Sub-linear Algorithms	January 12, 2015

Homework 6

Lecturer: Ronitt Rubinfeld — *Due Date: January 19, 2015*

Homework guidelines: Same as for homework 1. Automatic extension of one week (to January 26) for whoever wants it.

1. Say that f_1, f_2, f_3, mapping from group G to H, are *linear consistent* if there exists a linear function $\phi : G \to H$ (that is $\forall x, y \in G, \phi(x) + \phi(y) = \phi(x + y)$) and $a_1, a_2, a_3 \in H$ such that $a_1 + a_2 = a_3$ and $f_i(x) = \phi(x) + a_i$ for all $x \in G$. A natural choice for a test of linear consistency is to verify that

$$Pr_{x,y\in_r G}[f_1(x) + f_2(y) \neq f_3(x + y)] \leq \delta$$

for some small enough choice of δ.

   - Assume G, H are Abelian. Show that f, g, h are linear-consistent iff for every $x, y \in G$ $f(x) + g(y) = h(x + y)$.
   - Let $G = \{+1, -1\}^n$ and $H = \{+1, -1\}$. First note that since $a_i \in \{+1, -1\}$, then linear consistent f_i must be linear functions or "negations" of linear functions. We refer to the union of linear functions and the negations of linear functions as the *affine functions*. In class we expressed the minimum distance of f to a linear function. Express the minimum distance of a function f to an affine function.
   - Show that if f_1, f_2, f_3 satisfy the above test, then for each $i \in \{1, 2, 3\}$, there is an affine function g_i such that $Pr_{x\in_r G}[f_i(x) \neq g_i(x)] \leq \delta$.
   - (Extra credit) Show that there are linear consistent functions g_1, g_2, g_3 such that for $i \in \{1, 2, 3\}$, $Pr_{x\in_r G}[f_i(x) \neq g_i(x)] \leq \frac{1}{2} - \frac{2\gamma}{3}$ where $\gamma = \frac{1}{2} - \delta$.

2. For function $f : \{1, -1\}^n \to \{1, -1\}$, the NAE test chooses $x, y, z \in \{1, -1\}^n$ by choosing, independently for each i, the triple (x_i, y_i, z_i) uniformly from the set of "not all equal" triples (that is, all 3-tuples from $1, -1$ except for $(1, 1, 1)$ and $(-1, -1, -1)$). Then the test accepts iff the three outcomes $(f(x), f(y), f(z))$ are not all equal. Show that the probability that the NAE test passes a function f is

$$\frac{3}{4} - \frac{3}{4} \sum_{S\subseteq[n]} \left(\frac{-1}{3}\right)^{|S|} \hat{f}(S)^2$$

3. Show that for any monotone function $f : \{+1, -1\}^n \to \{+1, -1\}$, the influence of the i^{th} variable is equal to the value of the Fourier coefficient of $\{i\}$, that is $\inf_i(f) = \hat{f}(\{i\})$.

4. Show that the majority function $f(x) = \text{sign}(\sum_i x_i)$ maximizes the total influence among n-variable monotone functions mapping $\{+1, -1\}^n$ to $\{+1, -1\}$, for n odd.

5. You are given $n \times n$ matrices A, B, C whose elements are from $\mathcal{Z}_2$ (integers mod 2). Show a (randomized) algorithm running in $O(n^2)$ time which verifies $A \cdot B = C$. The algorithm should always output "pass" if $A \cdot B = C$ and should output "fail" with probability at least 3/4 if $A \cdot B \neq C$. Assume the field operations $+, \times, -$ can be done in $O(1)$ steps.

Useful definitions:

1. For $x = (x_1, \dots, x_n) \in \{+1, -1\}^n$, $x^{\oplus i}$ is x with the i-th bit flipped, that is,

$$x^{\oplus i} = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n).$$

   The *influence of the i-th variable on* $f : \{+1, -1\}^n \to \{+1, -1\}$ is

$$\mathrm{Inf}_i(f) = \Pr_x \left[f(x) \neq f\left(x^{\oplus i}\right) \right].$$

   The *total influence* of f is

$$\mathrm{Inf}(f) = \sum_{i=1}^{n} \mathrm{Inf}_i(f).$$

2. A function $f : \{+1, -1\}^n \to \{+1, -1\}$ is *monotone* if for all $x, y \in \{+1, -1\}^n$ such that $x_i \leq y_i$ for each i, $f(x) \leq f(y)$. Assume that $-1 < +1$.