Sub-linear Algorithms December 28, 2015

Homework 5

Lecturer: Ronitt Rubinfeld *Due Date: January 11, 2016*

Turn in directly to the grader.

1. Say that f_1, f_2, f_3, mapping from group G to H, are *linear consistent* if there exists a linear function $\phi : G \to H$ (that is $\forall x, y \in G, \phi(x) + \phi(y) = \phi(x + y)$) and $a_1, a_2, a_3 \in H$ such that $a_1 + a_2 = a_3$ and $f_i(x) = \phi(x) + a_i$ for all $x \in G$. A natural choice for a test of linear consistency is to verify that

$$Pr_{x,y \in_r G}[f_1(x) + f_2(y) \neq f_3(x + y)] \leq \delta$$

for some small enough choice of δ.

- Assume G, H are Abelian. Show that f, g, h are linear-consistent iff for every $x, y \in G$ $f(x) + g(y) = h(x + y)$.
- Let $G = \{+1, -1\}^n$ and $H = \{+1, -1\}$. First note that since $a_i \in \{+1, -1\}$, then linear consistent f_i must be linear functions or "negations" of linear functions. We refer to the union of linear functions and the negations of linear functions as the *affine functions*. In class we expressed the minimum distance of f to a linear function. Express the minimum distance of a function f to an affine function.
- Show that if f_1, f_2, f_3 satisfy the above test, then for each $i \in \{1, 2, 3\}$, there is an affine function g_i such that $Pr_{x \in_r G}[f_i(x) \neq g_i(x)] \leq \delta$.
- (Extra credit) Show that there are linear consistent functions g_1, g_2, g_3 such that for $i \in \{1, 2, 3\}$, $Pr_{x \in_r G}[f_i(x) \neq g_i(x)] \leq \frac{1}{2} - \frac{2\gamma}{3}$ where $\gamma = \frac{1}{2} - \delta$.

2. For function $f : \{1, -1\}^n \to \{1, -1\}$, the NAE test chooses $x, y, z \in \{1, -1\}^n$ by choosing, independently for each i, the triple (x_i, y_i, z_i) uniformly from the set of "not all equal" triples (that is, all 3-tuples from $1, -1$ except for $(1, 1, 1)$ and $(-1, -1, -1)$). Then the test accepts iff the three outcomes $(f(x), f(y), f(z))$ are not all equal. Show that the probability that the NAE test passes a function f is

$$\frac{3}{4} - \frac{3}{4} \sum_{S \subseteq [n]} \left(\frac{-1}{3}\right)^{|S|} \hat{f}(S)^2$$