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On the approximation on finite intervals and local spline extrapolation

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Abstract – A procedure for constructing and analysing the third- and fifth-degree local splines is proposed, which is based on the connection between the splines and interpolation polynomials. The procedure allowed us to obtain extremely easy-to-implement and stable algorithms for constructing the splines both on inner subintervals and on subintervals adjacent to the endpoints. In some cases an explicit representation of remainder terms of approximation of functions and their derivatives, as well as an exact estimate of the approximation error, has been derived. Splines have also been constructed which extrapolate an approximated function to the exterior of the interval on which its values were calculated.

The n-th degree local spline of defect 1 on a grid $\{x_i\}$ is understood to be a polynomial spline which can be written as a linear combination of B-splines $B_n^k(x)$ of degree n, namely

$$S_n(f,x) = \sum_k l_k(f) B_n^k(x)$$

where $l_k(f)$ are finite linear combinations of grid values $\{f(x_r)\}$ which have been constructed by a definite algorithm.

Almost all the authors contributing to the theory of local splines point out that, unlike global interpolating and smoothing splines, local splines are easy and fast to implement and with these splines it is possible to keep in memory only the grid values $\{f(x_r)\}$ that belong to the interval on which the spline is calculated and the closest adjacent grid values. Local splines make it possible to attain the highest possible approximation order, however, in this case constants in the estimates are somewhat larger than the corresponding ones for interpolating splines. Note also such a possibility offered by local splines as real-time information processing.

It should also be mentioned that general formulae are available that allow arbitrary-degree local splines, which exactly reproduce polynomials of appropriate degrees, to be expressed in terms of B-splines [4]. However, these formulae are rather bulky and require complex algorithms for implementing even low-degree splines, especially on subintervals close to the endpoints of the approximation interval. With such a representation the study of remainder approximation terms presents difficulties.

In this paper, we suggest a procedure for constructing and analysing the third- and fifth-degree local splines. The procedure is based on the connection between the splines and interpolation polynomials. It makes it possible to obtain extremely easy-to-implement, stable algorithms for constructing the splines both on inner subintervals and on subintervals adjacent to the endpoints of the approximation interval.

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In some cases we have obtained an explicit representation of remainder terms of approximation of functions and their derivatives as well as an exact estimate of the approximation error. This approach has been first applied by one of the authors of the present paper in [10]. We present also splines which extrapolate approximated function to the exterior of the interval on which its values were calculated.

1. AUXILIARY INFORMATION

In what follows, $\{x_k\}$, $k = -\infty, \dots, \infty$, is an arbitrary grid on the x-axis, with $h_k = x_{k+1} - x_k$. Let f be a continuous function, $f_k = f(x_k)$. The basis of B-splines is most frequently used as a basis in the spline space. Following the terminology in [6], the following function will be called a normalized B-spline of degree n:

$$B_n^k(x) = (x_{k+n+1} - x_k)\varphi_n[x; x_k, \dots, x_{k+n+1}] \tag{1.1}$$

where $\varphi_n[x; x_k, \dots, x_{k+n+1}]$ is the divided difference of order $n+1$ of the function $\varphi_n(x;y) = (-1)^{n+1}(x-y)_+^n$ with respect to the variable y, and $x_+ = (x + |x|)/2$. In general, the symbol $g[x_l, \dots, x_{l+r}]$ denotes the divided difference of order r of the function g. We can write a B-spline in the form:

$$B_n^k(x) = (-1)^{n+1}(x_{k+n+1} - x_k) \sum_{p=k}^{k+n+1} \frac{(x - x_p)_+^n}{\omega'_{n+1,k}(x_p)}$$

$$= (x_{k+n+1} - x_k) \sum_{p=k}^{k+n+1} \frac{(x_p - x)_+^n}{\omega'_{n+1,k}(x_p)} \tag{1.2}$$

$$\omega_{r,k}(x) = \prod_{p=k}^{k+r} (x - x_p). \tag{1.3}$$

Note that $\operatorname{supp} B_n^k(x) = (x_k, x_{k+n+1})$.

If a spline $S(f,x)$ is constructed by the values f_k of a function $f(x)$ at points x_k, by its span is usually meant the set of points $\{x_k\}$ such that the values $\{f_k\}$ at these points are required in order to calculate the value of the spline S at the point x. Note that spans of interpolating splines of defect 1 and of degree higher than first, which have been constructed on a given interval $[a,b]$, always consist of all the grid points $\{x_k\}$ belonging to this interval.

2. LOCAL CUBIC SPLINES ON INNER SUBINTERVALS

In this section we suggest a convenient computational formula for constructing a local cubic spline with a minimum span (SMS) of defect 1. This spline, as well as an interpolating spline, reproduces the third-degree polynomials on subintervals $[x_n, x_{n+1}]$ of the interval $[a,b]$ which are far from its endpoints.

Thus, let $a = x_0 < x_1 < \dots < x_N = b$, $x \in [x_n, x_{n+1}]$, $n = 2, \dots, N-3$. Then every cubic spline can be represented as

$$S^3(f,x) = \sum_{k=n-3}^{n} F_{k+2}^3 B_3^k(x). \tag{2.1}$$

Evidently, we have

$$\omega_{3,n-1}(x) = (x - x_{n-1})(x - x_n)(x - x_{n+1})(x - x_{n+2}) > 0$$

at $x \in [x_n, x_{n+1}]$, as well as the remaining coefficients of the divided differences. Therefore we can use the following mean-value theorem.

