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# A Generalized Courant-Fischer Minimax Theorem 

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# A GENERALIZED COURANT-FISCHER MINIMAX THEOREM 

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## 1. Introduction

A useful tool for analyzing the spectrum of an Hermitian matrix is the CourantFischer Minimax Theorem [2].

Theorem 1. (Courant-Fischer Minimax Theorem) Suppose that $S \in \mathbb{C}^{n \times n}$ is an Hermitian matrix, then

$$
\lambda_{k}(S)=\min _{\operatorname{dim}(U)=k} \max _{\substack{x \in U \\ x \neq 0}} \frac{x^{*} S x}{x^{*} x}
$$

and

$$
\lambda_{k}(S)=\max _{\operatorname{dim}(V)=n-k+1} \min _{\substack{x \in V \\ x \neq 0}} \frac{x^{*} S x}{x^{*} x}
$$

where $\lambda_{k}(S)$ is the $k$ 'th largest eigenvalue of $S$.
The goal of this short communication it to present a generalization of Theorem 1, which we refer to as the Generalized Courant-Fischer Minimax Theorem. We now state the theorem, and we give a proof in the next section.

Theorem 2. (Generalized Courant-Fischer Minimax Theorem) Suppose that $S \in$ $\mathbb{C}^{n \times n}$ is an Hermitian matrix and that $T \in \mathbb{C}^{n \times n}$ is an Hermitian positive semidefinite matrix such that $\operatorname{null}(T) \subseteq \operatorname{null}(S)$. For $1 \leq k \leq \operatorname{rank}(T)$ we have

$$
\begin{aligned}
\lambda_{k}(S, T)= & \min _{\operatorname{dim}(U)=k} \max _{x \in U} \frac{x^{*} S x}{x^{*} T x} \\
& U \perp \operatorname{null}(T)
\end{aligned}
$$

and

$$
\lambda_{k}(S, T)=\max _{\operatorname{dim}(V)=\operatorname{rank}(T)-k+1} \min _{x \in V} \frac{x^{*} S x}{x^{*} T x}
$$

## 2. Proof

We begin by stating and proving a generalization of the Courant-Fischer Theorem for pencils of Hermitian positive definite matrices.

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Theorem 3. Let $S, T \in \mathbb{C}^{n \times n}$ be Hermitian matrices. If $T$ is also positive definite then

$$
\lambda_{k}(S, T)=\min _{\operatorname{dim}(U)=k} \max _{x \in U} \frac{x^{*} S x}{x^{*} T x}
$$

and

$$
\lambda_{k}(S, T)=\max _{\operatorname{dim}(V)=n-k+1} \min _{x \in V} \frac{x^{*} S x}{x^{*} T x}
$$

Proof. Let $T=L^{*} L$ be the Cholesky factorization of $B$. Let $U$ be some $k$ dimensional subspace of $\mathbb{C}^{n}$, let $x \in U$, and let $y=L x$. Since $T$ is Hermitian positive definite (hence nonsingular), the subspace $W=\{L x: x \in U\}$ has dimension $k$. Similarly, for any $k$-dimensional subspace $W$, the subspace $U=\left\{L^{-1} x: x \in W\right\}$ has dimension $k$. We have

$$
\frac{x^{*} S x}{x^{*} T x}=\frac{x^{*} L^{*} L^{-*} S L^{-1} L x}{x^{*} L^{*} L x}=\frac{y^{*} L^{-*} S L^{-1} y}{y^{*} y}
$$

By applying the Courant-Fischer to $L^{-*} S L^{-1}$, we obtain

$$
\begin{aligned}
\lambda_{k}\left(L^{-*} S L^{-1}\right) & =\min _{\operatorname{dim}(W)=k} \max _{y \in W} \frac{y^{*} L^{-*} S L^{-1} y}{y^{*} y} \\
& =\min _{\operatorname{dim}(U)=k} \max _{x \in S} \frac{x^{*} S x}{x^{*} T x}
\end{aligned}
$$

The generalized eigenvalues of $(S, T)$ are exactly the eigenvalues of $L^{-*} S L^{-1}$ so the first equality of the theorem follows. The second equality can be proved using a similar argument.

Before proving the generalized version of the Courant-Fischer Minimax Theorem we show how to convert an Hermitian positive semidefinite problem to an Hermitian positive definite problem.

Lemma 4. Let $S, T \in \mathbb{C}^{n \times n}$ be Hermitian matrices. Assume that $T$ is also $a$ positive semidefinite and that $\operatorname{null}(T) \subseteq \operatorname{null}(S)$. For any $Z \in \mathbb{C}^{n \times \operatorname{rank}(T)}$ with $\operatorname{range}(Z)=\operatorname{range}(T)$, the determined generalized eigenvalues of $(S, T)$ are exactly the generalized eigenvalues of $\left(Z^{*} S Z, Z^{*} T Z\right)$.

Proof. We first show that $Z^{*} T Z$ has full rank. Suppose that $Z^{*} T Z v=0$. We have $T Z v \in \operatorname{null}\left(Z^{*}\right)$. Therefore, $T Z v \perp \operatorname{range}(Z)=\operatorname{range}(T)$. Obviously $T Z v \in$ range $(T)$, so we must have $v=0$. Since $\operatorname{null}\left(Z^{*} T Z\right)=\{0\}$, the matrix $Z^{*} T Z$ has full rank.

