

# Computation of interpolatory splines via triadic subdivision

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**Abstract** We present an algorithm for the computation of interpolatory splines of arbitrary order at triadic rational points. The algorithm is based on triadic subdivision of splines. Explicit expressions for the subdivision symbols are established. These are rational functions. The computations are implemented by recursive filtering.

**Keywords** Triadic subdivision · Splines

**Mathematics Subject Classifications (2000)** 65D17 · 65D07 · 93E11

## 1 Introduction

Denote by  $\mathfrak{S}^p$  the space of polynomial splines  $S(x)$  of order  $p$  defined on the uniform grid  $\mathbf{g}^0 \triangleq \{k\}$ ,  $k \in \mathbb{Z}$ , such that the arrays  $\{S(k)\}$ ,  $k \in \mathbb{Z}$ , belong to  $l_1$ . We propose to compute values of the splines via a triadic subdivision. The insertion rule for a spline  $S(x) \in \mathfrak{S}^p$  is

**Triadic Insertion Rule:** Let  $\mathbf{f}^0 \triangleq \{f_k^0 = S(k)\}$ ,  $k \in \mathbb{Z}$ . For  $j = 0, 1, \dots$ , we construct the spline  $S_j(x)$  on the grid  $\mathbf{g}^j \triangleq \{k3^{-j}\}$ ,  $k \in \mathbb{Z}$ , which interpolates the sequence  $\mathbf{f}^j \triangleq \{f_k^j\}$  on the grid  $\mathbf{g}^j$ . Then,  $f_k^{j+1} = S_j(k3^{-(j+1)})$ ,  $k \in \mathbb{Z}$ .

Note that the value of a spline at any point can be expressed as a linear combination of its values at grid points. In other words, any value  $f_k^{j+1}$  can be

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derived by some filtering of the sequence  $\mathbf{f}^j$ . We present explicit expressions for these filters for splines of arbitrary order. Moreover, we prove that for any  $j \in \mathbb{N}$ ,  $k \in \mathbb{Z}$  we have  $f_k^j = S_0(k3^{-j})$ . Thus, we obtain a fast algorithm that computes the values of the spline  $S_0(x)$  from the space  $\mathfrak{S}^p$ , which interpolates the sequence  $\mathbf{f}^0$  on the grid  $\mathbf{g}^0$  at the triadic rational points  $\{k3^{-j}\}$ ,  $k \in \mathbb{Z}$ . A similar fact holds for the Dyadic Insertion Rule when the order  $p$  of splines is even. This is not true for the Dyadic Insertion Rule when  $p$  is odd. In this case, when  $p$  is odd, the spline-based subdivision no longer converges to a spline but rather to a function, which is smoother than the generating spline (see [6]).

The transfer functions of the presented filters are rational functions. These filters have infinite impulse response (masks). Nevertheless, the computations are implemented in a fast way by a recursive filtering. Note that recursive filtering should be applied only once to the initial data array  $\mathbf{f}^0$ . Further steps of the subdivision are implemented using simple filters, which have finite (and short) impulse response. Therefore, the presented scheme is competitive to standard schemes that calculates the interpolatory splines.

The paper has the following structure: in Section 2 we describe several spline filters. The triadic subdivision is described in Section 3. Remarks on the computational aspects of the spline subdivision is given in Section 4.

## 2 Spline filters

### 2.1 B-splines

The centered B-spline of first order is the characteristic function of the interval  $[-1/2, 1/2]$ . The centered B-spline of order  $p$  can be expressed as the convolution  $M^p(x) = M^{p-1}(x) * M^1(x)$ ,  $p \geq 2$ . Note that the B-spline of order  $p$  is supported on the interval  $(-p/2, p/2)$ . It is positive within its support and symmetric about zero. The B-spline  $M^p$  consists of pieces of polynomials of degree  $p - 1$  that are linked to each other at the nodes, such that  $M^p \in C^{p-2}$ . Nodes of B-splines of even order are located at points  $\{k\}$  and of odd order at points  $\{k + 1/2\}$ ,  $k \in \mathbb{Z}$ .