Proposition 2.1. If a function F is continuous on the interval $[a,b]$ and the numbers α and β are of the same sign, then

$$\alpha F(a) + \beta F(b) = (\alpha + \beta) F(c), \qquad c \in [a,b].$$

Finally, we obtain the following representation for the remainder approximation term:

$$\rho_n(x) = f[x_{n-1}, \ldots, x_{n+2}, \xi]\Omega_n(x), \qquad \xi \in [x_{n-2}, x_{n+3}] \tag{2.5}$$

$$\begin{aligned}\Omega_n(x) &= \omega_{3,n-1}(x) + (1-t)^3 e_n + t^3 d_n \\ &= h_n^2(h_{n-1} + h_n t)t(1-t)(h_{n+1} + h_n(1-t)) \\ &\quad + (1-t)^3 h_n^2 h_{n-1}^2 (h_{n-2} + h_{n-1} + h_n + h_{n+1})/[3(h_n + h_{n-1})].\end{aligned}$$

The representation (2.4) is convenient for calculating the spline because it permits the use of simple standard algorithms for constructing interpolation polynomials. Relation (2.5) also makes it possible to derive exact estimates of the remainder approximation terms.

Let us denote $\bar{h}_n = \max h_s$, $|n - s| < 3$,

$$\|g\|_\infty^n = \max_\xi |g(\xi)|, \qquad \xi \in [x_{n-2}, x_{n+3}].$$

We shall say that $g \in_n W_\infty^r$ if $\|g^{(r)}\|_\infty^n < A_n < \infty$. It is easy to verify that the inequalities are valid:

$$0 < \Omega_n(x) \leqslant \bar{h}_n^4((t^2 - t)^2 + \tfrac{2}{3}).$$

This results in estimates for the remainder approximation term, both pointwise and interval ones.

Theorem 2.2. The following estimates hold for continuous functions f:

$$|\rho_n(x)| \leqslant \bar{h}_n^4 \|f[x_{n-1}, \ldots, x_{n+2}, \xi]\|_\infty^n ((t^2 - t)^2 + \tfrac{2}{3}) \leqslant \bar{h}_n^4 \tfrac{35}{48} \|f[x_{n-1}, \ldots, x_{n+2}, \xi]\|_\infty^n. \tag{2.6}$$

If $f \in_n W_\infty^4$, then

$$|\rho_n(x)| \leqslant \tfrac{1}{24} \bar{h}_n^4 \|f^{(4)}\|_\infty^n ((t^2 - t)^2 + \tfrac{2}{3}) \leqslant \tfrac{35}{1152} \bar{h}_n^4 \|f^{(4)}\|_\infty^n. \tag{2.7}$$

Remark 2.1. Estimates (2.6) and (2.7) retain the validity for a uniform grid and cannot be improved in this case. The equalities are attained on the function $f(x) = (x - x_n)^4$. Estimates (2.7) then coincide with the known estimates given in [6,9]. We would like to call attention to pointwise estimates.

If we select $F_k^3 = \beta_{k,-1} f_{k-1} + \beta_{k,0} f_k + \beta_{k,1} f_{k+1}$,

$$\beta_{k,-1} = \frac{-h_k^2}{3h_{k-1}(h_{k-1}+h_k)}, \qquad \beta_{k,1} = \frac{-h_{k-1}^2}{3h_k(h_{k-1}+h_k)} \tag{2.2}$$

$$\beta_{k,0} = 1 - \beta_{k,-1} - \beta_{k,1}$$

then we obtain the SMS that exactly reproduces the third-degree polynomials [2,6].

It is easily seen that the span of the spline $S^3(f,x)$ at $x \in [x_n, x_{n+1}]$ consists of six points $\{x_{n-2}, \ldots, x_{n+3}\}$. Let the symbol $P_r(x)$ denote a cubic polynomial that interpolates the function f at the points $\{x_{r-1}, \ldots, x_{r+2}\}$. Then $f(x) = P_n(x) + R_n(x)$. The remainder terms is

$$R_n(x) = \omega_{3,n-1}(x) f[x, x_{n-1}, \ldots, x_{n+2}]\,. \tag{2.3}$$

The spline $S^3(f,x)$ exactly reproduces the third-degree polynomials in the sense that $S^3(P_n, x) = P_n(x)$. Therefore

$$S^3(f,x) = P_n(x) + S^3(R_n, x)\,.$$

According to (2.1) and (2.2), we have

$$S^3(R_n,x) = \beta_{n-1,-1} R_n(x_{n-2}) B_3^{n-3}(x) + \beta_{n+2,1} R_n(x_{n+3}) B_3^n(x) + \sum_{k=n-1}^{n+2} \psi_k(x) R_n(x_k)$$

where $\psi_k(x)$ are some polynomials of the third degree. However, $R_n(x_k) = 0$ if $k = n-1, \ldots, n+2$. Therefore

$$S^3(f,x) = P_n(x) + \beta_{n-1,-1} R_n(x_{n-2}) B_3^{n-3}(x) + \beta_{n+2,1} R_n(x_{n+3}) B_3^n(x)$$

and substituting the values of B-splines and $R_n(x_k)$, we get an explicit representation for $S^3(f,x)$.

Theorem 2.1. Let $x \in [x_n, x_{n+1}]$, $t = (x - x_n)/h_n$, $t \in [0,1]$. Then a cubic SMS can be written in the form:

$$\begin{aligned} S^3(f,x) &= P_n(x) - a_n R_n(x_{n-2})(1-t)^3 - b_n R_n(x_{n+3}) t^3 \\ &= P_n(x) - (1-t)^3 f[x_{n-2}, \ldots, x_{n+2}] e_n - t^3 f[x_{n-1}, \ldots, x_{n+3}] d_n \end{aligned} \tag{2.4}$$

$$d_n = h_n^2 h_{n+1}^2 p_n\,, \qquad e_n = h_n^2 h_{n-1}^2 p_{n-1}\,, \qquad a_n = h_n^2 h_{n+1}^2/(3h_{n+2} q_{n+2})$$

$$b_n = h_n^2 h_{n-1}^2/(3h_{n-2} q_n)\,, \qquad p_n = (x_{n+3} - x_{n-1})/[3(x_{n+2} - x_n)]$$

$$q_n = (x_{n+1} - x_{n-1})(x_{n+1} - x_{n-2})(x_n - x_{n-2})\,.$$

Using formula (2.4), we can write the remainder terms as

$$\begin{aligned} \rho_n(x) = f(x) - S^3(f,x) &= f[x_{n-1}, \ldots, x_{n+2}, x]\,\omega_{3,n-1}(x) \\ &\quad + (1-t)^3 f[x_{n-2}, \ldots, x_{n+2}] e_n + t^3 f[x_{n-1}, \ldots, x_{n+3}] d_n\,. \end{aligned}$$

