Lawrence Berkeley National Laboratory

Technical Report LBNL-6393E

# A Generalized Courant-Fischer Minimax Theorem

Haim Avron School of Computer Science, Tel-Aviv University

Esmond Ng Computational Research Division Lawrence Berkeley National Laboratory

Sivan Toledo School of Computer Science, Tel-Aviv University

August 2008

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

This manuscript has been authored by an author at Lawrence Berkeley National Laboratory under Contract No. DE-AC02-05CH11231 with the U.S. Department of Energy. The U.S. Government retains, and the publisher, by accepting the article for publication, acknowledges, that the U.S. Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for U.S. Government purposes.

## A GENERALIZED COURANT-FISCHER MINIMAX THEOREM

HAIM AVRON, ESMOND NG, AND SIVAN TOLEDO

## 1. INTRODUCTION

A useful tool for analyzing the spectrum of an Hermitian matrix is the *Courant-*Fischer Minimax Theorem [2].

**Theorem 1.** (Courant-Fischer Minimax Theorem) Suppose that  $S \in \mathbb{C}^{n \times n}$  is an Hermitian matrix, then

$$\lambda_k(S) = \min_{\dim(U)=k} \max_{\substack{x \in U \\ x \neq 0}} \frac{x^*Sx}{x^*x}$$

and

$$\lambda_k(S) = \max_{\dim(V)=n-k+1} \min_{\substack{X \in V \\ x \neq 0}} \frac{x^*Sx}{x^*x}$$

where  $\lambda_k(S)$  is the k'th largest eigenvalue of S.

The goal of this short communication it to present a generalization of Theorem 1, which we refer to as the *Generalized Courant-Fischer Minimax Theorem*. We now state the theorem, and we give a proof in the next section.

**Theorem 2.** (Generalized Courant-Fischer Minimax Theorem) Suppose that  $S \in \mathbb{C}^{n \times n}$  is an Hermitian matrix and that  $T \in \mathbb{C}^{n \times n}$  is an Hermitian positive semidefinite matrix such that  $\operatorname{null}(T) \subseteq \operatorname{null}(S)$ . For  $1 \leq k \leq \operatorname{rank}(T)$  we have

$$\lambda_k(S,T) = \min_{\substack{\dim(U) = k \\ U \perp \operatorname{null}(T)}} \max_{x \in U} \frac{x^* S x}{x^* T x}$$

and

$$\lambda_k(S,T) = \max_{\substack{\dim(V) = \operatorname{rank}(T) - k + 1 \\ V \perp \operatorname{null}(T)}} \min_{x \in V} \frac{x^* S x}{x^* T x}.$$

### 2. Proof

We begin by stating and proving a generalization of the Courant-Fischer Theorem for pencils of Hermitian positive definite matrices.

Date: August 2008.

**Theorem 3.** Let  $S, T \in \mathbb{C}^{n \times n}$  be Hermitian matrices. If T is also positive definite then

$$\lambda_k(S,T) = \min_{\dim(U)=k} \max_{x \in U} \frac{x^* S x}{x^* T x}$$

and

$$\lambda_k(S,T) = \max_{\dim(V)=n-k+1} \min_{x \in V} \frac{x^* S x}{x^* T x}$$

*Proof.* Let  $T = L^*L$  be the Cholesky factorization of B. Let U be some k-dimensional subspace of  $\mathbb{C}^n$ , let  $x \in U$ , and let y = Lx. Since T is Hermitian positive definite (hence nonsingular), the subspace  $W = \{Lx : x \in U\}$  has dimension k. Similarly, for any k-dimensional subspace W, the subspace  $U = \{L^{-1}x : x \in W\}$  has dimension k. We have

$$\frac{x^*Sx}{x^*Tx} = \frac{x^*L^*L^{-*}SL^{-1}Lx}{x^*L^*Lx} = \frac{y^*L^{-*}SL^{-1}y}{y^*y} \,.$$

By applying the Courant-Fischer to  $L^{-*}SL^{-1}$ , we obtain

$$\lambda_k(L^{-*}SL^{-1}) = \min_{\dim(W)=k} \max_{y \in W} \frac{y^*L^{-*}SL^{-1}y}{y^*y}$$
$$= \min_{\dim(U)=k} \max_{x \in S} \frac{x^*Sx}{x^*Tx} .$$

The generalized eigenvalues of (S,T) are exactly the eigenvalues of  $L^{-*}SL^{-1}$  so the first equality of the theorem follows. The second equality can be proved using a similar argument.

