03668.416701 Sublinear Time Algorithms	December 21,2009

Lecture 10

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1 Lecture Outline

- Testing Dictator functions
- Juntas

2 Recall (from lecture 9)

- A function f is Boolean if f : $\{\pm 1\}^n$ -> {±1}
- A Boolean function f is linear if f(x) f(y) = f(x y)
- χ_y(x) = $\prod_{i=1}^{n} x_i y_i$ [if y =∅ then $\chi_y(x) = 1$]
- Let S⊆ $[n]$ then $\chi_S(x) \equiv \prod_{i \in S} x_i$
- For all linear functions :

a) $\chi_s(x)\chi_s(y) = \chi_s(xy)$

b) f(x) = $\sum_{s \subseteq [n]} \hat{f}(s)\, \chi_s$(x) , where $\hat{f}(z) = \frac{1}{2^n} \sum_x f(x)\, \chi_s(x)$

c) $\chi_s(x)\chi_t(x) = \chi_{s \Delta t}(x)$

d) If f(x) =$\chi_s(x)$ then $\hat{f}(S) = 1$

$\hat{f}(T) = 0$ for all T ≠ S

e) $\hat{f}$(s) = 1- 2 pr[f(x) ≠ $\chi_s(x)$]

f) Plancherel : <f,g> = $\sum_{s \subseteq [n]} \hat{f}(s)\hat{g}(s)$

g) Boolean parseval : $\sum_{s \subseteq [n]} \hat{f}(s)^2 = 1$ (f Boolean)

h) $E_x[\chi_s(x)] = \begin{cases} 1 & if\ s = \emptyset \\ 0 & \text{otherwise} \end{cases}$

3) Testing Dictator functions

the dictator function f: $\{\pm 1\}^n$ → $\{\pm 1\}$, f ∈ [n] are $\chi_{\{1\}}, \chi_{\{2\}}, \dots, \chi_{\{n\}}$

We will drop the set notation and denote them by f(x)= χ_i

Def: H$\tilde{\breve{a}}$stad Test (δ)

- pick x,y $\in_R \{\pm 1\}^{n}$
- pick w $\in_R \{\pm 1\}$ with δ biased distribution ($\mathrm{pr}[\mathrm{w_i} = -1] = \delta$ and $\mathrm{pr}[\mathrm{w_i} = 1] = 1 - \delta$)
- Z $\leftarrow$ X * Y * W (* is coordinate wise multiplication)
- Accept if f(x) f(y) f(z)=1
- Reject otherwise

Thm:

Pr[H$\tilde{\breve{a}}$stad Test (δ) accepts] = $\frac{1}{2} + \frac{1}{2}\sum_{s\subseteq[n]}(1-2\delta)^{|s|}\ \hat{f}(s)^3$

Proof:

Indicator for H$\tilde{\breve{a}}$stad's Accept

1_H(x,y,z) = $\frac{1}{2} + \frac{1}{2}$f(x)f(y)f(z)

Pr[H$\tilde{\breve{a}}$stad Test (δ) accepts] = $\mathrm{E}_{x,y,z}[1_H(x,y,z)]$ =

$= \frac{1}{2} + \frac{1}{2}\mathrm{E}_{x,y,w}[f(x)f(y)f(z)] \equiv$ A

To calculate the value of A, we evaluate the expectation $\mathrm{E}_{x,y,w}[f(x)f(y)f(z)]$

$\mathrm{E}_{x,y,w}[f(x)f(y)f(z)] =$ ($\leftarrow$ by "b" from Recall)

$=\mathrm{E}_{x,y,w}[\sum_{S\subseteq[n]}\hat{f}(S)\chi_S(x)\ \sum_{T\subseteq[n]}\hat{f}(T)\chi_T(y)\ \sum_{U\subseteq[n]}\hat{f}(U)\chi_U(z)] =$

$=\sum_{S\subseteq[n],T\subseteq[n],U\subseteq[n]}\hat{f}(S)\hat{f}(T)\hat{f}(U)\mathrm{E}_{x,y,w}[\chi_S(x)\chi_T(y)\chi_U(z)] \equiv B$

To calculate the value of B, we evaluate the $\mathrm{E}_{x,y,w}[\chi_S(x)\chi_T(y)\chi_U(z)]]$

When $S \subseteq [n], T \subseteq [n]$ and $U \subseteq [n]$.