Now let us consider the derivatives. According to formulae (2.4), (2.5) we have

$$S^3(f,x)' = P_n'(x) + (3/h_n)\{(1-t)^2 a_n R_n(x_{n-2}) - b_n R_n(x_{n+3}) t^2\}$$

$$= P_n'(x) - (3/h_n)\{(1-t)^2 f[x_{n-2},\dots,x_{n+2}]e_n - t^2 f[x_{n-1},\dots,x_{n+3}]d_n\}$$

$$\rho_n'(x) = f'(x) - S^3(f,x)' = R_n'(x) - (3/h_n)\{(1-t)^2 a_n R_n(x_{n-2}) - b_n R_n(x_{n+3}) t^2\}.$$

For the second derivative we have

$$S^3(f,x)'' = P_n''(x) - (6/h_n^2)\{(1-t) a_n R_n(x_{n-2}) + b_n R_n(x_{n+3}) t\}$$

$$= P_n''(x) + (6/h_n^2)\{(1-t) f[x_{n-2},\dots,x_{n+2}]e_n + t f[x_{n-1},\dots,x_{n+3}]d_n\}$$

$$\rho_n''(x) = f''(x) - S^3(f,x)'' = R_n''(x) - (6/h_n^2)\{(1-t) a_n R_n(x_{n-2}) + b_n R_n(x_{n+3}) t\}.$$

In order to estimate the remainder approximation terms, we take advantage of the inequalities established in [5]. If $f \in {}_n W_\infty^4$, $x \in [x_n, x_{n+1}]$, then

$$|R^{(r)}(x)| \leqslant K_r h_n^{-4-r} \|f^{(4)}\|_\infty^n, \quad K_1 = \tfrac{1}{12}, \quad K_2 = \tfrac{5}{24}. \tag{2.8}$$

The constants K_r in these inequalities cannot be improved. As a result, we conclude that for $f \in {}_n W_\infty^4$ the following estimates are valid:

$$|\rho_n'(x)| \leqslant \tfrac{1}{6}\bar{h}_n^3 \|f^{(4)}\|_\infty^n, \quad |\rho_n''(x)| \leqslant \tfrac{3}{8}\bar{h}_n^2 \|f^{(4)}\|_\infty^n.$$

3. CUBIC LOCAL SPLINES NEAR THE ENDPOINTS OF THE APPROXIMATION INTERVAL

3.1. Interpolation at nodes close to the endpoints

As before, let $a = x_0 < x_1 \dots < x_N = b$. If $x \in [x_2, x_3]$ then the representation

$$S^3(f,x) = P_2(x) + S^3(R_2, x)$$

holds true; however, at this point it is more convenient for us to write this spline in the form:

$$S^3(f,x) = P_1(x) + s(x), \quad s(x) = S^3(R_1, x). \tag{3.1}$$

Let us take a further look at the spline $s(x)$ near the point $x_2 + 0$. The function $R_1(x)$ vanishes at the points $\{x_0,\dots,x_3\}$. Therefore, taking into account (1.2) and (1.3), we can write

$$s(x) = q_3 B_3^1(x) + q_4 B_3^2(x) = \beta_{3,1} R_1(x_4) B_3^1(x) + q_4 B_3^2(x)$$

where q_3 and q_4 are some coefficients. For $x \in [x_2, x_3]$, B-splines can be represented in the form:

$$B_3^1(x) = \alpha(x - x_1)^3 + \beta(x - x_2)^3, \quad B_3^2(x) = \gamma(x - x_2)^3.$$

We have

$$\alpha = 1/[h_1(h_1 + h_2)(h_1 + h_2 + h_3)]$$

while the other coefficients β and γ are of no interest to us. Thus,

$$s(x) = A(x - x_1)^3 + C(x - x_2)^3$$
$$A = \alpha\beta_{3,1}R_1(x_4) = -f[x_0,\dots,x_4]h_2^2(h_0 + h_1 + h_2 + h_3)/(3(h_1 + h_2)h_1) . \tag{3.2}$$

It is necessary to stress that

$$A = s(x_2 + 0)/h_1^3 = (S^3(f,x_2 + 0) - f_2)/h_1^3 .$$

Thus, we have

$$s(x_2 + 0) = Ah_1^3 , \quad s(x_2 + 0)' = 3Ah_1^2 , \quad s(x_2 + 0)'' = 6Ah_1 . \tag{3.3}$$

Now let us pass on to the interval $[x_0,x_2]$ and define the spline $S^3(f,x)$ on this interval as

$$S^3(f,x) = P_1(x) + A(x - x_1)_+^3$$
$$= P_1(x) + (x - x_1)_+^3(S^3(f,x_2 + 0) - f_2)/h_1^3 . \tag{3.4}$$

In view of (3.3), such a definition ensures that the spline, as well as its first and second derivatives, is continuous at the point x_2. The spline $S^3(f,x)$ possesses the same property at the point x_1. At the points x_1 and x_0, the spline $S^3(f,x)$ interpolates the corresponding values of the function f, and on the interval $[x_0,x_1]$ it coincides with the interpolation polynomial $P_1(x)$. Note that the idea of extending local splines to the endpoints of the approximation interval by using interpolation at grid points close to the endpoints is not new. It has been proposed, for example, in [7]. However, the procedure we suggest for such an extension differs essentially from [7].