The Fourier transform of the B-spline of order  $p$  is

$$\widehat{M^p}(\omega) \triangleq \int_{-\infty}^{\infty} e^{-i\omega x} M^p(x) dx = \left( \frac{\sin \omega/2}{\omega/2} \right)^p. \quad (2.1)$$

The time domain representation of the B-spline is

$$M^p(x) = \frac{1}{(p-1)!} \sum_{k=0}^{p-1} (-1)^k \binom{p}{k} \left(x + \frac{p}{2} - k\right)_+^{p-1}, \quad x_+ \triangleq (x + |x|)/2.$$

Shifts of B-splines form a basis in the space  $\mathfrak{S}^p$ . Namely, any spline  $S(x) \in \mathfrak{S}^p$  has the following representation:

$$S(x) = \sum_{k \in \mathbb{Z}} q_k M^p(x - k).$$

Equation 2.1 implies that for any  $p \geq 2$

$$\begin{aligned} M^p(x - k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(x-k)} \left( \frac{\sin \omega/2}{\omega/2} \right)^p d\omega \\ &= \sum_{l \in \mathbb{Z}} e^{2\pi ilx} \int_0^1 e^{2\pi i\omega(x-k)} \frac{(\sin \pi \omega)^p (-1)^{lp}}{(\pi(l + \omega))^p} d\omega \\ &= \int_0^1 e^{-2\pi i\omega k} m_x(\omega) d\omega, \end{aligned}$$

where  $m_x(\omega) \triangleq e^{2\pi i\omega x} (\sin \pi \omega) \sum_{l \in \mathbb{Z}} e^{2\pi ilx} \frac{(-1)^{lp}}{(\pi(l + \omega))^p}$ . (2.2)

The relation (2.2) means that  $M^p(x - k)$  is a Fourier coefficient of the 1-periodic function  $m_x(\omega)$  and this function can be represented as the sum:  $m_x(\omega) = \sum_{k \in \mathbb{Z}} e^{2\pi ik\omega} M^p(x - k)$ .

The functions  $m_x(\omega)$  are 1-periodic with respect to the variable  $\omega$  and are splines from  $\mathfrak{S}^p$  with respect to  $x$ . Their shifts along the grid are

$$m_{x+l}(\omega) = \sum_{k \in \mathbb{Z}} e^{2\pi ik\omega} M^p(x - (k-l)) = e^{2\pi il\omega} \sum_{k \in \mathbb{Z}} e^{2\pi ik\omega} M^p(x - k) = e^{2\pi il\omega} m_x(\omega).$$

Following Schoenberg [4], we call the splines  $m_x(\omega)$  the exponential splines.

### 2.2 Interpolatory splines

Equation 2.2 implies that a spline  $S(x) \in \mathfrak{S}^p$  can be represented via the exponential splines:

$$S(x) = \sum_{k \in \mathbb{Z}} q_k M^p(x - k) = \sum_{k \in \mathbb{Z}} q_k \int_0^1 e^{-2\pi i\omega k} m_x(\omega) d\omega = \int_0^1 \hat{q}(\omega) m_x(\omega) d\omega, \tag{2.3}$$

where  $\hat{q}(\omega) \triangleq \sum_{k \in \mathbb{Z}} e^{-2\pi i\omega k} q_k$  is the discrete Fourier transform (DFT) of the coefficient sequence  $\mathbf{q}^0 \triangleq \{q_k\}$ ,  $k \in \mathbb{Z}$ . Integral representation (2.3) was established in [5].

Denote

$$u(\omega) \triangleq m_0(\omega) = \sum_{k \in \mathbb{Z}} e^{-2\pi ik\omega} M^p(k). \tag{2.4}$$

Due to the compact support and symmetry of the B-splines, the functions  $u(\omega)$  are cosine polynomials. They were extensively studied in [1]. In particular, [1] established that  $u(\omega) > 0$  for all  $\omega \in \mathbb{R}$  and  $p \in \mathbb{N}$ . In addition, recurrent relations for computation of the polynomials of any order were presented in

[1]. However,  $u(\omega)$  can be easily calculated directly from (2.4). Equation 2.2 provides the following representation of the functions  $u(\omega)$  :

$$u(\omega) = (\sin \pi \omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l + \omega))^p}.$$

The values of the spline  $S(x)$  at the grid points and their DFT are

$$S(n) = \int_0^1 \hat{q}(\omega)m_n(\omega) d\omega = \int_0^1 e^{2\pi i\omega n} \hat{q}(\omega)u(\omega) d\omega \iff \widehat{S}(\omega) = \hat{q}(\omega)u(\omega).$$