$\mathrm{E}_{x,y,w}[\chi_S(x)\chi_T(y)\chi_U(z)] =$ ($\leftarrow$ by definition of z)

$= \mathrm{E}_{x,y,w}[\chi_S(x)\chi_T(y)\chi_U(x)\chi_U(y)\chi_U(w)] =$ ($\leftarrow$ by "c" from Recall)

$=\mathrm{E}_{x,y,w}[\chi_{S\Delta U}(x)\chi_{T\Delta U}(y)\chi_{U}(w)] =$

$= \mathrm{E}_x[\chi_{S\Delta U}(x)]\ \mathrm{E}_y[\chi_{T\Delta U}(y)]\ \mathrm{E}_w[\chi_U(w)] \equiv C$

Because $\mathrm{E}_x[\chi_{S\Delta U}(x)] = \begin{cases} 1 & \text{if } S = U \\ 0 & \text{otherwise} \end{cases}$ and $\mathrm{E}_y[\chi_{T\Delta U}(y)] = \begin{cases} 1 & \text{if } t = U \\ 0 & \text{otherwise} \end{cases}$

therefore,

$\mathrm{E}_x[\chi_{S\Delta U}(x)]\ \mathrm{E}_y[\chi_{T\Delta U}(y)] = \begin{cases} 1 & \text{if } S = T = U \\ 0 & \text{otherwise} \end{cases}$ and

$\mathrm{E}_w[\chi_U(w)] = \mathrm{E}_w[\prod_{i\in U} w_i] = \prod_{i\in U} \mathrm{E}_w[w_i]$

$\mathrm{E}_w[w_i] = (-1)\,\delta + (+1)(1-\delta) = 1-2\delta$, therefore, placing them in C we get

⇨ $C = \begin{cases} (1-2\delta)^{|S|} & \text{if } S = T = U \\ 0 & o.w \end{cases}$, and conclude that

⇨ $A = \frac{1}{2} + \frac{1}{2}\sum_{S\subseteq[n]}(1-2\delta)^{|S|}\ \hat{f}(S)^3$

Theorem : "Almost Dictator test"

C = { dictator } $\cup \{1\}$

There is test that makes $O(\frac{1}{\varepsilon^2})$ queries

And if $f \in C$,Pr [$T^f_{(\beta)}$] accepts $\geq 1-\beta$

If f $\varepsilon - far$ from C , Pr[$T^f_{(\beta)}$] reject $\geq 1-\beta$

(For simplicity can think of $\beta = \frac{1}{4}$)

Proof:

Plan:

- Given ε, f
- $\varepsilon \leftarrow \min(\varepsilon, 0.1)$
- Run Hăstad Test (δ) with $\delta = 0.75\,\varepsilon$

Case 1: $f = \chi_i$

Pr[f passes] =(by d from recall)= $\frac{1}{2} + \frac{1}{2}(1-2\delta)\ 1^3 = 1 - \delta = 1 - 0.75\,\varepsilon = 0.25\varepsilon$

Case 2: (proof of "counter positive")

Suppose

$1 - \varepsilon \leq$ Pr[f passes] $= \frac{1}{2} + \frac{1}{2}\sum_{s\subseteq[n]}(1-2\delta)^{|s|}\ \hat{f}(s)^3$ therefore,

⇨ $1 - 2\varepsilon \leq \sum_{s\subseteq[n]}(1-2\delta)^{|s|}\ \hat{f}(s)^3$
$\leq \max_s((1-2\delta)^{|s|}\hat{f}(s))\ \ \sum_{s\subseteq[n]}\hat{f}(s)^2 \equiv$ D
Because $\sum_{s\subseteq[n]}\hat{f}(s)^2 = 1$ (by Boolean Parseval from recall)

D $\leq \max_s((1-2\delta)^{|s|}\hat{f}(s)) \equiv$ K (remember that $(1-2\delta) < 1$)

∃ $\hat{f}$(s) such that $\hat{f}$(s) ≥ 1 – 2 ε

such that dist (f , χ_s) ≤ $\frac{1-(1-2\varepsilon)}{2}$ (by e from recall)

when dist (f , χ_s) ≡ Pr[f(x) ≠ χ_s(x)]

recall that $\delta = 0.75\varepsilon$ therefore,

K = $\max_s(\left(1-\frac{3\varepsilon}{2}\right)^{|s|}\ \hat{f}(s))$

Let denote that |s|≥ 2 , So because of $\hat{f}$(s) ≤ 1

1-2 ε $\leq (1-\frac{3\leq}{2})^2 = 1 - 3\varepsilon + \frac{9}{4}\varepsilon^2$ and that is a contradiction , therefore

$\exists s$ *such that* $|s| \leq 1$ *and* Pr[$f(x) = \chi_s(x)$] ≥ 1 - ε

For conclusion the Test is:

- Given ε, f
- $\varepsilon \leftarrow \min(\varepsilon, 0.1)$
- Run Hăstad Test (δ) with δ = 0.75 ε
- Accept if ≥ 1 – 0.8 ε *fraction of runs accept*
- Reject other wise

For checking dictator without "almost" can be done by few simple checks that it's not "1" by equation h from recall

<u>4) Juntas</u>

<u>Def</u>: f is a k-junta if depends on $\leq$ K vars .

How to find a relevant variable:

- pick X,Y
- if f(X) $\neq$ f(Y)

lets define $X = X_0$=(x_0,…, x_n)

X_1=(y_0,…, x_n)

…

Y= X_n=(y_0,…, y_n)

Therefore, there is i such that $f(X_i) \neq f(X_{i+1})$

And X_i and X_{i+1} differ by only one bit . Therefore, that bit must be the relevant bit.