Let us write the remainder approximation term

$$\begin{aligned}\rho_1(x) &= f(x) - S^3(f,x)\\ &= R_1(x) - A(x - x_1)_+^3/h_1^3\\ &= f[x_0,x_1,x_2,x_3,x]\omega_{3,0}(x)\\ &\quad + (x - x_1)_+^3 f[x_0,\dots,x_4]h_2^2(h_0 + h_1 + h_2 + h_3)/(3(h_1 + h_2)h_1) .\end{aligned}$$

On the interval $[x_1,x_2]$, both coefficients of the differences are positive and hence putting $t = (x - x_1)/h_1$ we have

$$\begin{aligned}\rho_1(x) &= |f[x_0,x_1,x_2,x_3,\xi](x - x_1)|\\ &\quad \times [(x - x_0)(x - x_2)(x - x_3) + (x - x_1)^2 h_2^2(h_0 + h_1 + h_2 + h_3)/(3(h_1 + h_2)h_1)]\\ &= f[x_0,x_1,x_2,x_3,\xi]\{h_1^2(h_0 + h_1t)t(1 - t)(h_2 + h_1(1 - t))\\ &\quad + t^3h_1^2h_2^2(h_0 + h_1 + h_2 + h_3)/(3(h_1 + h_2))\} .\end{aligned}$$

No we are prepared to formulate the result.

Theorem 3.1. The following estimates hold for continuous functions f:

$$|\rho_1(x)| \leqslant \bar{h}_1^4 \|f[x_0,x_1,x_2,x_3,\xi]\|_\infty^1 t(t^3 - 4t^2/3 - t + 2)$$

$$\leqslant \frac{16 - 3\sqrt{2}}{12\sqrt{2}} \bar{h}_1^4 \|f[x_0,x_1,x_2,x_3,\xi]\|_\infty^1 . \qquad (3.5)$$

If $f \in {}_1W_\infty^4$ then

$$|\rho_1(x)| \leqslant \frac{1}{24} \bar{h}_n^4 \|f^{(4)}\|_\infty^1 t(t^3 - 4t^2/3 - t + 2)$$

$$\leqslant \frac{16 - 3\sqrt{2}}{288\sqrt{2}} \bar{h}_1^4 \|f^{(4)}\|_\infty^1 . \qquad (3.6)$$

Remark 3.1. Estimates (3.5) and (3.6) retain their validity for a uniform grid and cannot be improved in this case. The equalities are attained on the function $f(x) = (x - x_1)^4$.

Now let us consider derivatives on the interval $[x_1,x_2]$. For the remainder term of approximation we have

$$\rho_1(x)' = f'(x) - S^3(f,x)'$$

$$= R_1(x)' + 3(x - x_1)^2 f[x_0,\dots,x_4] h_2^2(h_0 + h_1 + h_2 + h_3)/(3(h_1 + h_2)h_1) .$$

Now, using formulae (2.8), we obtain for $f \in {}_1W_\infty^4$

$$|\rho_1'(x)| \leqslant \tfrac{1}{6} \bar{h}_1^3 \|f^{(4)}\|_\infty^1 .$$

In a similar manner, for the second derivative we have

$$\rho_1(x)'' = f''(x) - S^3(f,x)''$$

$$= R_1(x)'' + 6(x - x_1) f[x_0,\dots,x_4] h_2^2(h_0 + h_1 + h_2 + h_3)/(3(h_1 + h_2)h_1)$$

$$|\rho_1''(x)| \leqslant \tfrac{11}{24} \bar{h}_1^2 \|f^{(4)}\|_\infty^1 .$$

On the interval $[x_0,x_1]$ the estimates coincide with the corresponding estimates for interpolation polynomials.

Now we shall dwell briefly on the ways of extending to the interval endpoint.

3.2. Extending a spline to an endpoint with a given slope

Assume that, in addition to the values of the approximated function $f_k = f(x_k)$, the value $f'(x_0) = m_0$ is given. Let $P(x)$ denote a cubic polynomial that interpolates the function f at the points $\{x_0,x_1,x_2\}$ with $P'(x_0) = m_0$. Such a polynomial can be written [1] in the form:

$$P(x) = f(x_0) + f[x_0,x_0](x - x_0) + f[x_0,x_0,x_1](x - x_0)^2 + f[x_0,x_0,x_1,x_2](x - x_0)^2(x - x_1) .$$

Here

$$f[x_0,x_0] = \lim_{\varepsilon \to 0} f[x_0 + \varepsilon, x_0] = f'(x_0) = m_0 .$$

Subsequent differences with a multiple node x_0 are defined in a similar manner. Then $f(x) = \tilde{P}(x) + \tilde{R}(x)$. The remainder term is

$$\tilde{R}(x) = \tilde{\omega}_{3,0}(x) f[x, x_0, x_0, x_1, x_2]$$
$$\tilde{\omega}_{3,0}(x) = (x - x_0)^2 (x - x_1)(x - x_2) . \tag{3.7}$$

Let $S^3_\varepsilon(f,x)$ be the spline constructed by the procedure described in Section 2 and Subsection 3.1 by using the values of the function f at the points $\{x_0, x_0 + \varepsilon, x_1, \ldots\}$. Let us define the spline $S^3(f,x)$ on the interval $[x_0, x_2]$ as

$$S^3(f,x) = \lim_{\varepsilon \to 0} S^3_\varepsilon(f,x) .$$

Then at $x \in [x_0, x_1]$, $x = x_0 + th_0$, the following representation is valid:

$$S^3(f,x) = \tilde{P}(x) - t^3 f[x_3, x_0, x_0, x_1, x_2] d$$
$$d = h_0^2 h_1^2 (x_3 - x_1)(x_2 - x_0)^{-1}/3 . \tag{3.8}$$

