6.893 Sub-linear Algorithms	December 8, 2014

Homework 3B

Lecturer: Ronitt Rubinfeld — *Due Date: December 22, 2014*

Turn in your solution to each *problem on a separate sheet of paper, with your name on each one.*

1. In class we gave an algorithm $CC(G, d, \epsilon)$, which outputs an additive estimate of the number of connected components of G to within $\epsilon \cdot n$, assuming that G has maximum degree d. In this problem, we show how to use it to construct an algorithm $CC'(G, \bar{d}, \epsilon)$ which outputs an additive estimate of the number of connected components of G to within $\epsilon \cdot n$, assuming that G has *average* degree $\bar{d}$.
   (a) Show that in $O(\bar{d}/\epsilon)$ expected time, we can compute a number d^* such that d^* is the k^{th} largest vertex degree for $\epsilon n/C \le k \le \epsilon n/4$ for some constant C.
   (b) Show that d^* is $O(\bar{d}/\epsilon)$
   (c) If one removes components containing *any* node of degree larger than d^*, how much can that change the number of connected components?
   (d) Briefly explain how to modify CC in order to construct CC'.

2. This problem is about testing monotonicity of functions defined over a directed graph G. The function maps nodes into binary values (i.e., $f : V \to \{0, 1\}$), and we say that it is *monotone* if for all directed edges (u, v), we have that $f(u) \le f(v)$. We say that f is ϵ-close to monotone if there is a monotone function g such that g and f differ on at most $\epsilon|V|$ entries.
   (a) Let $V = \{v_1, \dots, v_n\}$. For each directed graph $G = (V, E)$, let $B_G = (V', E')$ be the bipartite graph where $V' = \{v_1, \dots, v_n\} \bigcup \{v'_1, \dots, v'_n\}$, and $(v_i, v'_j) \in E'$ iff v_j is reachable from v_i in G.
   Show that a q-query testing algorithm for B_G with distance parameter $\epsilon/2$ yields a q-query testing algorithm for G with distance parameter ϵ.
   (b) Let f be a function on V which is ϵ-far from monotone over graph G. Then $TC(G)$ has a matching of violated edges of size at least $\epsilon|V|$. (Recall previous homework set).
   (c) Show that if f is a function over bipartite graph G, there is a test for monotonicity of f with query complexity at most $O(\sqrt{|V|/\epsilon})$.

3. Prove claim 2 from class on 8/12

4. Say that $f : \{0, 1\}^n \to \{0, ..., n\}$ is *monotone* if for all x, y such that $x_i \le y_i$ for $i = 1, \dots, n$, then $f(x) \le f(y)$. Show that distinguishing whether f is monotone from the case that f is ϵ-far from monotone (i.e., there is no monotone g such that f and g differ on at most ϵ-fraction of the domain $\{0, 1\}^n$) requires $\Omega(n)$ queries. Hint: reduce from the communication complexity problem of disjointness. Another hint: Let $|x|$ be the number of 1's in x. Let Alice define $p(x)$ to be -1 if the parity of the input bits in her set is 1, and 1 if the parity is 0. Let Bob define $q(x)$ similarly. Let them compute $h(x) = 2|x| + p(x) + q(x)$.