considerable room for manoeuvre. This is especially useful when solving ill-posed problems.

As in [17], we will consider a specific probability model: the r.v.'s e_k will have a normal distribution. Then $R_\rho^{ps}(x) = S_\rho^{ps}(\mathbf{z}, x) - f^{(s)}(x)$ is a normally distributed r.v.,

$$\mathbf{E}(R_\rho^{ps}(x)) = S_\rho^{ps}(\mathbf{f}, x) - f^{(s)}(x) = e_\rho^{ps}(\mathbf{f}, x),$$

$$\mathbf{E}(|S_\rho^{ps}(\mathbf{e}, x)|) = a_\rho^{ps}(x) = \frac{2}{\pi}[d_\rho^{ps}(x)]^{1/2}.$$

As a measure of the deviation of the spline $S_\rho^{ps}(\mathbf{z}, x)$ from $f^{(s)}(x)$ we take the quantity $Q_\rho^{ps}(\mathbf{z}, x) = \mathbf{E}(|R_\rho^{ps}(x)|)$.

Proposition 2. If e_k are normally distributed r.v.'s, then

$$Q_\rho^{ps}(\mathbf{z}, x) \leqslant 1.303 \max\{|e_\rho^{ps}(\mathbf{f}, x)|, a_\rho^{ps}(x)\}.$$

Moreover, if

$$|e_\rho^{ps}(\mathbf{f}, x)| \leqslant 0.25 a_\rho^{ps}(x), \tag{5.1}$$

then

$$Q_\rho^{ps}(\mathbf{z}, x) \leqslant 1.02 a_\rho^{ps}(x), \tag{5.2}$$

and if

$$a_\rho^{ps}(x) \leq 2.5|e_\rho^{ps}(\mathbf{f}, x)|,$$

then

$$Q_\rho^{ps}(\mathbf{z}, x) \leq 1.004|e_\rho^{ps}(\mathbf{f}, x)|.$$

Information about the quantity $e_\rho^{ps}(\mathbf{f}, x)$ may be derived from the asymptotic formulas of Theorem 4, if information is available about the values of the derivatives occurring in those formulae,

$$d_\rho^{ps}(x) = \frac{2}{\pi}[d_\rho^{ps}(x)]^{1/s} \leq \frac{2}{\pi}[d_\rho^{ps}(x_k)]^{1/s}.$$

the quantities $d_\rho^{ps}(x_k)$ are given by (4.4) and (4.5).

6. THE CONNECTION BETWEEN PERIODIC SPLINES AND THE FAST FOURIER TRANSFORM

There is a two-way connection between splines and the FFT. On the one hand, it is natural to use FFT algorithms to construct splines, as they considerably reduce the necessary amount of computation. On the other hand, there are many situations in which one has to compute FFTs where the use of splines may reduce the size of the data blocks to be processed and, consequently, the computational load. The idea of using splines in such situations is obvious. The "decimated" data are used to construct splines, and the omitted data are then filled out in accordance with the values of the spline at the points in question. It has been found that the use of periodic splines of even degree for such purposes produces very "technological" formulae and guarantees that the missing points can be determined with a high degree of accuracy.

The spline may be used as follows. One of the basic FFT algorithms is the "decimation-in-time" algorithm [19]. It is used in situations where the dimension K of the block is a power of 2, and consists in successively splitting each DFT of length K into a combination of two DFTs of half that length: if $K=2N$, $\mathbf{z}=\{z_k\}_0^{2N-1}$, then

$$T_n^{2N}(\mathbf{z}) = \frac{1}{2N}\sum_{k=0}^{2N-1} w^{-nk/2} z_k = \tag{6.1}$$

$$= \frac{1}{2}\left(\frac{1}{N}\sum_k w^{-nk} z_{2k} + w^{-n/2}\frac{1}{N}\sum_k w^{-nk} z_{2k+1}\right) =$$

$$= \frac{1}{2}[T_n(\mathbf{z}^0) + w^{-n/2} T_n(\mathbf{z}^1)],$$

$$\mathbf{z}^0 = \{z_{2k}\}_0^{N-1}, \qquad \mathbf{z}^1 = \{z_{2k+1}\}_0^{N-1}, \qquad k=0, 1, \ldots, N-1.$$