Assume that the spline  $S(x)$  interpolates a sequence  $\mathbf{f}^0$  at the grid points

$$S(n) = f_n^0 \iff \widehat{f}^0(\omega) = \widehat{S}(\omega) \iff \hat{q}(\omega) = \frac{\widehat{f}^0(\omega)}{u(\omega)}. \tag{2.5}$$

According to Triadic Insertion Rule, which was formulated in Section 1,  $f_n^1 = S(n/3)$ . Then,

$$\widehat{f}^1(\omega) = \sum_{n \in \mathbb{Z}} e^{-2\pi i\omega n} S(n/3) = e^{2\pi i\omega} \sigma_{-1}(3\omega) + \sigma_0(3\omega) + e^{-2\pi i\omega} \sigma_1(3\omega), \tag{2.6}$$

$$\sigma_0(\omega) \triangleq \sum_{n \in \mathbb{Z}} e^{-2\pi i\omega n} S(n) = \hat{q}(\omega)u(\omega) = \widehat{f}^0(\omega), \quad \sigma_{\pm 1}(\omega) \triangleq \sum_{n \in \mathbb{Z}} e^{-2\pi i\omega n} S\left(n \pm \frac{1}{3}\right).$$

Due to (2.2),

$$S\left(n \pm \frac{1}{3}\right) = \int_0^1 \hat{q}(\omega)m_{n \pm 1/3}(\omega) d\omega = \int_0^1 e^{2\pi i\omega n} \hat{q}(\omega)v_{\pm 1}(\omega) d\omega,$$

$$v_{\pm 1}(\omega) \triangleq m_{\pm 1/3}(\omega) = e^{\pm 2\pi i\omega/3} (\sin \pi \omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp} e^{\pm 2\pi il/3}}{(\pi(l + \omega))^p}. \tag{2.7}$$

Hence, using (2.5), we obtain

$$\sigma_{\pm 1}(\omega) = \frac{v_{\pm 1}(\omega)}{u(\omega)} \widehat{f}^0(\omega). \tag{2.8}$$

### 3 Triadic subdivision

#### 3.1 Structure of filters

Substituting (2.8) into (2.6), we obtain

$$\widehat{f}^1(\omega) = \widehat{T}(\omega) \widehat{f}^0(3\omega), \quad \widehat{T}(\omega) \triangleq e^{2\pi i\omega} \frac{v_{-1}(3\omega)}{u(3\omega)} + 1 + e^{-2\pi i\omega} \frac{v_1(3\omega)}{u(3\omega)}. \tag{3.9}$$

Similarly,  $\widehat{f}^{j+1}(\omega) = \widehat{T}(\omega) \widehat{f}^j(3\omega)$ ,  $j \in \mathbb{N}$ . Thus, in order to produce the array  $\mathbf{f}^{j+1}$ , we need to filter the upsampled array  $\mathbf{f}^j$  with the filter  $\mathbf{T}$ , whose DFT (the frequency response)  $\widehat{T}(\omega)$  is given in (3.9). We establish now the structure of the filter  $\mathbf{T}$ .

**Theorem 3.1** *The DFT of the filter  $\mathbf{T}$ , which originates from the spline of order  $p$ , is*

$$\widehat{T}(\omega) = \frac{(1 + 2 \cos 2\pi\omega)^p u(\omega)}{3^{p-1}u(3\omega)}. \tag{3.10}$$

*Proof* We have from (2.7) and (3.9)  $\widehat{T}(\omega) = G(\omega)/u(3\omega)$ , where

$$\begin{aligned} G(\omega) &= e^{2\pi i\omega} v_{-1}(3\omega) + u(3\omega) + e^{-2\pi i\omega} v_{-1}(3\omega) \\ &= (\sin 3\pi\omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp} (1 + e^{2\pi il/3} + e^{-2\pi il/3})}{(\pi(l + 3\omega))^p} \\ &= (1 + 2 \cos 2\pi\omega)^p (\sin \pi\omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l + 3\omega))^p} (1 + 2 \cos 2\pi l/3). \end{aligned}$$

