Is there a Solution?
Goal

- Undirected Connectivity
- In Random LOGSPACE

Plan

- Introduce Random Walks
Undirected Connectivity

**Instance:**

- An *undirected* graph $G=(V,E)$ and two vertices $s,t \in V$

**Decision Problem:**

- Is there a path in $G$ from $s$ to $t$

**Theorem:**

- $\text{CONN} \in \text{NL}$

**Proof:**

- Nondeterministic walk
Nondeterministic vs. Random

All-powerful guesses
- which neighbor to go to next

Randomly guess
- which neighbor to go to next
Random Walks

0 • Add a **self loop** to each vertex

Start • at \( s \)

Let • \( d_i \) be the **degree** of the current node.

Jump • to each neighbor with probability \( \frac{1}{d_i} \)

Stop • if reach \( t \)
Notation:

• Let $v_t$ denote the vertex visited at time $t$ ($v_0 = s$)
• Let $p_t(i) = Pr[v_t = i]$
Lemma:

• Let $G = (V, E)$ be a connected graph, then for any $i \in V$

$$\lim_{t \to \infty} \pi_t(i) = \frac{d}{2|E|}$$
Lemma:

• If for some $t$, for all $i \in V$

$$p_t(i) = \frac{d_i}{2|E|}$$

then for all $i \in V$

$$p_{t+1}(i) = \frac{d_i}{2|E|}$$
Proof:

\[ \sum d_i = 2|E| \]

**Lemma:** Vertex \( i \) has probability \( d_i \) at time \( t \) \( \Rightarrow \) Vertex \( i \) has the same probability at time \( t+1 \)

Proof:

- Animated
Proof:

• $\Sigma d_i = 2|E|$

**Lemma:** Vertex $i$ has probability $d_i$ at time $t \Rightarrow$ Vertex $i$ has the same probability at time $t+1$.

Proof:

• Animated
Using the Asymptotic Estimate

**Corollary:**

- Starting from any vertex $i$, the expected time till revisit $i$ is $2|E|/d_i$.

**Proof:**

- If expected time were longer, we could not have seen $i$ as many times as we should.
Note that if the right answer is 'NO', we clearly answer 'NO'.

Hence, this random walk algorithm has one-sided error.

Such algorithms are called "Monte-Carlo" algorithms.
The probability we head in the right direction is \( 1/d_s \). But every time we get here, we get a second chance!

**Note:**

- If there is a path, in how many steps do we expect to arrive at \( \dagger \)?
Since expectedly we return to each vertex within $|E|/d_i$ steps.

The walk expectedly heads in the right direction within $2|E|$ steps.

By linearity of the expectation, it is expected to reach $t$ within $d(s,t) \cdot 2 |E| \leq 2|V| \cdot |E|$ steps.
1. Run the random walk from $s$ for $2|V| \cdot |E|$ steps.
2. If node $t$ is ever visited, answer “there is a path from $s$ to $t$”.
3. Otherwise, reply “there is probably no path from $s$ to $t$”.

Complexity
**Theorem**: The above algorithm
- uses logarithmic space
- always right for 'NO' instances.
- errs with probability at most \( \frac{1}{2} \) for 'YES' instances.

To maintain the current position we only need \( \log|V| \) space.

Markov: \( \Pr(X > 2E[X]) < \frac{1}{2} \)
We explored the undirected connectivity problem.
We saw a log-space randomized algorithm for this problem.
We used an important technique called random walks.