It will suffice to compute $T_n^{2N}(\mathbf{z})$ for $n=0, 1, \ldots, N-1$, and then, for the other n values, to use the fact that $T_{n+N}(\mathbf{z}^0) = T_n(\mathbf{z}^0)$, $T_{n+N}(\mathbf{z}^1) = T_n(\mathbf{z}^1)$.

Now let $x_k = k/N$, $x_{k+0.5} = (2k+1)/(2N)$, $k=0, 1, \ldots, N-1$. Using $\{x_k\}$ as a grid, construct a spline of even degree from the data $\mathbf{z}^0$:

$$S_\rho^{2m-1}(\mathbf{z}^0, x) = \frac{1}{N}\sum_k q_k M^{2m-1}(x - x_k).$$

Let $m_k = M^{2m-1}(x_{k+0.5})$, $\mathbf{m} = \{m_k\}_0^{N-1}$, $s_{2l+1} = S_\rho^{2m-1}(\mathbf{z}^0, x_{l+0.5})$, $s_{2l} = S_\rho^{2m-1}(\mathbf{z}^0, x_l)$, $\mathbf{s}^0 = \{s_{2l}\}_0^{N-1}$, $\mathbf{s}^1 = \{S_{2l+1}\}_0^{N-1}$. We have

$$s_{2l+1}=\frac{1}{N}\sum_k q_k m_{l-k}.$$

Hence

$$T_n(\mathbf{s}^1)=T_n(\mathbf{q})T_n(\mathbf{m})=T_n(\mathbf{m})T_n(\mathbf{z}^0)/u_n^{2m-1}(\rho),$$

$$u_n^{2m-1}(\rho)=\rho|v_n|^{2m}u_n^{2(m-1)}/u_n^{2m-1}+u_n^{2m-1},$$

$$T_n(\mathbf{m})=\frac{1}{N}\sum_k q_k M^{2m-1}(x_{k+0.5}).$$

Lemma 3. $T_n(\mathbf{m})=w^{n/2}t_n$, where $\{t_n\}_0^{N-1}$ are real numbers with $t_{n+N}=-t_n$.

Proof. It follows from (4.6) that

$$T_n(\mathbf{m})=m_n^{2m-1}\left(\frac{1}{2N}\right)=w^{n/2}\sin^{2m-1}(\pi n/N)\times$$

$$\times\sum_{l=-\infty}^{\infty}\frac{(-1)^l}{[\pi(n+Nl)/N]^{2m-1}}.$$

Since $T_{n+N}(\mathbf{m})=T_n(\mathbf{m})$, $w^{(N+n)/2}=-w^{n/2}$, we have $t_{N+n}=-t_n$.

Lemma 4. Let $z_1=f(x_{1/2})$. If $f\in\mathbb{C}^{2m}$, then

$$T_n(\mathbf{s}^1)-T_n(\mathbf{z}^1)=N^{-2m}T_n(\mathbf{F}^1)\alpha_{2m}+o(N^{-2m}),$$

$$\alpha_{2m}=(2m-1)[b_{2m}(0)-b_{2m}(1/2)]+(-1)^{m+1}\rho,$$

$$\mathbf{F}^1=\{f^{(2m)}(x_{k+0.5})\}_0^{N-1}.$$

The proof follows directly from (3.3). We recall that the points $x_{k+0.5}$ are quasi-interpolation points for splines of even degree; hence the spline reproduces the value of f at these points with very high accuracy.