We split the sum in the last equation into three sub-sums along  $l = 3m$ ,  $l = 3m + 1$  and  $l = 3m - 1$ . Respectively, the function  $G(\omega)$  splits as  $G(\omega) = (1 + 2 \cos 2\pi\omega)^p (\gamma_0(\omega) + \gamma_{-1}(\omega) + \gamma_1(\omega))$ , where

$$\begin{aligned} \gamma_0(\omega) &\triangleq \frac{(\sin \pi\omega)^p}{3^p} \sum_{m \in \mathbb{Z}} \frac{(-1)^{mp}}{(\pi(m + \omega))^p} (1 + 2 \cos 2\pi m) = 3^{-p+1}u(\omega), \\ \gamma_{\pm 1}(\omega) &\triangleq \frac{(\sin \pi\omega)^p}{3^p} \sum_{m \in \mathbb{Z}} \frac{(-1)^{(m+1)p}}{(\pi(m + \omega) \pm \pi/3)^p} (1 + 2 \cos 2\pi(m \pm 1/3)) = 0. \end{aligned}$$

Hence, (3.10) follows. □

### 3.2 Convergence

The spline  $L(x) \in \mathfrak{S}^p$ , which interpolates the Kronecker delta sequence  $\delta_k^0$ , is called the fundamental spline of the space  $\mathfrak{S}^p$ . The spline  $S(x) \in \mathfrak{S}^p$ , which interpolates the data sequence  $\mathbf{f}^0 = \{f_k^0\}$ , can be represented as

$$S(x) = \sum_{k \in \mathbb{Z}} f_k^0 L(x - k). \tag{3.11}$$

$L(x)$  decays exponentially when  $x$  grows. Therefore, (3.11) represents the spline, which interpolates the sequence  $\{f_k^0\}$ ,  $k \in \mathbb{Z}$ , even in the case when the sequence has a power growth. In this case, the spline  $S(x)$  also has a power growth. It was proved in [3] that the spline of power growth, interpolating the data of power growth, is unique.

We prove that the values of the spline  $S(x)$  at any set of triadic rational points  $S(k3^{-j})$  can be calculated by successive application of Triadic Insertion Rule, which was formulated in Section 1, to the array  $\mathbf{f}^0$ .

**Theorem 3.2** *Assume the spline  $S_0(x)$  belongs to  $\mathfrak{S}^p$ ,  $S_0(k) = f_k^0$  and  $\mathbf{f}^0 = \{f_k^0\} \in l_1$ . Let  $\mathbf{T}$  be the filter defined by (3.10). Assume that for  $j = 1, 2, \dots$ ,  $\mathbf{f}^j =$*

$\{f_k^j\}$  is the array, whose DFT is found from the relation  $\widehat{f^{j+1}}(\omega) = \widehat{T}(\omega)\widehat{f^j}(3\omega)$ . Then,

$$S_0(k3^{-j}) = f_k^j, \quad j = 1, 2, \dots, \tag{3.12}$$

*Proof* We have  $S_0(k3^{-1}) = f_k^1$  from the definition of Triadic Insertion Rule. Next step of subdivision consists in the construction of the spline  $S_1(x)$  on the grid  $\mathbf{g}^1 = \{k3^{-1}\}$ , such that  $S_1(k3^{-1}) = f_k^1$ . Then,  $f_k^2 = S_1(k3^{-2})$ . We prove that  $f_k^2 = S_0(k/9)$ . The subsequent relations in (3.12) are derived by a simple induction.

The array  $\mathbf{f}^2$  is obtained by repeated application of the filter  $\mathbf{T}$  to the array  $\mathbf{f}^0$ :

$$\begin{aligned} \widehat{f^2}(\omega) &= \widehat{T}(\omega)\widehat{f^1}(3\omega) = \widehat{T}(\omega)\widehat{T}(3\omega)\widehat{f^0}(9\omega) \\ &= \frac{(1 + 2 \cos \pi \omega)^p u(\omega)}{3^{p-1}u(3\omega)} \frac{(1 + 2 \cos 3\pi \omega)^p u(3\omega)}{3^{p-1}u(9\omega)} \widehat{f^0}(9\omega) \\ &= \frac{H(\omega)}{u(9\omega)} \widehat{f^0}(9\omega), \end{aligned}$$

where

$$H(\omega) \triangleq 9^{1-p} ((1 + 2 \cos 2\pi \omega)(1 + 2 \cos 6\pi \omega))^p u(\omega).$$