The remainder approximation term is

$$\begin{aligned}\rho(x) &= f(x) - S^3(f,x) \\ &= \tilde{\omega}_{3,0}(x) f[x, x_0, x_0, x_1, x_2] + t^3 f[x_3, x_0, x_0, x_1, x_2] d \\ &= f[\xi, x_0, x_0, x_1, x_2](dt^3 + \tilde{\omega}_{3,0}(x))\end{aligned}$$

because $\tilde{\omega}_{3,0}(x) \geqslant 0$ on this interval. It is easy to verify that the inequalities are valid:

$$\tilde{\omega}_{3,0}(x) \leqslant \bar{h}_1^4 t^2 (1 - t)(2 - t) , \qquad d \leqslant \bar{h}_1^4/2 .$$

Theorem 3.2. The following unimprovable estimates hold for differentiable functions f at $x \in [x_0, x_1]$, $x = x_0 + th_0$:

$$|\rho(x)| \leqslant \bar{h}_1^4 \| f[x_0, x_0, x_1, x_2, \xi] \|_\infty^1 t^2 (t^2 - 5t/2 + 2) \leqslant \tfrac{1}{2} \bar{h}_1^4 \| f[x_0, x_0, x_1, x_2, \xi] \|_\infty^1 .$$

If $f \in {}_1W^4_\infty$ then $|\rho(x)| \leqslant \frac{1}{48} \bar{h}_n^4 \| f^{(4)} \|_\infty^1$.

For the derivatives we have

$$|\rho_1'(x)| \leqslant K_1 \bar{h}_1^3 \| f^{(4)} \|_\infty^1 \, |\rho_1''(x)| \leqslant K_2 \bar{h}_1^2 \| f^{(4)} \|_\infty^1$$

with $K_1 = 7/48$ and $K_2 = 1/3$.

Note that although the constants in the estimates for the derivatives are not unimprovable, they are still less than the unimprovable constants $1/6$ and $3/8$, respectively, in the estimates for the splines constructed in Subsection 3.1.

On the interval $[x_1, x_2]$ the spline can be written as follows:

$$S^3(f,x) = P_1(x) - (1 - t)^3 f[x_3, x_0, x_0, x_1, x_2] d - t^3 f[x_3, x_4, x_0, x_1, x_2] e_1$$
$$t = (x - x_1)/h_1 . \tag{3.9}$$

The quantities e_n have been defined in (2.4).

The remainder approximation term is

$$\rho(x) = f[\xi, x_0, x_3, x_1, x_2]((1-t)^3 d + t^3 e_1 + \omega_{3,0}(x)), \qquad x_0 \leqslant \xi \leqslant x_4$$

because $\omega_{3,0}(x) \geqslant 0$ on this interval.

Theorem 3.3. The following unimprovable estimates hold for differentiable functions f at $x \in [x_1, x_2]$, $x = x_1 + th_1$:

$$|\rho(x)| \leqslant \bar{h}_1^4 \| f[x_3, x_0, x_1, x_2, \xi] \|_\infty^1 ((t^2 - t)^2 + \tfrac{2}{3} - \tfrac{1}{6}(1-t)^3)$$

$$\leqslant 0.636 \bar{h}_1^4 \| f[x_3, x_0, x_1, x_2, \xi] \|_\infty^1, \qquad x_0 \leqslant \xi \leqslant x_4.$$

If $f \in {}_1W_\infty^4$ then $|\rho(x)| \leqslant 0.0265 \bar{h}_n^4 \| f^{(4)} \|_\infty^1$.

It can be shown that the derivatives obey the same estimates as do the estimates of splines constructed in Subsection 3.1.

3.3. Extending a spline to an endpoint with a fictitious slope

With some modifications, the algorithm in the preceding subsection can also be employed in the case in which the value $f'(x_0)$ is unknown. Let $P_4(x)$ denote the fourth-degree polynomial that interpolates a function f at the points $\{x_0, \ldots, x_4\}$. Let us put

$$m_0 = P_4'(x_0) = f[x_0, x_1] + f[x_0, x_1, x_2](x_0 - x_1)$$
$$+ f[x_3, x_0, x_1, x_2](x_0 - x_1)(x_0 - x_2)$$
$$+ f[x_3, x_0, x_1, x_2, x_4](x_0 - x_1)(x_0 - x_2)(x_0 - x_3)$$

and construct a spline $S^3(f,x)$ on the interval $[x_0, x_2]$ by the formula in Subsection 3.2, assuming that $f[x_0, x_0] = m_0$ and defining correspondingly subsequent differences with a multiple node x_0. Then we can prove the following propositions.

Theorem 3.4. If $f \in {}_1W_\infty^5$, then the following estimates are valid for $x \in [x_0, x_1]$:

$$|\rho(x)| \leqslant \tfrac{1}{48} \bar{h}_n^4 \| f^{(4)} \|_\infty^1 + 0.03026 \bar{h}_n^5 \| f^{(5)} \|_\infty^1.$$

If $f \in {}_1W_\infty^4$, then at $x \in [x_1, x_2]$ we also have

$$|\rho(x)| \leqslant 0.0265 \bar{h}_n^4 \| f^{(4)} \|_\infty^1.$$

It can be shown that for $x \in [x_1, x_2]$ the derivatives obey the same estimates as do the derivatives of the splines constructed in Subsection 3.1. As for the interval $[x_0, x_1]$, we give here only the result for the first derivative on a uniform grid; namely if $f \in {}_1W_\infty^6$, then

$$|\rho'(x)| \leqslant 0.0274 h_n^3 \| f^{(4)} \|_\infty^1 + 0.2 h_n^4 \| f^{(5)} \|_\infty^1 + 0.0051 h_n^5 \| f^{(6)} \|_\infty^1.$$

Though this estimate is not optimum, it favours the use of the spline considered here rather than the spline from Subsection 3.1 in approximating the derivative.