Corollary 1. The following estimate holds to within $o(N^{-2m})$:

$$|T_n(\mathbf{s}^1)-T_n(\mathbf{z}^1)|\leqslant N^{-2m}\max_x|f^{(2m)}(x)|\,|\alpha_{2m}|.$$

Obviously, $T_n(\mathbf{s}^0)=T_n(\mathbf{z}^0)u_n^{2m-1}/u_n^{2m-1}(\rho)$. If $\rho=0$, then $T_n(\mathbf{s}^0)=T_n(\mathbf{z}^0)$.

Lemma 5. Let $z_l=f(x_{1/2})$. If $f\in\mathbb{C}^{2m}$, then

$$T_n(s^0)-T_n(\mathbf{z}^0)=N^{-2m}T_n(\mathbf{F}^0)(-1)^{m+1}\rho+$$

$$+o(N^{-2m}),\quad \mathbf{F}^0=\{f^{(2m)}(x_k)\}_0^{N-1}.$$

The proof follows directly from Theorem 4.

Corollary 2. The following estimate holds to within $o(N^{-2m})$:

$$|T_n(\mathbf{s}^0)-T_n(\mathbf{z}^0)|\leqslant N^{-2m}\max_x|f^{(2m)}(x)|\rho.$$

Now let $\mathbf{s}=\{S_p^{2m}(\mathbf{z}^0,x_{l/2})\}_{l=0}^{2N-1}$. By (6.1) and Lemma 3, we can write

$$T_n^{2N}(\mathbf{s})=\frac{1}{2}T_n(\mathbf{z}^0)(u_n^{2m-1}+t_n)/u_n^{2m-1}(\rho). \tag{6.2}$$

If $\rho=0$ (an interpolating spline), then

$$T_n^{2N}(\mathbf{s})=\frac{1}{2}T_n(\mathbf{z}^0)[1+t_n/u_n^{2m-1}(0)]. \tag{6.3}$$

We can now state the following theorem.

Theorem 6. Let $z_l = f(x_{l/2})$. If $f \in \mathbb{C}^{2m}$, then

$$T_n^{2N}(\mathbf{s}) = T_n^{2N}(\mathbf{z}) + N^{-2m}\{(2m-1)[b_{2m}(0) - b_{2m}(1/2)]T_n(\mathbf{F}^1) + \\ +(-1)^{m+1}\rho T_n^{2N}(\mathbf{F})\} + o(N^{-2m}f(x)^{(2m)}), \tag{6.4}$$

$$\mathbf{F} = \{f^{(2m)}(x_{l/2})\}_0^{2N-1}.$$

The following estimate holds to within $o(N^{-2m}f(x)^{(2m)})$:

$$|T_n^{2N}(\mathbf{s}) - T_n^{2N}(\mathbf{z})| \leqslant N^{-2m} \max_x (|f(x)^{(2m)}|\{(2m-1)|[b_{2m}(0) - \\ - b_{2m}(1/2)]| + \rho\}). \tag{6.5}$$

Formulae (6.4) and (6.5) show that if $z_l = f(x_{l/2})$, $l = 0, 1, \ldots, 2N-1$, and f is sufficiently smooth, then the DFT of the spline of even degree given by (6.2), (6.3) is an excellent approximation to $T_n^{2N}(\mathbf{z})$. In order to compute $T_n^{2N}(\mathbf{s})$, however, we have to evaluate only $T_n^{2N}(\mathbf{z}^0)$ and, therefore, to process a data block only half as large as for $T_n^{2N}(\mathbf{z})$. If there are no reading errors, it is natural to use an interpolating spline and use formula (6.3). When that is done, as in the computation of the FFT, it suffices to compute $T_n^{2N}(\mathbf{s})$ for $n = 0, 1, \ldots, N-1$ and to determine the rest from the formulae

$$T_{n+N}^{2N}(\mathbf{s}) = \frac{1}{2} T_n(\mathbf{z}^0)(u_n^{2m-1} - t_n)/u_n^{2m-1}(\rho).$$

If $\rho = 0$ (an interpolating spline), then

$$T_{n+N}^{2N} = \frac{1}{2} T_n(\mathbf{z}^0)[1 - t_n/u_n^{2m-1}(0)].$$

The coefficients t_n are readily determined for splines of any degree. The algorithm we have proposed may be called the spline-FFT algorithm (or SFFT). To compute $T_n(\mathbf{z}^0)$ one can use FFTs or, if the solvability conditions permit, SFFTs again.