Denote  $s_k = S_0(k/9)$ ,  $k \in \mathbb{Z}$ . Then, the DFT of this array is  $\widehat{s}(\omega) = \widetilde{H}\widehat{f^0}(9\omega)(\omega)/u(9\omega)$ , where

$$\widetilde{H}(\omega) \triangleq \sum_{\nu=1}^4 (e^{-2\nu\pi i\omega} w_{\nu}(9\omega) + e^{2\nu\pi i\omega} w_{-\nu}(9\omega)) + u(9\omega), \quad w_{\pm\nu}(\omega) \triangleq m_{\pm\nu/9}(\omega).$$

We can write, using (2.7),

$$\begin{aligned} \widetilde{H}(\omega) &= (\sin 9\pi \omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l + 9\omega))^p} \left( 1 + \sum_{\nu=1}^4 e^{2\nu\pi il/9} + e^{-2\nu\pi il/9} \right) \\ &= ((1 + 2 \cos 6\pi \omega)(1 + 2 \cos 2\pi \omega))^p \\ &\quad \times (\sin \pi \omega)^p \sum_{l \in \mathbb{Z}} \frac{(-1)^{lp}}{(\pi(l + 9\omega))^p} \left( 1 + 2 \sum_{\nu=1}^4 \cos 2\pi \nu l/9 \right) \\ &= ((1 + 2 \cos 6\pi \omega)(1 + 2 \cos 2\pi \omega))^p \sum_{\mu=-4}^4 \kappa_{\mu}(\omega), \end{aligned}$$

where

$$\kappa_{\mu}(\omega) \triangleq \left( 1 + 2 \sum_{\nu=1}^4 \cos 2\pi \nu \mu/9 \right) \frac{(\sin \pi \omega)^p}{9^p} \sum_{m \in \mathbb{Z}} \frac{(-1)^{(m+1)p}}{(\pi(m + \omega) + \mu\pi/9)^p}.$$

It is easily verified that for  $\mu = \pm 1, \dots, \pm 4$   $\kappa_\mu(\omega) \equiv 0$ , while  $\kappa_0(\omega) = 9^{1-p}u(\omega)$ . Hence, it follows that

$$\tilde{H}(\omega) = H(\omega) \iff f_k^2 = S_0(k3^{-2}), \quad k \in \mathbb{Z}.$$

Repeating our reasoning with the initial data set  $\mathbf{f}^1$  instead of  $\mathbf{f}^0$ , we prove that  $f_k^3 = S_1(k3^{-3})$  and so on. □

*Remark* If the initial data is the delta sequence  $f_k^0 = \delta_k^0$  then we get  $f_k^j = L(k3^{-j})$ , where  $L(x)$  is the fundamental spline of the space  $\mathfrak{S}^p$ . Therefore, we can extend the assertion of Theorem 3.2 from the splines belonging to  $\mathfrak{S}^p$  to the splines that interpolate sequences of power growth.

### 4 Remarks on the computational aspects of the spline subdivision

It was stated in Section 3.1 that in order to produce the array  $\mathbf{f}^{j+1}$  we need to filter the upsampled array  $\mathbf{f}^j$  with the filter  $\mathbf{T}$ . In the frequency domain, filtering is presented by (3.9). The frequency response of the filter is given by (3.10). For computational purposes, we present filtering using the  $z$ -transform. We recall that the  $z$ -transform of a sequence  $\mathbf{a} = \{a_k\} \in l_1$  is  $A(z) \triangleq \sum_{k \in \mathbb{Z}} z^{-k} a_k$ . The  $z$ -transform of the shifted sequence is  $\sum_{k \in \mathbb{Z}} z^{-k} a_{k+l} = z^l A(z)$ .

Denote  $z \triangleq e^{2\pi i \omega}$ . Then, the DFT of a sequence  $\mathbf{a}$  becomes its  $z$ -transform  $\hat{a}(\omega) = A(z)$ . The DFT of a filter  $\mathbf{T}$  becomes its  $z$ -transform  $\hat{T}(\omega) = T(z)$  and it is called the transfer function. The subdivision is represented as

$$F^{j+1}(z) = T(z)F^j(z^3), \quad T(z) = \frac{(z^{-1} + 1 + z)^p U(z)}{U(z^3)}, \quad U(z) \triangleq \sum_{k \in \mathbb{Z}} z^{-k} M^p(k). \tag{4.13}$$

The function  $U(z)$  is a Laurent polynomial and the transfer function  $T(z)$  is a rational function. Such filters are called the infinite impulse response (IIR) filters as opposite to the finite impulse response (FIR) filters, whose transfer functions are Laurent polynomials.