Before proving the generalized version of the Courant-Fischer Minimax Theorem we show how to convert an Hermitian positive semidefinite problem to an Hermitian positive definite problem.

**Lemma 4.** Let  $S, T \in \mathbb{C}^{n \times n}$  be Hermitian matrices. Assume that T is also a positive semidefinite and that  $\operatorname{null}(T) \subseteq \operatorname{null}(S)$ . For any  $Z \in \mathbb{C}^{n \times \operatorname{rank}(T)}$  with  $\operatorname{range}(Z) = \operatorname{range}(T)$ , the determined generalized eigenvalues of (S,T) are exactly the generalized eigenvalues of  $(Z^*SZ, Z^*TZ)$ .

*Proof.* We first show that  $Z^*TZ$  has full rank. Suppose that  $Z^*TZv = 0$ . We have  $TZv \in \text{null}(Z^*)$ . Therefore,  $TZv \perp \text{range}(Z) = \text{range}(T)$ . Obviously  $TZv \in \text{range}(T)$ , so we must have v = 0. Since  $\text{null}(Z^*TZ) = \{0\}$ , the matrix  $Z^*TZ$  has full rank.

Suppose that  $\lambda$  is a determined eigenvalue of (S, T). We will show that it is a determined eigenvalue of  $(Z^*SZ, Z^*TZ)$ . The pencil  $(Z^*SZ, Z^*TZ)$  has exactly rank $(Z^*TZ)$  determined eigenvalues. We will show that  $Z^*TZ$  is full rank, so the pencil  $(Z^*SZ, Z^*TZ)$  has exactly rank(T) eigenvalues. Since the pencil (S,T) has exactly rank(T) determined eigenvalues, each of them an eigenvalue of  $(Z^*SZ, Z^*TZ)$ , this will conclude the proof.

Now let  $\mu$  be an eigenvalue of  $(Z^*SZ, Z^*TZ)$ . It must be determined, since  $Z^*TZ$  has full rank. Let y be the corresponding eigenvector,  $Z^*SZy = \mu Z^*TZy$ , and let x = Zy. Now there are two cases. If  $\mu = 0$ , then SZy = Sx = 0 (since  $Z^*$  has full rank and at least as many columns as rows). The vector x is in range $(Z) = \operatorname{range}(T), Tx \neq 0$ . This implies that  $\mu = 0$  is also a determined eigenvalue of (S, T).

If  $\mu \neq 0$ , the analysis is a bit more difficult. Clearly,  $TZy \in \operatorname{range}(T) = \operatorname{range}(Z)$ . But  $\operatorname{range}(Z) = \operatorname{range}(Z^{*+})$  [1, Proposition 6.1.6.vii], so  $Z^{*+}Z^*TZy = TZy$  [1, Proposition 6.1.7]. We claim that  $SZy \in \operatorname{range}(Z)$ . If it is not, it Zy must be in  $\operatorname{null}(T) \subseteq \operatorname{null}(S)$ , but  $\mu$  would have to be zero. Therefore, we also have  $Z^{*+}Z^*SZy = SZy$ , so by multiplying  $Z^*SZy = \mu Z^*TZy$  by  $Z^{*+}$  we see that  $\mu$  is an eigenvalue of (S,T).