(*)

- find that i by O(logn) queries (by binary search)

if $f(X_0) \neq f(X_{n/2})$ then

recurse on 0...$\frac{n}{2}$

else recurse on $\frac{n}{2}$+1…n

but that too much queries …

algorithm:

- given k, ε
- randomly partition 1…n into s parts $I_1, \dots, I_s$ (where s=poly(k,$\frac{1}{\varepsilon}$))
- R ← $\emptyset$
- Repeat up to r= O($\frac{k}{\varepsilon}$) times
- Generate (x,y) randomly

  Such that $X_r = Y_r$ ← agree on indices in R

- if f(X) $\neq f(y)$ use binary search to find relevant I_j
- R← R ∪ I_j
- if R has > K relevant parts reject
- pass

Notation:

$X_s \equiv ordered\ list\ (x_i : i\ \in s)$

$X_s Y_s\ \equiv Z = (z_1, \dots, z_n)$ such that $z_s = x_s\ and\ z_{\bar{s}} = y_{\bar{s}}$

Def: "influence" of S ⊆ [n] on f is

$\inf_f(s) = 2\Pr_{x,y}$[f(X) $\neq f(y)$] such that $x_{\bar{s}} = y_{\bar{s}}$

Prop(homework)

$\inf_f(s) = \sum_{T\ s.t\ S\cap T\ \neq\ \theta} \hat{f}(T)^2$

Analysis

- If f is k-junta (pass)
  Because never in more than k relevant parts
- If f $\varepsilon - far$?

Like in (*) but for groups:

Given partition of 1…n into groups

Define: "relevant group" group that contains relevant variable

O(log # groups) queries enough to find a relevant group.

Lemma: (warm up) f $\varepsilon - far$ from k-junta ➔

$\forall \bar{J}$ s.t $|\bar{J}| \leq K$ $\quad 2\Pr_{\substack{x,y \\ \text{s.t } x_{\bar{J}}=y_{\bar{J}}}}[f(X) \neq f(y)] = \inf([n] \setminus \bar{J}) \geq 2\varepsilon$

Proof:

Fix $\bar{J}$ *such that* $|\bar{J}| \leq K$

Define h such that $h(x) \equiv \text{majority}_Z \, f(X_j Z_{\bar{J}}) = \text{sign}(E_z[f(X_j Z_{\bar{J}})])$

- h(x) only depends on X_j
- h is the junta on the variables J that has the best agreement with f.

$2\Pr_x[f(x) \neq h(x)] = 1 - E_x[f(x)h(x)] \equiv D$

$E_x[f(x)h(x)] = (+1)(\Pr[f(x) = h(x)] + (-1)\Pr[f(x) \neq h(x)] =$

$= 1 - 2\alpha$ when $\Pr[f(x)=h(x)] = 1 - \alpha$ → $\Pr[f(x) \neq h(x)] = \alpha$

Therefore, $D = 1 - (1 - 2\alpha) = 2\alpha$

$D = 1 - E_x E_z[f(X_j Z_{\bar{J}}) h(X_j Z_{\bar{J}})] =$ (← by construction of h)

$= 1 - E_x[E_z[f(X_j Z_{\bar{J}})] h(X_j)]$ and $h(X_j) = sign(E[f(X_j Z_{\bar{J}})])$

And because $|g(x)| = g(x)\,\text{sign}(g(x))$ therefore, D = 1 – E_x [| E_z [f$(X_j Z_{\bar{J}})$] |] $\leq$ 1— E_x [E_z $[f(X_j Z_{\bar{J}})\ ]^2]$ ($\leftarrow$ if g<1then g^2< g)

= 1— E_x [E_z $[\sum_s \hat{f}(s)\, \chi_s(x)\ ]^2]$ =

= 1— E_x [$\sum_{s\subseteq J}$ $\sum_{T\subseteq J}$ $\hat{f}(s)\hat{f}(T)$ E_z [$\chi_s(x)\ \chi_T(x)$ $]^2]$ $\equiv$ S

To calculate the value of S, lets evaluate the expectation $E_z[\ f(X_j Z_{\bar{J}})\ ]$

$E_z[\ f(X_j Z_{\bar{J}})\ ]$ = $E_z\ [\sum_s \hat{f}(s)\, \chi_s(X_j Z_{\bar{J}})\ ]$ = $\sum_s \hat{f}(s)$ E_z $[\chi_s(X_j Z_{\bar{J}})$ $\equiv$ T

But $\chi_s(X_j Z_{\bar{J}}) = \begin{cases} X_s\ (X_j) & if\ s \subseteq J \\ 0 & o.w \end{cases}$ therefore,

T = $\sum_{s\subseteq J}$ $\hat{f}(s)\, X_s(x)$ therefore,

S = 1 - $\sum_{s\subseteq J}$ $\hat{f}(s)^2 = \sum_{s\subseteq[n]} \hat{f}(s)^2 - \sum_{s\subseteq J} \hat{f}(s)^2$ = $\sum_{s:s\cap([n]\setminus J)\neq\emptyset} \hat{f}(s)^2$ =

(by home work→) = Inf([n]\j) .

Continue in the next lesson .