**Proposition 4.1** ([2]) *The roots of the Laurent polynomials  $U(z)$  are all simple and negative. Each root  $\zeta$  can be paired with a dual root  $\theta$  such that  $\zeta \theta = 1$ . Thus, if  $p = 2r - 1$ ,  $p = 2r$  then  $U(z)$  can be represented by:*

$$U(z) = \prod_{n=1}^r \frac{1}{\gamma_n} (1 + \gamma_n z) (1 + \gamma_n z^{-1}), \quad 0 < |\gamma_1| < |\gamma_2| < \dots < |\gamma_r| = e^{-g} < 1, \quad g > 0.$$

The above properties of the polynomials  $U(z)$  enable to implement filtering via a recursive algorithm, which is widely used in signal processing.

There are two ways to implement the described subdivision scheme.

**Polyphase filtering:** One way is to implement the filter  $\mathbf{T} = \{T_k\}$ ,  $k \in \mathbb{Z}$ , using the so-called polyphase representation of the filter:

$$T(z) = zT_{-1}(z^3) + T_0(z^3) + z^{-1}T_1(z^3), \quad T_\nu(z) \triangleq \sum_{k \in \mathbb{Z}} z^{-k} T_{3k+\nu}, \quad \nu = -1, 0, 1.$$

Then, the polyphase representation of the array  $\mathbf{f}^{j+1}$  is

$$F^{j+1}(z) = zF_{-1}^{j+1}(z^3) + F_0^{j+1}(z^3) + z^{-1}F_1^{j+1}(z^3),$$

$$F_\nu^{j+1}(z) \triangleq \sum_{k \in \mathbb{Z}} z^{-k} f_{3k+\nu}^{j+1} = T_\nu(z)F^j(z), \quad \nu = -1, 0, 1.$$

Thus, in order to retrieve the sub-arrays  $\{f_{3k\pm 1}^{j+1}\}$  we have to filter the array  $\{f_k^j\}$  with the filters, whose transfer functions are  $T_{\pm 1}(z)$ , respectively. Recall that  $\{f_{3k}^{j+1} = f_k^j\}$ . The polyphase components of the transfer function  $T(z)$  can be derived from the expression for  $T(z)$  given in (4.13). Obviously,  $T_1(z) = T_{-1}(z^{-1})$ . However, it is easily seen from (3.9) that  $T_{\pm 1}(e^{2\pi i \omega}) = v_{\pm 1}(\omega)/u(\omega)$ .

**Direct filtering:** Equation 4.13 suggests that, when several steps of the subdivision must be carried out, a direct application of the filter  $\mathbf{T}$  is preferable. It follows from (4.13) that

$$F^{j+1}(z) = T(z) \cdot F^j(z^3) = \prod_{l=1}^{j-1} T(z^{3^l}) \cdot F^0(z^{3^j})$$

$$= \frac{U(z) \prod_{l=0}^{j-1} (z^{-3^l} + 1 + z^{3^l})^p}{U(z^{3^j})} \cdot F^0(z^{3^{j+1}}).$$

For example,

$$F^3(z) = \frac{U(z)(z^{-1} + 1 + z)^p(z^{-3} + 1 + z^3)^p(z^{-9} + 1 + z^9)^p}{U(z^{27})} \cdot F^0(z^{27}).$$

Thus, the subdivision is implemented via the following steps:

1. The IIR filter with the transfer function  $1/U(z)$  is applied to the data array  $\mathbf{f}^0$ .
2. The produced array is upsampled<sup>1</sup> and filtered with FIR filter, whose transfer function is  $(z^{-1} + 1 + z)^p$  (repeated  $j$  times).
3. The produced array is filtered with FIR filter whose transfer function is  $U(z)$ .

Note that the IIR filtering is applied only once.

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<sup>1</sup>Upsampling means replacing an array  $\{a_k\}$  by the array  $\{\tilde{a}_k\}$  such that  $\tilde{a}_{3k} = a_k$  and  $\tilde{a}_{3k\pm 1} = 0$ .



### 4.1 Examples of filters

**Linear spline:**  $T^2(z) = (z + 1 + z^{-1})^2 / 3$ ,  $T_1^2(z) = T_{-1}^2(z^{-1}) = (z + 2) / 3$ . This is a single FIR filter in the family. All the filters derived from splines of higher orders are IIR.