Suppose that $\lambda$ is a determined eigenvalue of $(S, T)$. We will show that it is a determined eigenvalue of $\left(Z^{*} S Z, Z^{*} T Z\right)$. The pencil $\left(Z^{*} S Z, Z^{*} T Z\right)$ has exactly $\operatorname{rank}\left(Z^{*} T Z\right)$ determined eigenvalues. We will show that $Z^{*} T Z$ is full rank, so the pencil $\left(Z^{*} S Z, Z^{*} T Z\right)$ has exactly $\operatorname{rank}(T)$ eigenvalues. Since the pencil $(S, T)$ has exactly $\operatorname{rank}(T)$ determined eigenvalues, each of them an eigenvalue of $\left(Z^{*} S Z, Z^{*} T Z\right)$, this will conclude the proof.

Now let $\mu$ be an eigenvalue of $\left(Z^{*} S Z, Z^{*} T Z\right)$. It must be determined, since $Z^{*} T Z$ has full rank. Let $y$ be the corresponding eigenvector, $Z^{*} S Z y=\mu Z^{*} T Z y$, and let $x=Z y$. Now there are two cases. If $\mu=0$, then $S Z y=S x=0$ (since $Z^{*}$ has full rank and at least as many columns as rows). The vector $x$ is in range $(Z)=\operatorname{range}(T), T x \neq 0$. This implies that $\mu=0$ is also a determined eigenvalue of $(S, T)$.

If $\mu \neq 0$, the analysis is a bit more difficult. Clearly, $T Z y \in \operatorname{range}(T)=$ range $(Z)$. But range $(Z)=\operatorname{range}\left(Z^{*+}\right)[1$, Proposition 6.1.6.vii $]$, so $Z^{*+} Z^{*} T Z y=$ $T Z y$ [1, Proposition 6.1.7]. We claim that $S Z y \in \operatorname{range}(Z)$. If it is not, it $Z y$ must be in $\operatorname{null}(T) \subseteq \operatorname{null}(S)$, but $\mu$ would have to be zero. Therefore, we also have $Z^{*+} Z^{*} S Z y=S Z y$, so by multiplying $Z^{*} S Z y=\mu Z^{*} T Z y$ by $Z^{*+}$ we see that $\mu$ is an eigenvalue of $(S, T)$.

We are now ready to prove Theorem 2, the generalization of the Courant-Fischer Minimax Theorem. The technique is simple: we use Lemma 4 to reduce the problem to a smaller-sized full-rank problem, apply Theorem 3 to characterize the determined eigenvalues in terms of subspaces, and finally show a complete correspondence between the subspaces used in the reduced pencil and subspaces used in the original pencil.

Proof. (Theroem 2) Let $Z \in \mathbb{C}^{n \times \operatorname{rank}(T)}$ have range $(Z)=\operatorname{range}(T)$. We have
and

$$
\lambda_{k}(S, T)=\lambda_{k}\left(Z^{*} S Z, Z^{*} T Z\right)=\quad \max \quad \operatorname{dim}(W)=\operatorname{rank}(T)-k+1 \min _{x \in W} \frac{x^{*} Z^{*} T Z x}{x^{*} Z^{*} T Z x}
$$

The leftmost equality in each of these equations follows from Lemma 4 and the rightmost one follows from Theorem 3.

We now show that for every $k$-dimensional subspace $U \subseteq \mathbb{C}^{n}$ with $U \perp \operatorname{null}(T)$, there exists a $k$-dimensional subspace $W \subseteq \mathbb{C}^{\operatorname{rank}(T)}$ such that

$$
\left\{\frac{x^{*} S x}{x^{*} T x}: x \in U\right\}=\left\{\frac{y^{*} Z^{*} S Z y}{y^{*} Z^{*} T Z y}: y \in W\right\}
$$

and vice versa. The validity of this claim establishes the min-max side of the theorem.

We first need to show that $k \leq \operatorname{rank}(T)$. This is true because every vector in $U$ is in range $(T)$, so its dimension must be at most $\operatorname{rank}(T)$.

Define $W=\left\{y \in \mathbb{C}^{\operatorname{rank}(T)}: Z y \in U\right\}$. Let $b_{1}, \ldots, b_{k}$ be a basis for $U$. Because $U \perp \operatorname{null}(T), b_{j} \in \operatorname{range}(T)$, so there is a $y_{j}$ such that $Z y_{j}=b_{j}$. Therefore, dimension of $W$ is at most $k$. Now let the vectors $y_{i}$ 's be a basis of $W$ and define $b_{i}=Z y_{i}$. The $b_{i}$ 's span $U$, so there are at most $k$ of them, so the dimension of $W$ is at least $k$. Therefore, it is exactly $k$.

Every $x \in U$ is orthogonal to null $(T)$, so it must be in range $(T)$. There exist a $y \in$ $\mathbb{C}^{\operatorname{rank}(T)}$ such that $Z y=x$. So we have $x^{*} S x / x^{*} T x=y^{*} Z^{*} S Z y / y^{*} Z^{*} T Z y$. Combining with the fact that $y \in W$, we have shown inclusion of one side. Now suppose $y \in W$. Define $x=Z y \in U$. Again we have $x^{*} S x / x^{*} T x=y^{*} Z^{*} S Z y / y^{*} Z^{*} T Z y$, which shows the other inclusion.

Now we will show that for every $k$-dimensional subspace $W$ there is a subspace $U$ that satisfies the claim. Define $U=\{Z y: y \in W\}$. Because $Z$ has full rank, $\operatorname{dim}(U)=k$. Also, $U \subseteq \operatorname{range}(Z)=\operatorname{range}(T)$ so $U \perp \operatorname{null}(T)$. The equality of the Raleigh-quotient sets follows from taking $y \in W$ and $x=Z y \in U$ or vice versa.

## References

[1] Denis S. Bernstein. Matrix Mathematics: Theory, Facts, and Formulas with Applications to Linear Systems Theory. Princeton University Press, 2005.
[2] Gene H. Golub and Charles F. Van Loan. Matrix Computations (3rd ed.). Johns Hopkins University Press, Baltimore, MD, USA, 1996.

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