0368.4163 Randomness and Computation	May 6, 2009

Homework 3

Lecturer: Ronitt Rubinfeld | *Due Date: May 20, 2009*

1. You are given a 2-SAT formula $\phi(x_1, \ldots, x_n)$. Consider the following algorithm for finding a satisfying assignment:

   - Start with an arbitrary assignment. If it's satisfying, output it and halt.
   - Do s times:
     - Pick an arbitrary unsatisfied clause
     - Pick one of the two literals in it uniformly at random
     - Complement the setting of the chosen literal
     - If the new assignment satisfies ϕ, output the assignment and halt.

   Show that if you pick s to be $O(n^2)$, and ϕ is satisfiable, you will output a satisfying assignment with probability at least 3/4.

   Hint: Show that the cover time of the random walk on the n-node line is $O(n^2)$.

2. A d-regular graph has (K, A)-vertex expansion if $\forall S \subset V, |S| \leq K, |\lambda(S)| \geq A|S|$ where $\lambda(S) = |\{u | \exists v \in S s.t. (u, v) \in E\}|$. For a probability distribution π over $[n]$, the *collision probability* is

   $$||\pi||^2 = \sum_x \pi_x^2$$

   Show that the following is true:

   - $||\pi||^2 \geq \frac{1}{|S(\pi)|}$ where $S(\pi) = \{x | \pi_x > 0\}$.
   - $||\pi||^2 = ||\pi - u||^2 + \frac{1}{n}$ where u is the uniform distribution.
   - If there is a constant $\lambda < 1$ such that the transition matrix of G is such that $|\lambda_2| \leq \lambda$ then for any $\alpha < 1$, G has vertex expansion $(\alpha n, \frac{1}{(1-\alpha)\lambda^2 + \alpha})$.

   Remark: We use the convention that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the random walk matrix.

3. We say that an undirected graph on n nodes is *labeled* if the edges adjacent to each vertex are labeled with numbers from 1 to n, and no two edges are labeled with the same number. An edge may be labeled differently on each of its endpoints.

   Given is a labeled graph G on n nodes, a node v in G, and a string $s = (s_1, \ldots, s_k) \in \{1, \ldots, n\}^\star$. Consider the following procedure. Our initial position is v. In the i-th step, if there is an edge adjacent to the current node, labeled with s_i, we follow that edge. Otherwise, we stay at the current node. We call s a (G, v)-*cover* if it can be used to visit all vertices of G by following to the above procedure.

   A string $s \in \{1, \ldots, n\}^\star$ is a *universal traversal sequence* for size n if for every labeled connected graph G on n nodes and every node v in G, s is a (G, v)-cover.

   Show that there exists a universal traversal sequence for size n of length $n^{O(1)}$.

4. Given graph G (regular, undirected and not bipartite). Let A be a random walker that is starting at the uniform (stationary) distribution on nodes. Let B be a random walker that starts at an arbitrary node u. Let M_G^u be the expected time for A to meet up with B. Give the best general upper and lower bounds that you can on M_G^u.