**Quadratic spline:**

$$T^3(z) = \frac{(z + 6 + z^{-1})(z + 1 + z^{-1})^3}{9(z^3 + 6 + z^{-3})},$$

$$T_1^3(z) = T_{-1}^3(z^{-1}) = \frac{25z + 46 + z^{-1}}{9(z + 6 + z^{-1})}. \tag{4.14}$$

**Cubic spline:**

$$T^4(z) = \frac{(z + 4 + z^{-1})(z + 1 + z^{-1})^4}{27(z^3 + 4 + z^{-3})},$$

$$T_1^4(z) = T_{-1}^4(z^{-1}) = \frac{z^2 + 60z + 93 + 8z^{-1}}{27(z + 4 + z^{-1})}.$$

**Spline of fourth degree:**

$$T^5(z) = \frac{(z^2 + 76z + 230 + 76z^{-1} + z^{-2})(z + 1 + z^{-1})^5}{81(z^6 + 76z^3 + 230 + 76z^{-3} + z^{-6})},$$

$$T_1^5(z) = T_{-1}^5(z^{-1}) = \frac{625z^2 + 11516z + 16566 + 2396z^{-1} + z^{-2}}{81(z^2 + 76z + 230 + 76z^{-1} + z^{-2})}.$$

**Spline of fifth degree:**

$$T^6(z) = \frac{(z^2 + 26z + 66 + 26z^{-1} + z^{-2})(z + 1 + z^{-1})^6}{243(z^6 + 26z^3 + 66 + 26z^{-3} + z^{-6})},$$

$$T_1^6(z) = T_{-1}^6(z^{-1}) = \frac{z^3 + 1018z^2 + 10678z + 14498 + 29336z^{-1} + 32z^{-2}}{243(z^2 + 26z + 66 + 26z^{-1} + z^{-2})}.$$

We now provide an example of a recursive implementation of an IIR filter. In [6] this is discussed in more details.

**Implementation of the filter  $T_1^3$ .** The  $z$ -transform of this filter is given in (4.14):

$$T_1^3(z) = \frac{25z + 46 + z^{-1}}{9(z + 6 + z^{-1})} = \frac{\alpha}{9} \frac{25z + 46 + z^{-1}}{(1 + \alpha z)(1 + \alpha z^{-1})},$$

where  $\alpha = 3 - 2\sqrt{2} \approx 0.172$ . Then, application of the filter  $\mathbf{T}_1^3$  to an array  $\mathbf{a} = \{a_k\}$ , whose  $z$ -transform is  $A(z)$ , is implemented as a subsequent application of three filters:  $\mathbf{b} = \mathbf{T}_1^3 \mathbf{a} = \mathbf{R}_l \mathbf{R}_r \mathbf{F} \cdot \mathbf{a}$ , which are defined by their  $z$ -transforms:

$$F(z) = \frac{\alpha}{9} (25z + 46 + z^{-1}), \quad R_r(z) = \frac{1}{1 + \alpha z^{-1}}, \quad R_l(z) = \frac{1}{1 + \alpha z}.$$

Thus, filtering is carried out in three steps:

$$A^1(z) = F(z)A(z) \iff a_k^1 = \frac{\alpha}{9} (25a_{k+1} + 46a_k + a_{k-1}),$$

$$A^2(z) = R_r(z)A^1(z) \iff (1 + \alpha z^{-1}) A^2(z) = A^1(z) \iff a_k^2 = a_k^1 - \alpha a_{k-1}^2,$$

$$B(z) = R_l(z)A^2(z) \iff (1 + \alpha z) B(z) = A^2(z) \iff b_k = a_k^2 - \alpha b_{k+1}.$$

The filter  $\mathbf{F}$  is FIR unlike the filters  $\mathbf{R}_l$  and  $\mathbf{R}_r$ . Application of the filter  $\mathbf{R}_r$  is called causal recursive filtering. Here, for the calculation of the term  $a_k^2$ , the previously derived term  $a_{k-1}^2$  is used. Application of the  $\mathbf{R}_l$  is called anti-causal recursive filtering. All these procedures are implemented in a fast way. Computation of splines of higher orders uses filters, which are factorized into longer cascades of the same structure.

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