Similar formulae are also true near the right-hand endpoint of the approximation interval.

4. EXTRAPOLATION WITH THE USE OF CUBIC LOCAL SPLINES

As previously, let us suppose that the values $f(x_k) = f_k$ are given at the points $a = x_0 < x_1 < \ldots < x_N = b$. Let $x_{-1} < x_0$ and the function f is defined on the interval $[x_{-1}, x_0]$. We need extend the spline $S^3(f,x)$ constructed by the data $\{f_k\}$, $k \geqslant 0$, in order to approximate the function f. We consider two methods for extrapolating the spline which coincides with the interpolation polynomial $P_1(x)$ on the interval $[x_0, x_1]$.

4.1. Extrapolation with a quasi-interpolation at a given point

Again we use the representation

$$S^3(f,x) = P_1(x) + s(x)\,. \tag{4.1}$$

Since $s(x) \equiv 0$ on the interval $[x_0, x_1]$, then in order to ensure the continuity of the spline and its derivatives, put $s(x) = A(x - x_0)^3$ on the interval $[x_{-1}, x_0]$. Let $f(x_{-1}) = f_{-1}$ be an unknown value. Then we have

$$P_1(x_{-1}) = f_{-1} - R_1(x_{-1}) = f_{-1} - f[x_{-1}, \ldots, x_3](x_{-1} - x_0)(x_{-1} - x_1)(x_{-1} - x_2)(x_{-1} - x_3)\,.$$

Thus,

$$\rho(x_{-1}) = f_{-1} - S(f, x_{-1}) = f[x_{-1}, \ldots, x_3]D(h) - Ah_{-1}^3$$

$$D(h) = h_{-1}(h_{-1} + h_0)(h_{-1} + h_0 + h_1)(h_{-1} + h_0 + h_1 + h_2)\,.$$

Now let us choose $A = f[x_0, \ldots, x_4]D(h)/h_{-1}^3$. Then

$$\rho(x_{-1}) = D(h)(f[x_{-1}, \ldots, x_3] - f[x_0, \ldots, x_4]) = -D(h)(f[x_{-1}, \ldots, x_3, x_4])(x_4 - x_{-1})\,.$$

Hence, if the fifth difference $f[x_{-1}, \ldots, x_3, x_4]$ is bounded, then $\rho(x_{-1}) = O(\bar{h}^5)$. If the approximated function $f = P^4(x)$ is the fourth-degree polynomial, then $S(f, x_{-1}) = P^4(x_{-1})$.

4.2. Extrapolation with a minimum integral of the error

The method of extending the spline to the interval $[x_{-1}, x_0]$ which is proposed below is especially efficient when it is used for solving differential equations by methods like Adams' one.

As before, we extend the spline according to (4.1), where the function $s(x) = A(x - x_0)^3$. In doing so, we have

$$\rho(x) = f(x) - S(f,x) = R_1(x) - A(x - x_0)^3\,.$$

First let us consider the integral

$$\int_{x_{-1}}^{x_0} R_1(x)\,dx = \int_{x_{-1}}^{x_0} f[x_0, x_1, x_2, x_3, x](x - x_0)(x - x_1)(x - x_2)(x - x_3)\,dx$$

$$= f[x_0, x_1, x_2, x_3, \xi]\, d(h)\,, \qquad \xi \in [x_{-1}, x_0]$$

$$d(h) = h_{-1}^5/5 + h_{-1}^4(3h_0 + 2h_1 + h_2)/4$$

$$+ h_{-1}^3(3h_0^2 + 4h_0h_1 + 2h_0h_2 + h_1h_2 + h_1^2)/3 + h_{-1}^2 h_0(h_0 + h_1)(h_0 + h_1 + h_2)/2\,.$$

Taking into account that

$$\int_{x_{-1}}^{x_0} A(x-x_0)^3\,dx = Ah_{-1}^4/4$$

we put $A = 4d(h)f[x_0,x_1,x_2,x_3,x_4]/h_{-1}^4$. Then we obtain

$$\int_{x_{-1}}^{x_0} \rho(x)\,dx = d(h)f[x_0,x_1,x_2,x_3,x_4,\xi](x_4-\xi).$$

The following estimates hold:

$$\left|\int_{x_{-1}}^{x_0} R_1(x)\,dx\right| \leqslant \tfrac{163}{30}\bar{h}_1^5\|f[x_0,x_1,x_2,x_3,\xi]\|_\infty^1 \leqslant \tfrac{163}{720}\bar{h}_1^5\|f^{(4)}\|_\infty^1$$

$$\left|\int_{x_{-1}}^{x_0} \rho(x)\,dx\right| \leqslant \tfrac{163}{6}\bar{h}_1^6\|f[x_0,x_1,x_2,x_3,x_4,\xi]\|_\infty^1 \leqslant \tfrac{163}{720}\bar{h}_1^6\|f^{(5)}\|_\infty^1$$

5. FIFTH-DEGREE LOCAL SPLINES ON A UNIFORM GRID

The techniques developed above for constructing and analysing cubic splines can also be applied to splines of other degrees. We consider here only the fifth-degree splines. Considering such splines on arbitrary grids involves bulky calculations, and we therefore restrict ourselves here to a uniform grid, i.e. $h_k = h$ for all k.