The smoother the function f, the higher the degree of the spline that must be used in the approximation. If $z_l = f(x_{l/2}) + e_l$, one should use smoothing splines.

We add here that, since all terms in (6.2), (6.3) are real, these formulae may be used directly to calculate sine and cosine DFTs. The method yields even greater computational economy in the multidimensional case, to which it is easily generalized.

As pointed out previously, the variance of a spline of second, fourth and probably all other degrees is minimal at $x = x_{k+0.5}$. Formula (6.2) yields a formula for the variance $\mathbf{D}(T_n^{2N}(\mathbf{s}))$ if $\mathbf{D}(e) = d$:

$$\mathbf{D}(T_n^{2N}(\mathbf{s})) = \frac{d}{4N}[(u_n^{2m-1} + t_n)/u_n^{2m-1}(\rho)]^2.$$

Note that $\mathbf{D}(T_n^{2N}(\mathbf{z})) = d/(4N)$. Numerical experimentation has shown that, subject to the optimal choice of the parameter ρ, the SFFT algorithm possesses very good filtering properties.

To illustrate this we present the DFT for interpolating splines of second and fourth degrees. Let $y = \cos(\pi n/N)$. Second degree:

$$T_n^{2N}(\mathbf{s}) = \frac{1}{2} T_n(\mathbf{z}^0)\frac{(1+y)^2}{1+y^2};$$

fourth degree:

$$T_n^{2N}(\mathbf{s}) = \frac{1}{2} T_n(\mathbf{z}^0)\frac{5+88y+18y^2+8y^3}{5+18y^2+y^4}.$$

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AN ALGORITHM FOR INVERTING DISCRETE CONVOLUTIONS BY SECTIONING†

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A method is proposed for inverting long discrete convolutions by the method of sectioning with overlap. Cyclic convolutions of short sections are converted into triangular Toeplitz systems of equations, thus permitting rapid solution in many situations of practical importance, without using Fourier transforms.

1. INTRODUCTION

THE INVERSION of convolutions is a key problem in applications arising in many branches of modern engineering: television, photography, optics, tomography, radio-astronomy, computer vision, etc.

The usual approaches rely on the general theory of the solution of ill-posed problems, in which the chief tool is Tikhonov's regularization method. There is a voluminous literature on this subject (see, e.g., [1–3]).

We will be concerned with convolution inversion in a special case: in the processing of long streams of data, containing hundreds and thousands of readings, we propose to use sectioning with overlap — a widely used method in practical work pertaining to the computation of long convolutions [4, 5]. In this method the initial flow of data is broken up into short sections, and each section is processed in turn using a cyclic convolution, for which many good and fast algorithms are available. A large part of these algorithms presuppose frequent use of fast Fourier transforms, as there is a direct connection between the eigenvalues and eigenvectors of the circulant matrix operator and the discrete Fourier transform.

Unfortunately, this of little use in convolution inversion, since in many important special cases the Fourier spectra of the transformed impulse responses of the filters contain zeros and it becomes necessary to deal with division by zero.

Direct inversion of the cyclic convolution matrix is rarely a feasible approach, since the circulant matrix operator frequently turns out to be singular.

In this paper we will use another property of the circulant: the connection with triangular Toeplitz matrices which, under the conditions of the problem, are always invertible. This enables us to devise fast algorithms for inverting long convolutions without having to use Fourier transforms.

The aim of the paper is to propose one such algorithm.

2. CONVERSION TO A TOEPLITZ EQUATION

Consider the equation of a cyclic convolution

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