We are now ready to prove Theorem 2, the generalization of the Courant-Fischer Minimax Theorem. The technique is simple: we use Lemma 4 to reduce the problem to a smaller-sized full-rank problem, apply Theorem 3 to characterize the determined eigenvalues in terms of subspaces, and finally show a complete correspondence between the subspaces used in the reduced pencil and subspaces used in the original pencil.

*Proof.* (Theorem 2) Let  $Z \in \mathbb{C}^{n \times \operatorname{rank}(T)}$  have  $\operatorname{range}(Z) = \operatorname{range}(T)$ . We have

$$\lambda_k(S,T) = \lambda_k(Z^*SZ, Z^*TZ) = \min_{\dim(W) = k} \max_{x \in W} \frac{x^*Z^*SZx}{x^*Z^*TZx}$$

and

$$\lambda_k(S,T) = \lambda_k(Z^*SZ, Z^*TZ) = \max_{\dim(W) = \operatorname{rank}(T) - k + 1} \min_{x \in W} \frac{x^*Z^*TZx}{x^*Z^*TZx}.$$

The leftmost equality in each of these equations follows from Lemma 4 and the rightmost one follows from Theorem 3.

We now show that for every k-dimensional subspace  $U \subseteq \mathbb{C}^n$  with  $U \perp \text{null}(T)$ , there exists a k-dimensional subspace  $W \subseteq \mathbb{C}^{\operatorname{rank}(T)}$  such that

$$\left\{\frac{x^*Sx}{x^*Tx}: x \in U\right\} = \left\{\frac{y^*Z^*SZy}{y^*Z^*TZy}: y \in W\right\}\,,$$

and vice versa. The validity of this claim establishes the min-max side of the theorem.

We first need to show that  $k \leq \operatorname{rank}(T)$ . This is true because every vector in U is in  $\operatorname{range}(T)$ , so its dimension must be at most  $\operatorname{rank}(T)$ .

Define  $W = \{y \in \mathbb{C}^{\operatorname{rank}(T)} : Zy \in U\}$ . Let  $b_1, \ldots, b_k$  be a basis for U. Because  $U \perp \operatorname{null}(T), b_j \in \operatorname{range}(T)$ , so there is a  $y_j$  such that  $Zy_j = b_j$ . Therefore, dimension of W is at most k. Now let the vectors  $y_i$ 's be a basis of W and define  $b_i = Zy_i$ . The  $b_i$ 's span U, so there are at most k of them, so the dimension of W is at least k. Therefore, it is exactly k.

Every  $x \in U$  is orthogonal to null(T), so it must be in range(T). There exist a  $y \in \mathbb{C}^{\operatorname{rank}(T)}$  such that Zy = x. So we have  $x^*Sx/x^*Tx = y^*Z^*SZy/y^*Z^*TZy$ . Combining with the fact that  $y \in W$ , we have shown inclusion of one side. Now suppose  $y \in W$ . Define  $x = Zy \in U$ . Again we have  $x^*Sx/x^*Tx = y^*Z^*SZy/y^*Z^*TZy$ , which shows the other inclusion.

Now we will show that for every k-dimensional subspace W there is a subspace U that satisfies the claim. Define  $U = \{Zy : y \in W\}$ . Because Z has full rank,  $\dim(U) = k$ . Also,  $U \subseteq \operatorname{range}(Z) = \operatorname{range}(T)$  so  $U \perp \operatorname{null}(T)$ . The equality of the Raleigh-quotient sets follows from taking  $y \in W$  and  $x = Zy \in U$  or vice versa.

\* 7\*777

### References

- Denis S. Bernstein. Matrix Mathematics: Theory, Facts, and Formulas with Applications to Linear Systems Theory. Princeton University Press, 2005.
- [2] Gene H. Golub and Charles F. Van Loan. Matrix Computations (3rd ed.). Johns Hopkins University Press, Baltimore, MD, USA, 1996.

*Current address*: Haim Avron and Sivan Toledo: School of Computer Science, Tel-Aviv University, Esmond Ng: Lawrence Berkeley National Laboratory