The span of the fifth-degree SMS that exactly reproduces the fifth-degree polynomials contains ten grid points; i.e. if $x \in [x_n, x_{n+1}]$, then the values $\{f_k\}_{n-4}^{n+5}$ are required in order to calculate $S^5(f,x)$. If $n \geqslant 4$ then the spline $S^5(f,x)$ can easily be written explicitly [9], namely, putting $t = (x-x_n)/h$, we get

$$S^5(f,x) = \sum_{k=-4}^{5} f_{k+n}Q_k(t), \quad Q_k(t) = \sum_{i=0}^{4} \frac{a_i(t+i+1-k)_+^5}{28800}, \tag{5.1}$$

$$a_0 = 9690, \quad a_1 = -4680, \quad a_2 = 1305, \quad a_3 = -190, \quad a_4 = 13$$

$$Q_{-k}(t) = Q_{k+1}(1-t).$$

In [9] an explicit representation of the remainder approximation term for functions $f \in C^6$ has been obtained in the form:

$$\rho_n(x) = f(x) - S^5(f,x) = -h^6 f(\xi)^{(6)}\left[\frac{\theta^2(\theta+1/2)}{720} + \frac{11}{960}\right], \qquad \theta = (1-t)t.$$

If $x \in [x_{n-4}, x_{n+5}]$, then $\xi \in [x_{n-4}, x_{n+5}]$. From this immediately follows the estimate unimprovable in the class C^6:

$$|\rho_n(x)| \leqslant Ch^6\|f^{(6)}\|_\infty^n, \qquad C = \frac{177}{15360} \approx 0.01152. \tag{5.2}$$

Now let us extend the spline $S^5(f,x)$ to the interval $[x_0,x_4]$. In [3] it has been proposed to extend the spline so that $S^5(f,x_k) = f_k$, $k = 0,1,2,3$, and corresponding formulae have been derived for representing such a spline as a combination of B-splines. The formulae are rather bulky and allow only order-of-magnitude estimates of the remainder approximation term to be found. Here we realise this idea in a simpler form and obtain an explicit representation and estimates for the remainder approximation term.

Let the symbol $P(x)$ denote the fifth-degree polynomial that interpolates the function f at the points $\{x_0,\ldots,x_5\}$. Then $f(x) = P(x) + R(x)$. The remainder term is

$$R(x) = \omega_{5,0}(x) f[x,x_0,\ldots,x_5]. \qquad (5.3)$$

The spline $S^5(f,x)$ exactly reproduces polynomials of the fifth-degree, therefore at $x \in [x_4,x_5]$ we have

$$S^5(f,x) = P(x) + S^5(R,x).$$

Let us write the expression for $S^5(R,x)$ on this interval, using representation (5.1). Since $R_k = R(x_k) = 0$ at $k = 0,\ldots,5$, we have

$$S^5(R,x) = \sum_{k=6}^{8} R_k Q_k(t) = h^{-5} \sum_{k=6}^{8} R_k Q_k((x-x_4)/h)$$

$$= \frac{h^{-5}}{28800}\big[(-4680(x-x_4)^5 + 1305(x-x_3)^5 - 190(x-x_2)^5 + 13(x-x_1)^5)R_6$$

$$+ (1305(x-x_4)^5 - 190(x-x_3)^5 + 13(x-x_2)^5)R_7$$

$$+ (-190(x-x_4)^5 + 13(x-x_3)^5)R_8\big].$$

Now, let us define the spline $S^5(f,x)$ at $x \in [x_0,x_4]$ as follows:

$$S^5(f,x) = P(x) + \frac{h^{-5}}{28800}\big[(1305(x-x_3)_+^5 - 190(x-x_2)_+^5 + 13(x-x_1)_+^5)R_6$$

$$+ (-190(x-x_3)_+^5 + 13(x-x_2)_+^5)R_7 + 13(x-x_3)_+^5 R_8\big].$$

Such a definition ensures that the function $S^5(f,x)$ is continuous, together with its derivatives of order up to fourth, on the interval $[x_0,x_4]$.

On the interval $[x_0,x_1]$ we have $S^5(f,x) = P(x)$. The remainder approximation term is

$$\rho(x) = f(x) - S^5(f,x) = R(x).$$

The following estimate holds:

$$|\rho(x)| \leqslant 16.9009h^6 \|f[x_0,x_1,x_2,x_3,x_4,x_5,\xi]\|_\infty^1 \leqslant 0.0235h^6 \|f^{(6)}\|_\infty^1.$$

On the interval $[x_1,x_2]$ we have

$$S^5(f,x) = P(x) + \frac{13h^{-5}}{28800}(x-x_1)^5 R_6.$$

The remainder approximation term is

$$\rho(x) = R(x) - \frac{13h^{-5}}{28800}(x-x_1)^5 R_6$$

$$|\rho(x)| \leqslant 5.374h^6 \|f[x_0,x_1,x_2,x_3,x_4,x_5,\xi]\|_\infty^1 \leqslant 0.0075h^6 \|f^{(6)}\|_\infty^1 .$$

On the interval $[x_2,x_3]$ we have

$$S^5(f,x) = P(x) + \frac{h^{-5}}{28800}\left[(-190(x-x_2)^5 + 13(x-x_1)^5)R_6 + 13(x-x_2)^5 R_7\right].$$

The remainder approximation term is

$$\rho(x) = R(x) - \frac{h^{-5}}{28800}\left[(-190(x-x_2)^5 + 13(x-x_1)^5)R_6 + 13(x-x_2)^5 R_7\right]$$

$$|\rho(x)| \leqslant 11.443h^6 \|f[x_0,x_1,x_2,x_3,x_4,x_5,\xi]\|_\infty^1 \leqslant 0.0159h^6 \|f^{(6)}\|_\infty^1 .$$

On the interval $[x_3,x_4]$ we have

$$S^5(f,x) = P(x) + \frac{h^{-5}}{28800}\left[(1305(x-x_3)^5 - 190(x-x_2)^5 + 13(x-x_1)^5)R_6\right.$$

$$\left. + (-190(x-x_3)^5 + 13(x-x_2)^5)R_7 + 13(x-x_3)^5 R_8\right].$$

The estimate for the remainder approximation term is

$$|\rho(x)| \leqslant 11.443h^6 \|f[x_0,x_1,x_2,x_3,x_4,x_5,\xi]\|_\infty^1 \leqslant 0.0159h^6 \|f^{(6)}\|_\infty^1 .$$

Note that unlike the estimates on the intervals $[x_1,x_2]$ and $[x_3,x_4]$, all the constants in the estimates of $|\rho(x)|$ on the intervals $[x_2,x_3]$ and $[x_0,x_1]$ cannot be improved.

6. EXTRAPOLATION WITH THE USE OF FIFTH-DEGREE SPLINES

We consider here only the scheme of extrapolation with a quasi-interpolation at the point $x_{-1} = -h$, given the values $f(x_k) = f_k$ at $r \geqslant 0$. As in Section 4, we write the spline

$$S^5(f,x) = P(x) + A(x-x_0)^5. \tag{6.1}$$

Let $f(x_{-1}) = f_{-1}$ be a quantity we do not know. Then

$$P(x_{-1}) = f_{-1} - R(x_{-1}) = f_{-1} - \Delta^6 f_{-1}(x_{-1}-x_0)\ldots(x_{-1}-x_5)/720h^6 .$$

Thus,

$$\rho(x_{-1}) = f_{-1} - S^5(f,x_{-1}) = \Delta^6 f_{-1} - Ah^5 .$$

Now we choose $A = \Delta^6 f_0 h^{-5}$. Then we have

$$\rho(x_{-1}) = -\Delta^7 f_{-1} .$$

Hence, if $f^{(7)}$ is continuous, then

$$\rho(x_{-1}) = -f^{(7)}(\xi)h^7, \qquad \xi \in [x_{-1},x_6] .$$

If the approximated function $f = P^6(x)$ is a polynomial of the sixth degree, then $S^5(f,x_{-1}) = P^6(x_{-1})$.

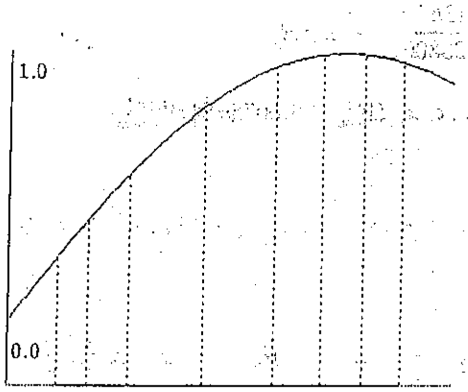

Figure 1. A plot of the function $F(x) = \sin 2x$ on a nonuniform grid.

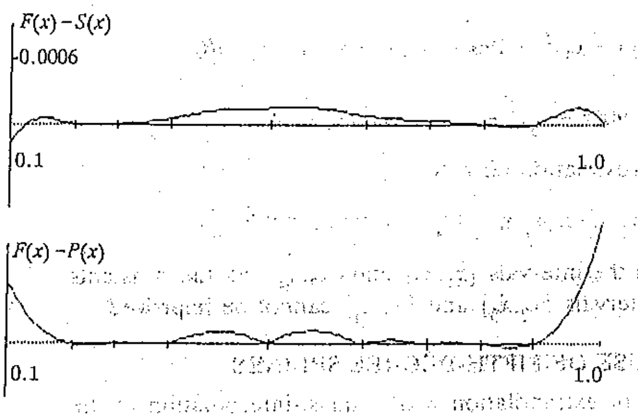

Figure 2. A plot of the error of approximation and extrapolation of the function $F(x) = \sin 2x$ by the local cubic spline $S(x)$ and sections of interpolation polynomials $P(x)$.

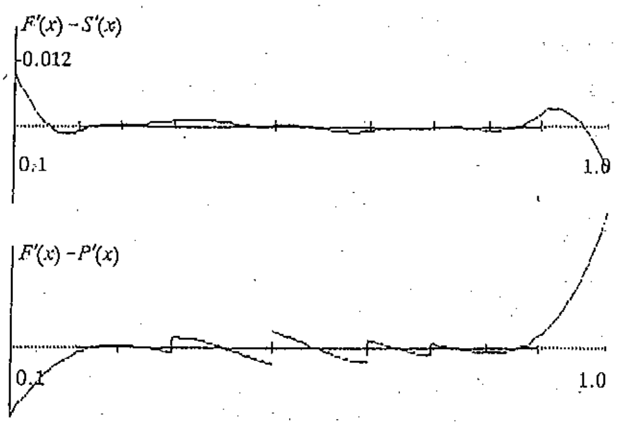

Figure 3. A plot of the error of approximation and extrapolation of the derivative F' by S' and P'.

CONCLUSION

In conclusion, we give some illustrations of what has been considered above. Approximation and extrapolation of a section of the function $y = \sin 2x$ is performed. Shown in Fig. 1 is the function together with the grid, on which its values are calculated. The dashed lines along the axis X indicate the extrapolation interval. The function is approximated and extrapolated by the grid data by two methods: (1) using local cubic splines $S(x)$ (for extending the spline to the endpoint with interpolation or for extrapolation with quasi-interpolation); (2) using sections of interpolation polynomials $P(x)$. Figure 2 shows the differences $f(x) - S(x)$ and $f(x) - P(x)$, and Fig. 3 shows $f'(x) - S'(x)$ and $f'(x) - P'(x)$.

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